

Edge-addition monotonicity of positive p -energy fails for every $p \geq 1$

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Abstract

For a graph G with adjacency eigenvalues λ_i , define the positive p -energy by

$$\mathcal{E}_p^+(G) = \sum_{\lambda_i > 0} \lambda_i^p.$$

At a 2021 AIM workshop, Guo conjectured that the positive square energy $s^+ = \mathcal{E}_2^+$ should inherit the familiar edge-addition monotonicity of the spectral radius, $\rho(G + uv) \geq \rho(G)$. That conjecture was subsequently shown to fail at $p = 2$. Tang, Liu, and Wang then introduced positive p -energy, proved nonmonotonicity for every $1 \leq p < 3$, and in version 3 of their preprint (26 March 2025) explicitly conjectured that monotonicity should hold for $p \geq 3$. We disprove this conjectured high-exponent extension completely: for every real $p > 2$ there are infinitely many connected graphs G and nonedges uv such that

$$\mathcal{E}_p^+(G + uv) < \mathcal{E}_p^+(G).$$

Together with the Tang–Liu–Wang counterexamples below 3, this shows that no exponent $p \geq 1$ restores the spectral-radius-style monotonicity: positive p -energy can decrease under the addition of an edge for every real $p \geq 1$.

The construction is a chain of clique blocks joined by regular bipartite graphs. An equitable quotient converges to $Q = I + cA(P_k)$. For noninteger p , a binomial-series sign argument for a fractional power of $I - cA(P_k)$ gives the required negative endpoint entry. At integer exponents, choosing c across the first spectral threshold leaves exactly one negative eigenvalue, and path locality forces the positive spectral contribution to have negative sign. We also give a fully rational 38-vertex certificate at $p = 4$ and determine the complete failure interval of a fixed 17-vertex counterexample at $p = 3$.

1 Introduction

For a simple graph G and a real number $p \geq 1$, let

$$\mathcal{E}_p^+(G) = \sum_{\lambda_i(G) > 0} \lambda_i(G)^p.$$

The surrounding literature begins with classical graph energy and its Schatten-norm extensions; see, for example, Nikiforov [11, 12] and Arizmendi–Guerrero [13]. The separate positive and negative square energies were developed in connection with the Elphick–Farber–Goldberg–Wocjan conjecture [5]; subsequent work includes Abiad et al. [4], Elphick–Linz [6], and the vertex-partitioning inequalities of Akbari et al. [7]. The square-energy conjecture itself was recently resolved by Liu, Tang, and Zhang [8]. Positive and negative p -energies have also appeared in recent extremal and coloring results [9, 10].

A more direct lineage for the present problem comes from spectral-radius monotonicity. If $uv \notin E(G)$, then Perron–Frobenius monotonicity gives $\rho(G + uv) \geq \rho(G)$. At the 2021 AIM

workshop on spectral graph and hypergraph theory, Guo proposed that the positive square energy should satisfy the analogous inequality

$$s^+(G + uv) \geq s^+(G),$$

explicitly as a strengthening of this spectral-radius phenomenon [3]. The conjecture is false: Abiad et al. gave an explicit graph for which deleting one edge increases s^+ [4, Example 2.6], equivalently an edge addition that decreases it.

Tang, Liu, and Wang [1] generalized the problem from $s^+ = \mathcal{E}_2^+$ to positive p -energy. They constructed counterexamples for every $1 \leq p < 3$. Importantly, version 3 of their arXiv preprint did more than pose an open question: after reporting that their constructions did not refute monotonicity for $p \geq 3$, they explicitly reformulated Guo's conjecture as Conjecture 3, asserting that

$$\mathcal{E}_p^+(G + uv) \geq \mathcal{E}_p^+(G) \quad (p \geq 3),$$

for every graph and every missing edge [2, Conjecture 3]. The published version phrases the remaining regime as an open monotonicity problem [1]. Thus the issue is a conjectured extension of spectral-radius monotonicity, not merely an isolated inequality question.

We disprove that conjectured extension completely. The local monotonicity problem is distinct from the global extremal problem of minimizing p -energy or positive p -energy among connected graphs; recent path-minimality results are given in [14, 15].

Theorem 1. *For every real $p > 2$, there are infinitely many connected simple graphs G and nonedges uv such that*

$$\mathcal{E}_p^+(G + uv) < \mathcal{E}_p^+(G).$$

Corollary 1. *For every real $p \geq 1$, positive p -energy is not monotone under edge addition.*

Proof. For $1 \leq p < 3$ this is the result of Tang–Liu–Wang [1]. For $p \geq 3$ it follows from Theorem 1; the overlap $2 < p < 3$ is harmless. $\square$

The key point is that the phenomenon has no upper threshold. In particular, $p = 4$ is not a monotonicity boundary, and neither is any larger exponent. This is compatible with recent global path-minimality results for positive p -energy [15]: a graph parameter can have a global minimum at a path while still decreasing under a particular local edge addition.

An earlier three-block family also gives counterexamples throughout $2 < p < 4$. To make the quantifiers explicit, that auxiliary argument has the form

$$\forall p \in (2, 4) \exists K = K(p) \exists N = N(p, K) \forall n \geq N : \Delta_{n,K}(p) < 0.$$

Thus the block-ratio parameter K is chosen *after* fixing p ; no single K works uniformly all the way up to 4. In fact the nonuniformity becomes acute as $p \uparrow 4$ (rather than as $p \downarrow 2$). The chain construction below makes that auxiliary family unnecessary for the main theorem.

2 The chain construction and the quotient limit

Fix an integer $k \geq 2$. Let $C_1, \dots, C_k$ be pairwise disjoint cliques of order n . Between C_i and C_{i+1} place any d -regular bipartite graph, for $1 \leq i < k$, and place no edges between nonconsecutive blocks. Add two vertices u, v , join u to every vertex of C_1 , and join v to every vertex of C_k . Let $H_{n,d,k}^{(0)}$ be the resulting graph, and set

$$H_{n,d,k}^{(1)} = H_{n,d,k}^{(0)} + uv.$$

For every integer $0 \leq d \leq n$, a d -regular bipartite graph with two parts of size n exists, for example as a union of d edge-disjoint perfect matchings of $K_{n,n}$. For $d \geq 1$ the graph $H_{n,d,k}^{(0)}$ is connected.

The partition

$$\{u\}, C_1, \dots, C_k, \{v\}$$

is equitable (see, e.g., [17] for equitable partitions and quotient spectra). In the normalized block-constant basis, and with $t \in [0, 1]$ temporarily denoting the weight of uv , the quotient matrix is

$$B_t(n, d) = \begin{pmatrix} 0 & \sqrt{n}e_1^\top & t \\ \sqrt{n}e_1 & (n-1)I_k + dA(P_k) & \sqrt{n}e_k \\ t & \sqrt{n}e_k^\top & 0 \end{pmatrix}. \quad (1)$$

The orthogonal complement of the block-constant subspace is invariant: a regular bipartite adjacency matrix sends zero-sum vectors to zero-sum vectors, clique adjacency acts as $-I$ there, and the endpoint vertices couple only to block-constant vectors. The restriction to this orthogonal complement is identical for $t = 0$ and $t = 1$. Hence it cancels from every energy difference.

For a real symmetric matrix $Q = \sum_j \theta_j w_j w_j^\top$ and a real s , define

$$Q_+^s := \sum_{\theta_j > 0} \theta_j^s w_j w_j^\top.$$

When $s = 0$, this is the orthogonal projector onto the positive eigenspace.

Lemma 1 (Quotient perturbation limit). *Fix $k \geq 2$. Let $d = d_n$ satisfy $d_n/n \rightarrow c \in (0, 1]$, and put*

$$Q = I_k + cA(P_k).$$

Assume that Q has no zero eigenvalue. For every fixed real $p > 2$,

$$\lim_{n \rightarrow \infty} \frac{\mathcal{E}_p^+(H_{n,d_n,k}^{(1)}) - \mathcal{E}_p^+(H_{n,d_n,k}^{(0)})}{n^{p-2}} = 2p(Q_+^{p-3})_{1k}. \quad (2)$$

Proof. Write $A = A(P_k)$ and

$$Q_n = \left(1 - \frac{1}{n}\right) I_k + \frac{d_n}{n} A, \quad \tilde{B}_t = \frac{1}{n} B_t(n, d_n), \quad S_n = 0 \oplus Q_n \oplus 0.$$

All estimates below are uniform for $t \in [0, 1]$; the implicit constants may depend on k and Q , but not on t or n . From (1), using the Frobenius norm to bound the operator norm,

$$\sup_{0 \leq t \leq 1} \|\tilde{B}_t - S_n\| \leq \sqrt{\frac{4}{n} + \frac{2}{n^2}} \leq \frac{3}{\sqrt{n}}. \quad (3)$$

Moreover $Q_n \rightarrow Q$ in operator norm. We shall use only the elementary Weyl eigenvalue bound and the Hellmann–Feynman identity for real symmetric matrices; see, e.g., Horn–Johnson [16].

Let $\theta_1, \dots, \theta_k$ be the eigenvalues of Q . They are simple because the eigenvalues of $A(P_k)$ are simple, and they are all nonzero by hypothesis. Set

$$\gamma = \frac{1}{4} \min \left\{ \min_j |\theta_j|, \min_{i \neq j} |\theta_i - \theta_j| \right\} > 0.$$

For all sufficiently large n , the eigenvalues $\theta_{j,n}$ of Q_n satisfy $|\theta_{j,n} - \theta_j| < \gamma/2$, and the perturbation in (3) is $< \gamma/2$. Weyl's inequality therefore gives, for every $t \in [0, 1]$, exactly one eigenvalue $\mu_{j,t}(n)$

of $\tilde{B}_t$ in a fixed γ -neighborhood of each θ_j , while the remaining two eigenvalues lie in a neighborhood of 0 whose radius tends to 0. In particular,

$$\sup_{t \in [0,1]} |\mu_{j,t}(n) - \theta_j| \longrightarrow 0. \quad (4)$$

The separated eigenvalues $\mu_{j,t}(n)$ are simple for all sufficiently large n .

Fix j , and let (x_t, y_t, z_t) be a unit eigenvector of $\tilde{B}_t$ for $\mu_{j,t}(n)$, where $y_t \in \mathbb{R}^k$. The endpoint and core equations are

$$\mu_{j,t} x_t = n^{-1/2} (y_t)_1 + (t/n) z_t, \quad (5)$$

$$(Q_n - \mu_{j,t} I) y_t = -n^{-1/2} (x_t e_1 + z_t e_k), \quad (6)$$

$$\mu_{j,t} z_t = n^{-1/2} (y_t)_k + (t/n) x_t. \quad (7)$$

By the definition of γ and (4), $|\mu_{j,t}| \geq 2\gamma$ for large n . Equations (5) and (7) thus give

$$|x_t| + |z_t| = O(n^{-1/2}) \quad (8)$$

uniformly in t .

The matrices Q_n and Q are affine functions of $A(P_k)$ and hence have the same orthonormal eigenvectors; write w_j for a unit eigenvector belonging to θ_j . By (6) and (8), the residual of y_t in the core equation is $O(n^{-1})$. Expanding y_t in the common eigenbasis of Q_n and Q , the spectral separation by γ implies that every component orthogonal to w_j is $O(n^{-1})$, uniformly in t . Since (8) and unit normalization give $\|y_t\| = 1 + O(n^{-1})$, we obtain

$$y_t = \sigma_{j,t,n} w_j + O(n^{-1}), \quad |\sigma_{j,t,n}| = 1 + O(n^{-1}), \quad (9)$$

uniformly in t . No choice of the sign of the eigenvector is needed below, because only the product of the two endpoint coordinates occurs.

Multiplying (5)–(7) by $\sqrt{n}$ and using (8) gives

$$\sqrt{n} x_t = \frac{(y_t)_1}{\mu_{j,t}} + O(n^{-1}), \quad \sqrt{n} z_t = \frac{(y_t)_k}{\mu_{j,t}} + O(n^{-1}).$$

Together with (4) and (9), this proves the uniform limit

$$\sup_{t \in [0,1]} \left| n x_t z_t - \frac{(w_j)_1 (w_j)_k}{\theta_j^2} \right| \longrightarrow 0. \quad (10)$$

Return to the unscaled eigenvalue $\lambda_{j,t}(n) = n \mu_{j,t}(n)$ of $B_t(n, d_n)$. Since it is simple, the Hellmann–Feynman formula applied to the uv entry gives

$$\frac{d}{dt} \lambda_{j,t}(n) = 2 x_t z_t.$$

If $\theta_j > 0$, then $\lambda_{j,t}(n) > 0$ for all t and all sufficiently large n , and (4)–(10) yield, uniformly in t ,

$$\begin{aligned} \frac{1}{n^{p-2}} \frac{d}{dt} \lambda_{j,t}(n)^p &= 2p \left(\frac{\lambda_{j,t}(n)}{n} \right)^{p-1} (n x_t z_t) \\ &\longrightarrow 2p \theta_j^{p-3} (w_j)_1 (w_j)_k. \end{aligned} \quad (11)$$

If $\theta_j < 0$, the corresponding large eigenvalue remains negative and contributes nothing to positive p -energy. Because the convergence in (11) is uniform on the compact interval $[0, 1]$, integration in t and summation over the positive eigenvalues of Q give

$$2p \sum_{\theta_j > 0} \theta_j^{p-3} (w_j)_1 (w_j)_k = 2p (Q_+^{p-3})_{1k}$$

from the k eigenvalues of order n .

It remains to justify that the two eigenvalues near the two zero eigenvalues of S_n are negligible. Let $\lambda = n\mu$ be either of these eigenvalues and let (x, y, z) be a corresponding unit eigenvector. Weyl's bound and (3) imply

$$\sup_{t \in [0, 1]} |\mu| \longrightarrow 0. \quad (12)$$

Put $\delta = \min_j |\theta_j| > 0$. For large n , uniformly in t and for either stray eigenvalue,

$$\|(Q_n - \mu I)^{-1}\| \leq \frac{2}{\delta}.$$

The unscaled core equation therefore gives the explicit estimate

$$\|y\| \leq \frac{2}{\delta \sqrt{n}} \sqrt{x^2 + z^2}. \quad (13)$$

Writing $q = \sqrt{x^2 + z^2}$, normalization and (13) give $q \geq 1/2$ for all sufficiently large n . The two endpoint equations, viewed as a two-dimensional vector equation, now imply

$$|\lambda|q \leq \sqrt{n} \sqrt{y_1^2 + y_k^2} + tq \leq \left(\frac{2}{\delta} + 1 \right) q.$$

Consequently

$$|\lambda| \leq \frac{2}{\delta} + 1 \quad (14)$$

for both stray eigenvalues, uniformly for $t \in [0, 1]$ and all sufficiently large n . Their total positive p -energy at $t = 0$ and at $t = 1$ is therefore $O(1)$, which is $o(n^{p-2})$ because $p > 2$. This completes the proof of (2). $\square$

3 Noninteger exponents

We first prove Theorem 1 when $p > 2$ is not an integer. Put

$$k = \lfloor p \rfloor \geq 2, \quad r = p - 3.$$

Then

$$k - 3 < r < k - 2. \quad (15)$$

Let $A = A(P_k)$ and choose a rational number

$$0 < c < \frac{1}{\rho(A)} = \frac{1}{2 \cos(\pi/(k+1))}. \quad (16)$$

Then both

$$Q = I + cA, \quad M = I - cA$$

are positive definite. Let

$$D = \text{diag}(1, -1, 1, -1, \dots).$$

Since $DAD = -A$, one has $Q = DMD$ and therefore, by functional calculus,

$$Q^r = DM^r D. \quad (17)$$

Because $c\rho(A) < 1$, the generalized binomial series converges absolutely in operator norm:

$$M^r = (I - cA)^r = \sum_{j=0}^{\infty} \binom{r}{j} (-cA)^j. \quad (18)$$

The distance between the endpoints of P_k is $k - 1$, so

$$(A^j)_{1k} = 0 \quad (j < k - 1),$$

while $(A^{k-1})_{1k} = 1$ and $(A^j)_{1k} \geq 0$ for all j .

For every integer $j \geq k - 1$, condition (15) implies that exactly $j - k + 2$ factors in

$$\binom{r}{j} = \frac{r(r-1) \cdots (r-j+1)}{j!}$$

are negative. Hence

$$\text{sgn} \left(\binom{r}{j} (-1)^j \right) = (-1)^{j-k+2+j} = (-1)^k.$$

Thus every nonzero summand in the $(1, k)$ entry of (18) has the same sign, and the first one is nonzero. Therefore

$$\text{sgn}(M_{1k}^r) = (-1)^k. \quad (19)$$

Since $D_{11}D_{kk} = (-1)^{k-1}$, equations (17) and (19) give the strict sign

$$(Q^r)_{1k} < 0. \quad (20)$$

As Q is positive definite, $Q_+^r = Q^r$.

Write $c = a/b$ in lowest terms and take n through multiples of b , with $d = cn$. Lemma 1 and (20) give

$$\lim_{n \rightarrow \infty} \frac{\mathcal{E}_p^+(H_{n,d,k}^{(1)}) - \mathcal{E}_p^+(H_{n,d,k}^{(0)})}{n^{p-2}} = 2p(Q^r)_{1k} < 0.$$

Hence every sufficiently large such n is a counterexample, and there are infinitely many.

4 Integer exponents

Now let $p = m \geq 3$ be an integer, and put

$$\vartheta = \frac{\pi}{m+1}.$$

Choose a rational $c \in (0, 1)$ such that

$$\frac{1}{2 \cos \vartheta} < c, \quad 1 - 2c \cos(2\vartheta) > 0. \quad (21)$$

Such a choice always exists. The first lower bound is below 1; for $m = 3$ the second inequality is automatic, while for $m \geq 4$ the strict inequality $\cos \vartheta > \cos(2\vartheta)$ leaves a nonempty interval before the second spectral threshold.

The eigenvalues of

$$Q = I_m + cA(P_m)$$

are

$$\theta_j = 1 + 2c \cos \frac{j\pi}{m+1}, \quad 1 \leq j \leq m.$$

By (21), $\theta_m < 0 < \theta_{m-1}$. Thus Q has exactly one negative eigenvalue and no zero eigenvalue.

Set $r = m - 3$. Since the endpoints of P_m have distance $m - 1$ and $r < m - 1$,

$$(Q^r)_{1m} = 0. \tag{22}$$

Let w_m be the normalized sine eigenvector corresponding to θ_m . Then

$$w_m(1)w_m(m) = \frac{2}{m+1}(-1)^{m+1} \sin^2 \vartheta.$$

Since $\theta_m < 0$ and $r = m - 3$,

$$\theta_m^r w_m(1)w_m(m) > 0.$$

Using the spectral decomposition in (22), and recalling that θ_m is the only negative eigenvalue, gives

$$0 = (Q_+^r)_{1m} + \theta_m^r w_m(1)w_m(m).$$

Therefore

$$(Q_+^{m-3})_{1m} < 0. \tag{23}$$

As in the noninteger case, take n through multiples of the denominator of c and put $d = cn$. Lemma 1 yields

$$\lim_{n \rightarrow \infty} \frac{\mathcal{E}_m^+(H_{n,d,m}^{(1)}) - \mathcal{E}_m^+(H_{n,d,m}^{(0)})}{n^{m-2}} = 2m(Q_+^{m-3})_{1m} < 0.$$

This proves Theorem 1 at every integer $m \geq 3$, and together with the previous section completes its proof for every real $p > 2$.

5 An explicit $p = 4$ certificate

Although the asymptotic theorem already settles $p = 4$, a concrete exact example is useful. Take four cliques $C_1, C_2, C_3, C_4 \cong K_9$, join consecutive cliques completely, add vertices u, v , join u to all of C_1 and v to all of C_4 , and omit uv . Call the resulting 38-vertex graph J_0 , and put $J_1 = J_0 + uv$.

An equitable-partition calculation gives

$$\chi_{J_0}(x) = (x+1)^{32}(x^3 - 25x^2 + 46x + 153)(x^3 - 7x^2 - 98x - 9), \tag{24}$$

$$\chi_{J_1}(x) = (x+1)^{34}(x+7)(x-14)(x^2 - 27x + 98). \tag{25}$$

Let $-a$ be the negative root of the first cubic in (24), and let $-b, -c$ be the two negative roots of the second, with $b > c > 0$. Their relevant negative spectra are

$$J_0 : \quad -1 \text{ (32 times)}, -a, -b, -c, \quad J_1 : \quad -1 \text{ (34 times)}, -7.$$

Direct rational evaluations give

$$a < \frac{421}{250}, \quad b < \frac{6939}{1000}, \quad c < \frac{93}{1000}. \quad (26)$$

Indeed a is the unique positive root of $a^3 + 25a^2 + 46a - 153$, while b, c are the two positive roots of $z^3 + 7z^2 - 98z + 9$; the derivatives have fixed signs on the stated isolating intervals.

Adding uv creates no new 4-cycle and both endpoint degrees are 9. From

$$\text{tr } A^4 = 2|E| + 4 \sum_w \binom{d(w)}{2} + 8C_4$$

we obtain

$$\text{tr } A(J_1)^4 - \text{tr } A(J_0)^4 = 2 + 4(9 + 9) = 74.$$

Since total fourth energy is the fourth spectral moment,

$$\mathcal{E}_4^+(J_1) - \mathcal{E}_4^+(J_0) = a^4 + b^4 + c^4 - 2329.$$

Using (26),

$$\mathcal{E}_4^+(J_1) - \mathcal{E}_4^+(J_0) < -\frac{1281106782111}{500000000000} < 0. \quad (27)$$

Numerically the difference is approximately -3.8556020556 .

6 A 17-vertex exact counterexample and its full failure interval

Take $n = m = 5$. Write $G_0 = G_{5,5}^{(0)}$ and $G_1 = G_{5,5}^{(1)}$. In this case

$$\begin{aligned} \det(xI - P_0) &= x^3 - 8x^2 - 39x + 20 =: f_0(x), \\ \det(xI - P_1) &= x^3 - 9x^2 - 31x + 54 =: f_1(x), \\ \det(xI - M_0) &= (x - 5)(x + 1), \\ \det(xI - M_1) &= x^2 - 3x - 9. \end{aligned}$$

Hence

$$\chi_{G_0}(x) = (x + 1)^{13}(x - 5)f_0(x), \quad (28)$$

$$\chi_{G_1}(x) = (x + 1)^{12}(x^2 - 3x - 9)f_1(x). \quad (29)$$

Because P_0 and P_1 are real symmetric, all roots of the two cubics are real. Their determinants are negative (the cubic constant terms are positive) while their traces are 8 and 9, respectively; hence each cubic has exactly one negative root. Write those roots as $-a$ and $-b$, respectively. Then

$$a^3 + 8a^2 - 39a - 20 = 0, \quad (30)$$

$$b^3 + 9b^2 - 31b - 54 = 0. \quad (31)$$

The negative eigenvalue of M_1 has absolute value

$$c = \frac{3(\sqrt{5} - 1)}{2}.$$

Because $N_{G_0}(u) \cap N_{G_0}(v) = \emptyset$, adding uv creates no triangle. Therefore

$$\text{tr } A(G_1)^3 = \text{tr } A(G_0)^3,$$

and

$$\Delta(3) := \mathcal{E}_3^+(G_1) - \mathcal{E}_3^+(G_0) = -55 + 27\sqrt{5} + b^3 - a^3. \quad (32)$$

On $[3, \infty)$ the polynomials in (30)–(31) are strictly increasing, and direct evaluation at 3 and 4 places both a and b in (3, 4). Exact evaluations then give

$$a > \frac{753}{200}, \quad b < \frac{454}{125}, \quad \sqrt{5} < \frac{2237}{1000}.$$

Consequently

$$\Delta(3) < -\frac{59470157}{10^9} < 0. \quad (33)$$

This alone gives a short exact 17-vertex counterexample at the endpoint $p = 3$ of the previously open range.

We now characterize all p for which this same graph pair is a counterexample.

Lemma 2 (Generalized Descartes rule). *Let*

$$F(s) = \sum_{i=1}^r c_i a_i^s, \quad 0 < a_1 < a_2 < \cdots < a_r, \quad c_i \neq 0.$$

Then the number of distinct real zeros of F is at most the number of sign changes in the sequence $(c_1, \dots, c_r)$.

Proof. Set $x = e^s$ and $r_i = \log a_i$. Multiplying by the positive factor x^{-r_1} reduces the claim to a generalized polynomial

$$f(x) = c_1 + \sum_{i=2}^r c_i x^{\rho_i}, \quad 0 < \rho_2 < \cdots < \rho_r, \quad x > 0.$$

We induct on r . The derivative has coefficient signs $(c_2, \dots, c_r)$, since all ρ_i are positive. If c_1 and c_2 have opposite signs, Rolle's theorem gives

$$Z(f) \leq Z(f') + 1,$$

matching the additional sign change at the front. If c_1 and c_2 have the same sign, then for sufficiently small $x > 0$ both $f(x)$ and $f'(x)$ have that sign. If f has a first positive zero, f' must therefore have a zero at or before it; together with the Rolle zeros between consecutive zeros of f , this gives

$$Z(f) \leq Z(f').$$

The induction hypothesis completes the proof. $\square$

Let $x_0 < X_0$ be the two positive roots of f_0 , and $x_1 < X_1$ the two positive roots of f_1 . The positive eigenvalues are therefore

$$\{x_0, 5, X_0\} \quad \text{for } G_0, \quad \left\{x_1, \frac{3+3\sqrt{5}}{2}, X_1\right\} \quad \text{for } G_1.$$

They satisfy the strict ordering

$$0 < x_0 < x_1 < \frac{3 + 3\sqrt{5}}{2} < 5 < X_0 < X_1. \quad (34)$$

Indeed, $f_0(0) > 0 > f_0(1)$, while $f_1(1) > 0 > f_1(2)$; also $(3 + 3\sqrt{5})/2 < 5$. Next,

$$f_0\left(\frac{56}{5}\right) = -\frac{1924}{125} < 0,$$

so $X_0 > 56/5$. Since

$$f_1(x) - f_0(x) = -x^2 + 8x + 34$$

is decreasing for $x > 4$ and is already negative at $56/5$, we have $f_1(X_0) < 0$, hence $X_1 > X_0$.

Thus

$$\Delta(p) = -x_0^p + x_1^p + \left(\frac{3 + 3\sqrt{5}}{2}\right)^p - 5^p - X_0^p + X_1^p. \quad (35)$$

The coefficient signs in increasing order of the bases are

$$-, +, +, -, -, +,$$

which have exactly three sign changes. Lemma 2 therefore implies that Δ has at most three distinct real zeros. One is the trivial zero

$$\Delta(0) = 0.$$

We next certify signs at $p = 2, 3, 4$. Since one edge is added,

$$\text{tr } A(G_1)^2 - \text{tr } A(G_0)^2 = 2.$$

Using the negative spectra from (28)–(29),

$$\Delta(2) = a^2 - b^2 + \frac{9\sqrt{5} - 21}{2}. \quad (36)$$

The additional exact bound

$$\sqrt{5} > \frac{559}{250}$$

gives

$$\Delta(2) > \frac{545801}{10^6} > 0. \quad (37)$$

Equation (33) gives $\Delta(3) < 0$.

Finally, the standard closed-walk identity

$$\text{tr } A^4 = 2|E| + 4 \sum_w \binom{d(w)}{2} + 8C_4$$

shows that

$$\text{tr } A(G_1)^4 - \text{tr } A(G_0)^4 = 42 :$$

the new edge contributes 2, the two endpoint degrees are both 5, and no new 4-cycle is created. Hence

$$\Delta(4) = 43 + a^4 - b^4 - c^4. \quad (38)$$

Using the same rational bounds as above yields

$$\Delta(4) > \frac{929118595792143}{16000000000000} > 0. \quad (39)$$

We can now state the complete characterization.

Theorem 2. *For the 17-vertex pair (G_0, G_1) there exist unique numbers*

$$\alpha \in (2, 3), \quad \beta \in (3, 4)$$

such that, for every $p \geq 1$,

$$\mathcal{E}_p^+(G_1) < \mathcal{E}_p^+(G_0) \iff \alpha < p < \beta.$$

Numerically,

$$\alpha \approx 2.4322519338217608578, \quad \beta \approx 3.0408359954191677907.$$

Proof. Continuity together with (37) and (33) produces a zero $\alpha \in (2, 3)$; continuity together with (33) and (39) produces a zero $\beta \in (3, 4)$. Along with $p = 0$, these are three distinct real zeros. By Lemma 2, there can be no others. The signs at 2, 3, 4 then force exactly the stated sign pattern on $p \geq 1$. $\square$

7 Discussion

Corollary 1 gives a complete negative resolution of the spectral-radius-style monotonicity program above. In particular, it disproves Conjecture 3 of Tang–Liu–Wang, arXiv v3 [2]: there is no exponent $p \geq 1$ for which positive p -energy is universally monotone under adding a missing edge. Thus, although the spectral radius itself satisfies $\rho(G + uv) \geq \rho(G)$, none of the positive p -energies inherits a universal analogue. The mechanism is different at integer and noninteger exponents. Away from the integers, a fractional-power binomial series has a rigid endpoint sign dictated by path distance. At an integer, that leading term vanishes by locality, and moving the quotient across its first spectral threshold creates one negative mode whose removal from the positive spectral power restores a strict negative sign.

The 38-vertex example shows that the failure at $p = 4$ can be certified with elementary rational arithmetic. The 17-vertex example gives a different perspective: for one fixed graph pair the failure occurs only on a bounded interval of exponents, even though the family construction produces counterexamples at every exponent by changing the graph.

Recent work proves path-minimality of total p -energy for every $p \geq 2$ [14] and path-minimality of positive p -energy in several regimes, including all connected graphs at $p = 4$ [15]. Those results sit alongside recent lower bounds and structural inequalities for positive and negative p -energies [7, 9, 10, 8]. Equation (27) emphasizes that such global extremal statements do not imply local edge-addition monotonicity.

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