

EXACT ZARANKIEWICZ VALUES ON TWO FINITE FRONTIER SLICES

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ABSTRACT. The Zarankiewicz number $Z(m, n, s, t)$ is the maximum number of edges in a bipartite graph with parts of orders m and n containing no copy of $K_{s,t}$. We give one combined, certificate-based computer-assisted proof for two finite slices and a corrected neighboring frontier:

$$\begin{aligned} Z(12, n, 3, 3) &= 6n \quad (18 \leq n \leq 22), & Z(13, 22, 3, 3) &= 137, \\ Z(13, 18, 3, 3) &= 116, & Z(14, 17, 3, 3) &= 118, \\ Z(14, 18, 3, 3) &= 124, & Z(15, 17, 3, 3) &= 126, \\ Z(15, 18, 3, 3) &= 132, & 132 \leq Z(16, 17, 3, 3) &\leq 133. \end{aligned}$$

The load-bearing new upper bounds are the exact 12×18 and 13×18 certificate packages. Their orbit certificates exclude every hypothetical matrix at the next edge count. Deletion lemmas and explicit witnesses close four neighboring cells, while the 16×17 entry is deliberately reported as an interval because only its 132-edge lower witness and the published 133 upper bound are certified here. Separately, the 13×22 proof excludes 138 ones by reducing to 83 degree profiles, rationally separating 77 of them, and eliminating the remaining six by marked-row congruences, leave enumeration, modular Gram tests, and exact Farkas certificates. All accepted claims are replayed by standard-library Python and exact integer/rational arithmetic; floating-point optimization is used only to discover certificates.

1. INTRODUCTION

The Zarankiewicz problem asks for the maximum number of edges in a bipartite graph avoiding a specified complete bipartite subgraph. It was posed by Zarankiewicz in 1951 [8] and was placed in its modern extremal form by Kővári, Sós and Turán [5]. Although the asymptotic theory is extensive, exact values for moderate finite parameters remain difficult. The difficulty is particularly pronounced for $K_{3,3}$ -free graphs: useful counting inequalities leave a small but highly structured set of integer configurations, while direct exhaustive search suffers from severe symmetry and proof-certification costs.

Write $Z(m, n, s, t)$ for the maximum number of edges in a subgraph of $K_{m,n}$ containing no $K_{s,t}$. Roman's linear-programming framework [6] and later strengthened relaxations [2] provide effective finite upper bounds. SAT-based methods have also been developed for exact finite cases [7]. The classical finite table of Collins, Riasanovsky, Wallace and Radziszowski [3] supplies, among other inputs, $Z(13, 17, 3, 3) \leq 110$ and $Z(16, 17, 3, 3) \leq 133$. On the lower-bound side, Bhan, Nobili, Raghuraman and Langer [1] reported the 12-by-22 value and a 137-edge construction for the 13-by-22 parameter, with upper bound 140.

$$137 \leq Z(13, 22, 3, 3) \leq 140.$$

The present paper adds the exact 12-by-18 certificate and its deletion consequences, and completes the 13-by-22 upper bound.

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Theorem 1.1.

$$\begin{aligned}
Z(12, n, 3, 3) &= 6n & (18 \leq n \leq 22), \\
Z(13, 22, 3, 3) &= 137, \\
Z(13, 18, 3, 3) &= 116, \\
Z(14, 17, 3, 3) &= 118, & Z(14, 18, 3, 3) = 124, \\
Z(15, 17, 3, 3) &= 126, & Z(15, 18, 3, 3) = 132, \\
132 \leq Z(16, 17, 3, 3) &\leq 133.
\end{aligned}$$

The values at $n = 18, 19, 20, 21$, the 13-by-22 upper bound, the four neighboring frontier equalities, and the 132-edge 16-by-17 construction are the claims assembled here. The $n = 22$ equality and its classical design are retained as prior material.

The proof is computer-assisted but certificate based. It is not a report of a mixed-integer solver returning “infeasible.” Every load-bearing numerical assertion is replayed by explicit finite data and exact arithmetic. The 12-by- n strip, the 13-by-22 theorem, and the corrected frontier package retain separate certificate directories under `proof/z12_18_21/`, `proof/`, and `proof/frontier_closure/`; the combined verification gate runs the first two and the frontier package’s own gate.

Theorem 1.2. *There is no 13×22 binary matrix with 138 ones and no all-one 3×3 submatrix. Consequently,*

$$\boxed{Z(13, 22, 3, 3) = 137}.$$

For the 13-by-22 component, the proof is computer-assisted but certificate based. It is not a report of a mixed-integer solver returning “infeasible.” Every load-bearing numerical assertion is replayed by short exact-arithmetic programs from explicit finite data. The architecture has three layers.

- (i) A global degree-profile argument reduces 138 edges to 83 profiles and eliminates 77 by exact rational Farkas certificates.
- (ii) A direct marked-row congruence argument excludes the exceptional profile $6^{18}7^39^1$.
- (iii) The remaining five profiles are reduced to finite residual block-design problems on twelve points. Exhaustive leave-multigraph enumeration, necessary Gram conditions, and exact Farkas certificates exclude all residual cases.

The source archive includes both certificate systems, all nested witnesses, independent profile enumerators, and a one-command verification gate. The result should nevertheless be regarded as a new computer-assisted proof pending independent reproduction and peer review.

2. THE EXACT $12 \times n$ STRIP

We first isolate the 12-row result. Identify each column with a support in $[12]$ and let λ_T be the number of columns containing a row triple T . The $K_{3,3}$ -free condition is $\lambda_T \leq 2$ for every $T \in \binom{[12]}{3}$.

The load-bearing upper bound is the exact certificate proof

$$Z(12, 18, 3, 3) \leq 108.$$

Its self-contained verifier enumerates 303 degree profiles for the neighboring 12×17 bound and 51 profiles for the 11×18 bound. Those bounds force any hypothetical 109-one 12×18 matrix into four normalized containment cases. The orbit trees in `proof/z12_18_21/` contain exact rational dual leaves and exhaustive integer branches for all four cases; the standard-library replay accepts every certificate.

The 18-column witness has 108 ones and row-triple histogram (6, 68, 146) for multiplicities (0, 1, 2). Four further witnesses are the first 19, 20, 21, and 22 blocks of the explicit classical 3-(12, 6, 2) Hadamard design listed in Appendix A; their histograms are (3, 54, 163), (1, 38, 181), (0, 20, 200), and (0, 0, 220), respectively. Thus the lower bounds are 108, 114, 120, 126, 132.

Lemma 2.1 (Minimum-column deletion). *For every m, n, s, t with $n \geq 2$,*

$$Z(m, n, s, t) \leq \left\lfloor \frac{n}{n-1} Z(m, n-1, s, t) \right\rfloor.$$

Proof. If an e -edge $m \times n$ matrix existed, some column would have degree at most $\lfloor e/n \rfloor$. Deleting it leaves at least $e - \lfloor e/n \rfloor$ edges in an admissible $m \times (n-1)$ matrix. Taking e one larger than the displayed floor makes this residual count exceed $Z(m, n-1, s, t)$, a contradiction. $\square$

Applying the lemma successively from $Z(12, 18, 3, 3) = 108$ gives

$$Z(12, 19, 3, 3) \leq 114, \quad Z(12, 20, 3, 3) \leq 120, \quad Z(12, 21, 3, 3) \leq 126, \quad Z(12, 22, 3, 3) \leq 132.$$

The nested witnesses attain all four bounds, proving the first line of Theorem 1.1. The $n = 22$ equality is included for a unified statement but is not claimed as a new value.

3. CORRECTED FINITE-FRONTIER CLOSURE

The second reproducibility package, under `proof/frontier_closure/`, proves the new exact value

$$Z(13, 18, 3, 3) = 116.$$

Its verifier reduces a hypothetical 117-edge matrix to 19 column-degree profiles. Thirteen profiles are separated by exact integer-scaled Farkas certificates; the remaining six are eliminated by marked-row deficit averaging, exact point-type enumeration, and 12 symmetry-orbit pair-packing certificates. The packaged 116-edge witness is checked exhaustively over all row triples. The package's optional dependency replay rechecks the earlier 12-by-17 and 12-by-18 certificates.

Four neighboring equalities follow by the one-row density lemma and explicit witnesses. Collins et al. [3] give the published bound $Z(13, 17, 3, 3) \leq 110$, so a 119-edge 14×17 matrix would leave at least

$$119 - \lfloor 119/14 \rfloor = 111 > 110$$

edges after deleting a row; hence $Z(14, 17) \leq 118$. Similarly,

$$125 - \lfloor 125/14 \rfloor = 117 > 116, \quad 127 - \lfloor 127/15 \rfloor = 119 > 118,$$

give the upper bounds $Z(14, 18) \leq 124$ and $Z(15, 17) \leq 126$. Finally,

$$133 - \lfloor 133/15 \rfloor = 125 > 124$$

gives $Z(15, 18) \leq 132$. The corresponding 118-, 124-, 126-, and 132-edge witnesses are stored and checked in the same package.

For 16×17 , the supplied 132-edge witness gives the lower bound $Z(16, 17) \geq 132$, while the published table gives $Z(16, 17) \leq 133$. We therefore record only

$$132 \leq Z(16, 17, 3, 3) \leq 133.$$

No SAT timeout, unsuccessful construction search, or unverified DRAT claim is used in this interval.

Remark 3.1 (Boundary of the retained frontier claims). *If c_{ij} is the number of columns containing both rows i and j , deleting those rows removes $d_i + d_j - c_{ij}$ edges, so the remaining edge count is $e - d_i - d_j + c_{ij}$. Earlier exploratory certificates based on this reduction are not retained here: the explicit 126-edge 15×17 and 132-edge 16×17 witnesses rule out the former 125- and 130-edge exact claims, and no independently replayed certificate for the associated upper-bound claim is included. The paper therefore records only the certified statements in Theorem 1.1.*

4. BLOCK FORMULATION AND THE LOWER BOUND

Let $X = [13]$. Identify the j th column of a binary 13×22 matrix with its support $E_j \subseteq X$, and put $d_j = |E_j|$. For each triple $T \in \binom{X}{3}$, define

$$\lambda_T = |\{j : T \subseteq E_j\}|.$$

The matrix is $K_{3,3}$ -free if and only if

$$(1) \quad \lambda_T \leq 2 \quad (T \in \binom{X}{3}).$$

Repeated columns are allowed throughout.

The supplementary file `proof/data/z13_22_137_blocks.json` contains 22 blocks with total size 137. Direct enumeration of all $\binom{13}{3} = 286$ triples verifies (1). Thus

$$(2) \quad Z(13, 22, 3, 3) \geq 137.$$

The same lower bound was first reported in [1]. A matrix plot of the independently checked witness is shown in Figure 1; its block supports are listed in Section B.

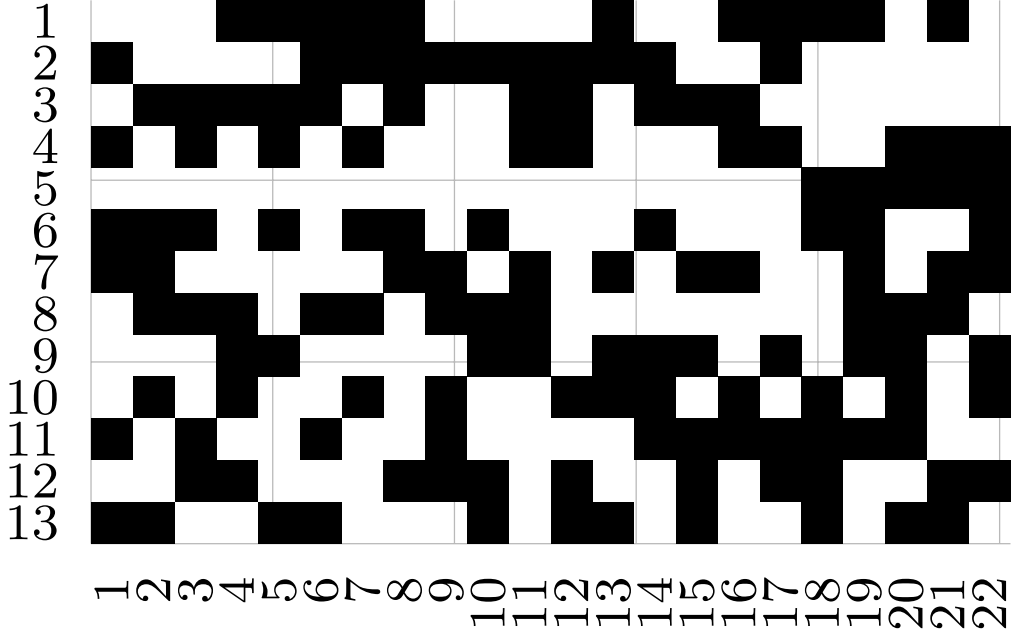


FIGURE 1. A checked 13×22 matrix with 137 ones. Black cells are ones. Every triple of rows is contained in at most two columns.

5. GLOBAL REDUCTION AT 138 EDGES

Assume for contradiction that blocks $E_1, \dots, E_{22}$ satisfy (1) and

$$(3) \quad \sum_{j=1}^{22} d_j = 138.$$

Double counting incidences between blocks and row triples gives

$$(4) \quad \sum_{j=1}^{22} \binom{d_j}{3} = \sum_{T \in \binom{X}{3}} \lambda_T \leq 2 \binom{13}{3} = 572.$$

Define the integer penalty

$$(5) \quad p(d) = \binom{d}{3} - 15d + 70, \quad 0 \leq d \leq 13.$$

Its values are

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$p(d)$	70	55	40	26	14	5	0	0	6	19	40	70	110	161.

Thus $p(d) \geq 0$, with equality precisely for $d = 6, 7$. Combining (3)–(5),

$$(6) \quad \sum_{j=1}^{22} p(d_j) = \sum_j \binom{d_j}{3} - 15 \cdot 138 + 70 \cdot 22 \leq 42.$$

Exact integer enumeration of all histograms $(n_0, \dots, n_{13})$ satisfying

$$\sum_d n_d = 22, \quad \sum_d d n_d = 138, \quad \sum_d p(d) n_d \leq 42$$

leaves exactly 83 degree profiles. This is independently regenerated by `enumerate138.py` and `verify_138_reduction.py`; no degree window is assumed.

5.1. The block-polytope relaxation. For a profile (n_d) , choose a degree f with $n_f > 0$ and, by row symmetry, fix one f -block as $F = \{1, \dots, f\}$. For every remaining allowed support $B \subseteq X$, let $x_B \geq 0$ denote its multiplicity. Every actual matrix yields an integral feasible point of the following relaxation:

$$(7) \quad \sum_{B \supseteq T} x_B \leq 2 - \mathbf{1}[T \subseteq F] \quad (T \in \binom{X}{3}),$$

$$(8) \quad \sum_{B \ni r} \binom{|B| - 1}{2} x_B \leq 132 - \mathbf{1}[r \in F] \binom{f - 1}{2} \quad (r \in X),$$

$$(9) \quad \sum_{B \supseteq P} (|B| - 2) x_B \leq 22 - \mathbf{1}[P \subseteq F] (f - 2) \quad (P \in \binom{X}{2}),$$

$$(10) \quad \sum_{|B|=d} x_B = n_d - \mathbf{1}[d = f] \quad (d \text{ active}).$$

The row and pair inequalities are nonnegative sums of triple-capacity inequalities; they are retained because they yield much shorter dual certificates.

Lemma 5.1 (Exact Farkas criterion). *Write (7)–(9) as $Ax \leq b$, and (10) as $Ex = e$, with $x \geq 0$. If rational vectors $\alpha \geq 0$ and unrestricted β satisfy*

$$A^T \alpha + E^T \beta \geq 0, \quad b^T \alpha + e^T \beta < 0,$$

then the profile is impossible.

Proof. For a feasible x ,

$$0 \leq x^T (A^T \alpha + E^T \beta) = \alpha^T Ax + \beta^T Ex \leq \alpha^T b + \beta^T e < 0,$$

a contradiction. This is the standard alternative theorem of Farkas [4]. $\square$

The verifier regenerates every subset B of each active degree and checks every dual coefficient using `fractions.Fraction`. Stored rational certificates exclude 77 of the 83 profiles. The six not separated by this global relaxation are

$$(11) \quad 6^{18} 7^3 9^1, \quad 6^{16} 7^6, \quad 5^1 6^{14} 7^7, \quad 5^2 6^{12} 7^8, \quad 4^1 6^{13} 7^8, \quad 5^3 6^{10} 7^9.$$

6. MARKED-ROW DEFICITS

For every triple T , put

$$\delta_T = 2 - \lambda_T \geq 0, \quad D_r = \sum_{T \ni r} \delta_T.$$

If

$$s = 572 - \sum_j \binom{d_j}{3}$$

is the unused triple capacity, then

$$(12) \quad \sum_{r \in X} D_r = 3s.$$

Since a fixed row belongs to $\binom{12}{2} = 66$ triples, each of capacity two,

$$(13) \quad D_r = 132 - \sum_{j: r \in E_j} \binom{d_j - 1}{2}.$$

This formula gives both congruence restrictions and a local design interpretation.

6.1. The profile $6^{18}7^39^1$. For this profile, $s = 23$, so $\sum_r D_r = 69$. Contributions from degree-six and degree-seven blocks in (13) are 10 and 15, both zero modulo five; the degree-nine contribution is $28 \equiv 3 \pmod{5}$. Thus

$$D_r \equiv \begin{cases} 2 \pmod{5}, & r \notin E_9, \\ 4 \pmod{5}, & r \in E_9. \end{cases}$$

The residue-minimal total is $4 \cdot 2 + 9 \cdot 4 = 44$. Since the actual total is 69, only five increments of size five are available, and at least eight rows have residue-minimal deficit.

For a row outside the degree-nine block with $D_r = 2$, if a, b count degree-six and degree-seven blocks through r , then (13) gives $2a + 3b = 26$, hence $(a, b) = (13, 0)$ or $(10, 2)$. Deleting r leaves blocks of sizes five and six on twelve points. Pair-capacity degrees imply, respectively, total size-five incidence at most $60 < 65$, or at most $48 < 50$. Both are impossible. Therefore all four rows outside E_9 consume an increment.

At least eight rows inside E_9 consequently have $D_r = 4$. For such a row, $2a + 3b = 20$. The case $(a, b) = (10, 0)$ is ruled out by the residual degree-nine block: its eight points permit at most three size-five incidences and the other four at most five, totaling $44 < 50$. Hence each of these at least eight rows lies in exactly two of the three degree-seven columns. Yet any fixed pair of degree-seven columns can share at most two such rows inside E_9 , since three shared rows together with E_9 would form a $K_{3,3}$. The three pairs account for at most six rows, a contradiction.

Proposition 6.1. *The profile $6^{18}7^39^1$ is impossible.*

7. THE MARKED-ROW LEAVE METHOD

It remains to exclude the five profiles in (11) containing only degrees four through seven. Fix a row r and delete it from every block containing it. The resulting residual blocks lie on the twelve-point set $X \setminus \{r\}$.

Definition 7.1. *The leave multigraph L_r has vertex set $X \setminus \{r\}$ and gives the pair $\{x, y\}$ multiplicity δ_{rxy} . Its total edge multiplicity is D_r and every edge multiplicity lies in $\{0, 1, 2\}$.*

Suppose the residual blocks have active sizes s and that point x occurs in $u_{x,s}$ blocks of size s . Pair capacity through $\{r, x\}$ gives

$$(14) \quad e_x = 22 - \sum_s (s - 1)u_{x,s},$$

where e_x is the degree of x in L_r . In addition,

$$(15) \quad \sum_x u_{x,s} = sn_s, \quad \sum_x e_x = 2D_r.$$

Equations (14)–(15) give a finite list of point-type multisets.

Let M be the point-by-residual-block incidence matrix. Its Gram matrix $Q = MM^T$ is determined by the point types and the leave:

$$(16) \quad Q_{xx} = \sum_s u_{x,s}, \quad Q_{xy} = 2 - \delta_{rxy} \quad (x \neq y).$$

Consequently,

$$(17) \quad \text{rank}_{\mathbb{F}_p}(Q) \leq q$$

for every prime p , where q is the number of residual blocks. When $q = 12$, M is square and

$$(18) \quad \det(Q) = \det(M)^2$$

must be a nonnegative integer square. The verifier uses (17) for $p = 101, 103, 107$ and (18) only as necessary conditions. Cases passing these screens are retained.

For every surviving typed leave, the possible residual blocks form a finite linear factorization system. It requires the prescribed incidence count of every point in every size class, the prescribed pair multiplicities $2 - \delta_{rxy}$, and the prescribed number of blocks of each size. An integral residual design would give a nonnegative integral solution. Each screened case is instead excluded by an integer-scaled rational Farkas vector whose inequalities are checked over every eligible residual block.

Proposition 7.2 (Certified local screen). *For each local parameter tuple invoked below, the program `local_screen_general.cpp` enumerates every point-type distribution satisfying (14)–(15), every loopless leave multigraph with edge multiplicities at most two and the prescribed degree sequence, and every case surviving (17) or (18). The Python verifier then checks an exact Farkas certificate for every retained factorization case.*

The proposition is a finite computational statement. Its implementation is deliberately small: the C++ source is approximately four kilobytes, and the certificate checker uses only the Python standard library. The complete enumeration counts used below are summarized in Table 1.

Branch	Local parameters	type multisets	leaves	certified survivors
Common $D = 7$	$(a, b) = (5, 5)$	19	8,641	0
Common $D = 7$	$(a, b) = (8, 3)$	37	6,733	4
$5^1 6^{14} 7^7$, inside	$(a, b) = (6, 4)$	508	43,715	0
$5^1 6^{14} 7^7$, inside	$(a, b) = (3, 6)$	6	534	0
$5^2 6^{12} 7^8$, $c = 2$	$(10, 1), (7, 3), (4, 5)$	–	–	195+217+28
$5^3 6^{10} 7^9$, $c = 3$	$(8, 2), (5, 4), (2, 6)$	–	–	4,746+656+3
$4^1 6^{13} 7^8$, inside	$(8, 3), (5, 5)$	41+48	367+275	12+3

TABLE 1. Selected exact local-enumeration counts. A dash indicates that only the final screened-case count is used in the proof report.

8. THE COMMON DEFICIT-SEVEN BRANCH

Four of the remaining profiles may force a row outside all degree-five blocks with $D_r = 7$. If a, b denote the numbers of degree-six and degree-seven columns through r , then (13) leaves

$$(a, b) = (5, 5), (8, 3), (11, 1).$$

The $(11, 1)$ branch has no point-type distribution. For $(5, 5)$, all 8,641 leave graphs fail the Gram-rank condition. The $(8, 3)$ branch has 37 point-type distributions and 6,733 leave graphs; four cases survive the Gram screen. Three have immediate exact local Farkas certificates.

The fourth case is fractionally feasible and therefore requires an integral refinement. The verifier enumerates all systems of its three residual size-six blocks, obtaining exactly 285 systems. The full automorphism group preserving the typed leave is $S_2 \times S_3 \times S_6$, of order 8,640, and partitions the systems into three orbits of sizes 15, 180, and 90. The first two representatives have no completion by eight residual size-five blocks; the third has exactly four completions. For each of these four local completions, and for each of the four relevant global profiles, a separate rational completion certificate excludes the remaining columns. Thus sixteen exact completion certificates close the branch.

Lemma 8.1. *No marked row outside all exceptional small blocks can realize the common deficit-seven configuration used in the profiles $6^{16}7^6$, $5^16^{14}7^7$, $5^26^{12}7^8$, or $5^36^{10}7^9$.*

9. ELIMINATION OF THE FIVE FINAL PROFILES

We now combine residue baselines with [Theorems 7.2](#) and [8.1](#).

9.1. The profile $6^{16}7^6$. Here $s = 42$, so $\sum_r D_r = 126$, and every $D_r \equiv 2 \pmod{5}$. If no row had deficit 2 or 7, then every deficit would be at least 12, giving a total at least 156. Every deficit-two local type is rejected by point-type or Gram screening, while every deficit-seven type is rejected by [Theorem 8.1](#). Hence the profile is impossible.

9.2. The profile $5^16^{14}7^7$. Let $c \in \{0, 1\}$ record membership in the unique degree-five block. Then $D_r \equiv 2 - c \pmod{5}$. The minimum cases $D = 2$ for $c = 0$ and $D = 1$ for $c = 1$ are locally impossible. The next baseline has deficit seven on the eight outside rows and six on the five inside rows, totaling

$$8 \cdot 7 + 5 \cdot 6 = 86.$$

The required total is 111, only five increments of size five larger, so some row attains this baseline. Outside rows are impossible by [Theorem 8.1](#); the inside possibilities

$$(a, b) = (3, 6), (6, 4), (9, 2), (12, 0)$$

are all rejected by exact local screening. Thus the profile is impossible.

9.3. The profile $5^26^{12}7^8$. Let $c \in \{0, 1, 2\}$ count the two degree-five blocks through the marked row. The initial cases $(c, D) = (0, 2), (1, 1), (2, 0)$ are impossible. The next residue baseline has total

$$7 \cdot 13 - 10 = 81,$$

whereas the required total is 96. Thus a baseline row exists. The $c = 0$ and $c = 1$ branches reduce to earlier screens. For $c = 2, D = 5$, the possible values are

$$(a, b) = (10, 1), (7, 3), (4, 5), (1, 7).$$

The last has no point types. The first three yield 195, 217, and 28 screened cases, respectively, all excluded by exact Farkas certificates.

9.4. The profile $5^36^{10}7^9$. The initial $c = 0, 1, 2$ cases are again impossible. The next baseline has total

$$7 \cdot 13 - 15 = 76,$$

while the required total is 81, so a baseline row exists. The branches $c = 0, 1, 2$ have already been excluded. For $c = 3, D = 4$, the possibilities

$$(a, b) = (8, 2), (5, 4), (2, 6)$$

produce 4,746, 656, and 3 screened local cases. Every one of the 5,405 factorization systems has an exact integer-scaled Farkas certificate.

9.5. **The profile $4^1 6^{13} 7^8$.** There are nine rows outside and four inside the degree-four block. Their minimum-residue baseline is

$$9 \cdot 2 + 4 \cdot 4 = 34,$$

whereas the required total is 84. If neither minimum occurred, the total would be at least 99, so a row of deficit two outside or deficit four inside must occur. Outside deficit-two rows are impossible. For an inside deficit-four row, the residual exceptional block has size three. The endpoint branches have no point types; the two nontrivial branches leave 12 and 3 screened factorization cases, all excluded by exact certificates.

Proposition 9.1. *None of the five profiles*

$$6^{16} 7^6, \quad 5^1 6^{14} 7^7, \quad 5^2 6^{12} 7^8, \quad 4^1 6^{13} 7^8, \quad 5^3 6^{10} 7^9$$

can occur in a $K_{3,3}$ -free 13×22 matrix with 138 ones.

10. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.2. The witness in Section B proves (2). Suppose a 138-one matrix existed. The penalty enumeration (6) places its degree multiset among 83 profiles. Exact global Farkas certificates exclude 77 profiles. Theorem 6.1 excludes $6^{18} 7^3 9^1$, and Theorem 9.1 excludes the five remaining profiles. Therefore no 138-one matrix exists. Deleting ones shows that no larger $K_{3,3}$ -free matrix exists either, so the checked 137-one construction is optimal. $\square$

11. VERIFICATION, REPRODUCIBILITY, AND TRUST BOUNDARY

The complete proof artifact is designed to separate discovery from verification.

11.1. **One-command verification.** From the repository root, run

```
cd proof
python3 verify_all.py
```

The verifier compiles `local_screen_general.cpp` with a C++17 compiler if needed, then performs the following checks:

1. validates the 137-one witness over all 286 row triples;
2. independently re-enumerates the 83 possible 138-edge degree profiles;
3. replays 77 global rational Farkas certificates;
4. checks the elementary exclusion of $6^{18} 7^3 9^1$;
5. exhaustively regenerates every marked-row point-type and leave case used for the five final profiles;
6. verifies every local and completion certificate coefficient exactly.
7. runs the nested 12-by-18 through 12-by-22 witness scans and deletion-chain checks in `proof/z12_18_21/`.

The expected final line is

```
ALL EXACT-VALUE VERIFICATION GATES PASSED
12-BY-18-THROUGH-22 GATE PASSED
```

The frontier gate is run separately from its directory:

```
cd proof/frontier_closure
python3 verify_all.py
python3 mutation_tests.py
python3 frontier_propagation.py
```

It verifies 13 global and 12 local frontier certificates, six explicit witnesses, the five exact equalities above, and the $132 \leq Z(16, 17) \leq 133$ interval. The optional dependency flag replays the bundled 12-row package.

11.2. Certificate counts. The complete development archive also retains an earlier certified 139-edge exclusion. Across all retained gates, the checker verifies 125,983 candidate-block coefficients for the historical 139 proof, 390,039 for the global 138 reduction, and 5,262,392 for the final local proof, totaling

$$5,778,414$$

exact nonnegativity checks. The 139 component is not logically needed for [Theorem 1.2](#); it is retained as an independent regression test.

11.3. What is and is not trusted. The proof depends on:

- exhaustive finite integer enumeration in the supplied C++ source;
- Python integer and `Fraction` arithmetic;
- explicit Farkas vectors stored in JSON;
- the explicit lower-bound witness.

Floating-point linear programming was used during discovery to locate dual vectors. Before inclusion, these vectors were rationalized or integer-scaled, and the final checker verifies their defining inequalities from scratch. No floating-point solver status, MILP infeasibility flag, tolerance, or randomized search result is a premise.

The only symmetry reduction in the integral common- $D = 7$ branch constructs all 8,640 group elements explicitly, verifies that each preserves point types and leave multiplicities, and checks orbit coverage of all 285 labeled systems. Repeated columns remain permitted throughout.

11.4. Epistemic status. The package has passed its internal exact-arithmetic gate and an adversarial audit targeting omitted degree profiles, hidden simplicity assumptions, incomplete leave enumeration, invalid rank or determinant implications, symmetry loss, and numerical tolerance. Independent reproduction, a second implementation of the local enumerator, and expert peer review remain desirable. Accordingly, this manuscript reports a complete reproducible computer-assisted proof, not a peer-reviewed consensus result.

12. DISCUSSION

The proof illustrates a useful division of labor for finite extremal problems. A coarse global relaxation disposes of most degree patterns efficiently. The few profiles surviving the relaxation are not random hard instances: they are nearly balanced around the zero-penalty degrees six and seven, and their small unused triple capacity forces strong congruences on marked-row deficits. Passing to the leave of a marked row converts those congruences into a compact residual packing problem whose Gram matrix is almost determined.

The method is potentially reusable. For nearby $K_{3,3}$ -free matrix problems, one may seek a supporting line for $d \mapsto \binom{d}{3}$, enumerate low-penalty degree profiles, apply global dual separation, and reserve exact leave enumeration for the integrality-only residue. The computational burden is then concentrated in small, transparent certificate families rather than a monolithic SAT or MILP run.

A literature and repository audit dated 3 August 2026 located no prior publication or public preprint reporting the exact values $Z(12, 18, 3, 3) = 108$, $Z(12, 19, 3, 3) = 114$, $Z(12, 20, 3, 3) = 120$, or $Z(12, 21, 3, 3) = 126$. The audit treats the $n = 22$ value and the 3-(12, 6, 2) design as prior/classical material, and it records the scope and limitation of the search in the accompanying 12-row package.

For $Z(13, 22, 3, 3)$, the 137-edge construction was previously reported while the exact upper bound is supplied here. These priority statements are necessarily provisional and should be rechecked during peer review.

The frontier audit dated 3 August 2026 located no prior indexed report of $Z(13, 18) = 116$, $Z(14, 17) = 118$, $Z(14, 18) = 124$, $Z(15, 17) = 126$, or $Z(15, 18) = 132$, nor of the 132-edge 16×17 construction. This is a dated search finding rather than a universal nonpublication guarantee.

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APPENDIX A. THE NESTED 12-ROW WITNESS FAMILY

The following 22 supports form the verified classical 3-(12, 6, 2) design. Every triple of row labels occurs in exactly two supports. The first n supports therefore give the lower-bound witness for each $18 \leq n \leq 22$. The machine-readable CSV files under `proof/z12_18_21/data/` are the load-bearing representation used by the verifier.

- 1 {2, 7, 8, 9, 11, 12}
- 2 {1, 3, 4, 5, 6, 10}
- 3 {1, 2, 3, 4, 8, 9}
- 4 {2, 4, 5, 8, 10, 11}
- 5 {1, 4, 6, 7, 8, 11}
- 6 {3, 5, 6, 8, 9, 11}
- 7 {1, 2, 3, 5, 7, 11}
- 8 {1, 2, 6, 9, 10, 11}
- 9 {1, 2, 5, 6, 8, 12}
- 10 {3, 4, 5, 7, 8, 12}
- 11 {1, 3, 6, 7, 9, 12}
- 12 {1, 2, 4, 7, 10, 12}
- 13 {2, 3, 5, 9, 10, 12}
- 14 {4, 6, 8, 9, 10, 12}
- 15 {2, 3, 4, 6, 11, 12}
- 16 {1, 4, 5, 9, 11, 12}
- 17 {5, 6, 7, 10, 11, 12}
- 18 {1, 3, 8, 10, 11, 12}
- 19 {1, 5, 7, 8, 9, 10}
- 20 {2, 3, 6, 7, 8, 10}
- 21 {2, 4, 5, 6, 7, 9}
- 22 {3, 4, 7, 9, 10, 11}

APPENDIX B. THE 137-EDGE WITNESS

Rows are labeled $1, \dots, 13$. The following 22 column supports have total size 137:

- | | |
|--------------------------------------|--|
| $B_1 = \{2, 4, 6, 7, 11, 13\}$ | $B_2 = \{3, 6, 7, 8, 10, 13\}$ |
| $B_3 = \{3, 4, 6, 8, 11, 12\}$ | $B_4 = \{1, 3, 8, 9, 10, 12\}$ |
| $B_5 = \{1, 3, 4, 6, 9, 13\}$ | $B_6 = \{1, 2, 3, 8, 11, 13\}$ |
| $B_7 = \{1, 2, 4, 6, 8, 10\}$ | $B_8 = \{1, 2, 3, 6, 7, 12\}$ |
| $B_9 = \{2, 7, 8, 10, 11, 12\}$ | $B_{10} = \{2, 6, 8, 9, 12, 13\}$ |
| $B_{11} = \{2, 3, 4, 7, 8, 9\}$ | $B_{12} = \{2, 3, 4, 10, 12, 13\}$ |
| $B_{13} = \{1, 2, 7, 9, 10, 13\}$ | $B_{14} = \{2, 3, 6, 9, 10, 11\}$ |
| $B_{15} = \{3, 7, 9, 11, 12, 13\}$ | $B_{16} = \{1, 3, 4, 7, 10, 11\}$ |
| $B_{17} = \{1, 2, 4, 9, 11, 12\}$ | $B_{18} = \{1, 5, 6, 10, 11, 12, 13\}$ |
| $B_{19} = \{1, 5, 6, 7, 8, 9, 11\}$ | $B_{20} = \{4, 5, 8, 9, 10, 11, 13\}$ |
| $B_{21} = \{1, 4, 5, 7, 8, 12, 13\}$ | $B_{22} = \{4, 5, 6, 7, 9, 10, 12\}$ |

The first seventeen blocks have size six and the final five have size seven. The accompanying verifier checks that each row triple occurs in at most two blocks.

REFERENCES

- [1] J. Bhan, N. Nobili, S. Raghuraman, and P. Langer, *New bounds for Zarankiewicz numbers via reinforced LLM evolutionary search*, arXiv:2605.01120, 2026.
- [2] S. Davies, P. Gill, and D. Horsley, *Improved upper bounds on Zarankiewicz numbers*, Discrete Math. **349** (2026), no. 5, 114924. doi:10.1016/j.disc.2025.114924.
- [3] A. F. Collins, A. W. N. Riasanovsky, J. C. Wallace, and S. P. Radziszowski, *Zarankiewicz numbers and bipartite Ramsey numbers*, arXiv:1604.01257, 2016. arXiv:1604.01257.
- [4] J. Farkas, *Theorie der einfachen Ungleichungen*, J. Reine Angew. Math. **124** (1902), 1–27. doi:10.1515/crll.1902.124.1.
- [5] T. Kővári, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math. **3** (1954), 50–57. doi:10.4064/cm-3-1-50-57.
- [6] S. Roman, *A problem of Zarankiewicz*, J. Combin. Theory Ser. A **18** (1975), no. 2, 187–198. doi:10.1016/0097-3165(75)90007-2.
- [7] Y. Tan, *An attack on Zarankiewicz’s problem through SAT solving*, arXiv:2203.02283, 2022.
- [8] K. Zarankiewicz, *Problem P 101*, Colloq. Math. **2** (1951), 301.