

Distributed Algorithms for α -Potential Functions in General-Sum Games

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Abstract— We study the problem of computing the tightest α -potential approximation of a general-sum game over continuous action spaces, within a prescribed class of potential functions and when each player has access only to its own utility function. The difficulty is twofold: the approximation error involves a worst-case search over an infinite set of unilateral deviations, and the required utility information is distributed across players.

For a linear-in-parameters potential class, we use an exact finite-tuple reformulation that separates the problem into a global outer search over deviation tuples and distributed convex inner problems. We develop a primal–dual inner oracle tailored to this structure and establish a uniform one-sided accuracy guarantee. This oracle can be combined with global outer search to obtain an end-to-end guarantee on the outer optimization error. We also develop a projected zeroth-order outer method as a computationally lighter alternative for higher-dimensional problems. Numerical experiments illustrate the accuracy–computation tradeoff between the two outer-search methods and show that the proposed optimization framework can improve upon analytical α -potential constructions.

I. INTRODUCTION

Strategic interactions with heterogeneous and coupled objectives arise in networked control, resource allocation, and multi-agent learning [1], [2]. Potential games simplify the analysis of such systems by representing players’ unilateral utility changes through a common scalar function [3]. Exact potential functions, however, need not exist in general-sum games. Near-potential games instead quantify the mismatch through an approximation parameter α ; smaller α yields tighter equilibrium and learning guarantees for the original game [4]–[6]. This motivates computing the smallest approximation error achievable within a prescribed class of potential functions.

For finite-action games, unilateral deviations can be enumerated and the nearest potential can be computed by linear programming [7]. The continuous-action setting is substantially more difficult: the deviation space is infinite, and evaluating the uniform full-domain approximation error requires a generally nonconcave global maximization over deviations, even when the objective is convex in the potential parameters. Moreover, in many multi-agent systems utility information is naturally distributed: each player can evaluate its own utility but need not know the utility functions of other players. A centralized computation would therefore require collecting player-specific utility information. We instead seek

a distributed procedure that keeps these utility evaluations local and coordinates the computation through communication among players.

Existing continuous-action approaches typically exploit additional game structure rather than directly optimizing the full-domain error. For example, explicit or analytical α -potential constructions have been developed for asymmetric network games [6], dynamic congestion games [5], and dynamic N -player games [8], among others. Such constructions provide useful analytical bounds but need not yield the tightest approximation for a particular game instance. NePPO [9] instead optimizes an equilibrium-local proxy rather than the uniform full-domain criterion. Motivated by this gap, we study the distributed computation of the tightest full-domain approximation over continuous actions when each player has access only to its own utility function.

We consider a potential class that is linear in its parameters, while allowing nonlinear dependence on the players’ actions through a prescribed feature map. Using the finite-tuple reformulation of [10], we express the original problem as a global outer maximization over finite collections of deviations, where evaluating each candidate tuple requires solving a convex optimization problem over the potential parameters. This yields a modular two-level architecture: the inner problem can be solved using distributed optimization with local utility evaluations, while different search procedures can be used for the outer problem.

For the inner problem, general distributed constrained-optimization methods could in principle be applied [11]–[13]. We instead develop an oracle tailored to the finite-maximum structure of our problem. Each player maintains a single scalar constraint multiplier, evaluates only its local utility, and communicates its primal iterate with neighboring players. The resulting network-averaged output provides a one-sided estimate of the finite-tuple value with an error bound that is uniform over all queried tuples. This uniform oracle property allows us to combine the distributed inner solver with global tree search method EXOTIC [10] and obtain an end-to-end guarantee for the outer objective. We also consider projected zeroth-order ascent as a computationally lighter alternative when hierarchical global search becomes expensive. Our main contributions are as follows.

(1) We formulate the distributed computation of the tightest near-potential approximation, within a prescribed linear-in-parameters potential class, for continuous-action general-sum games. Applying the finite-tuple reformulation of [10], we obtain an equivalent outer maximization over deviation tuples with convex inner problems. We also give sufficient conditions under which an optimizer of the finite-tuple problem

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recovers an optimizer of the original full-domain problem (Proposition 4).

(2) We develop a distributed primal–dual oracle tailored to these inner problems. By retaining their finite-maximum structure, each player uses one scalar constraint multiplier, accesses only its local utility, and exchanges only its primal iterate with neighboring players. For an inner iteration budget K , we establish a uniform one-sided oracle error of order $\tilde{O}(K^{-1/4})$ over all deviation tuples (Theorem 7).

(3) We combine the distributed oracle with EXOTIC and derive an end-to-end convergence guarantee for the optimal near-potential value (Theorem 12). Together with the structural recovery conditions of Proposition 4, this further yields asymptotic recovery of an optimal potential parameter. We additionally develop a projected zeroth-order outer method as a scalable heuristic that avoids hierarchical tree search. Numerical experiments compare the resulting accuracy–computation tradeoff and show that the optimization-based approach can improve upon analytical near-potential constructions.

a) Notation: For a scalar λ , let $[\lambda]_+ := \max\{\lambda, 0\}$. Given a nonempty closed convex set Ω , let $P_\Omega(x)$ denote the Euclidean projection of x onto Ω . For a convex function f , $\partial f(x)$ denotes its sub-differential at x , while $\nabla_x f$ denotes its gradient with respect to x whenever it exists. The communication network is represented by a connected undirected graph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$, where $\mathcal{V}_\mathcal{G}$ is identical to the player set $\mathcal{I}$ and $\mathcal{E}_\mathcal{G} \subseteq \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$ is the edge set. The neighbor set of player i is $\mathcal{N}_i := \{j \in \mathcal{I} : (i, j) \in \mathcal{E}_\mathcal{G}\}$. Let $A_\mathcal{G}$ be the adjacency matrix, with $[A_\mathcal{G}]_{ij} = 1$ if $(i, j) \in \mathcal{E}_\mathcal{G}$ and zero otherwise. Let $D_\mathcal{G} := \text{diag}(d_1, \dots, d_N)$, where $d_i := \sum_j [A_\mathcal{G}]_{ij}$, and define the graph Laplacian as $L_\mathcal{G} := D_\mathcal{G} - A_\mathcal{G}$.

II. PROBLEM FORMULATION

Consider a strategic-form game $\Gamma = (\mathcal{I}, \{\mathcal{A}_i\}_{i \in \mathcal{I}}, \{u_i\}_{i \in \mathcal{I}})$ with player set $\mathcal{I} := \{1, \dots, N\}$, nonempty compact action sets $\mathcal{A}_i \subseteq \mathbb{R}^{d_i}$, and Lipschitz continuous utility functions $u_i : \mathcal{A} \rightarrow \mathbb{R}$, where $\mathcal{A} := \prod_{i \in \mathcal{I}} \mathcal{A}_i$. For each $i \in \mathcal{I}$, write $a = (a_i, a_{-i})$, where a_{-i} denotes the actions of all other players. Let $\mathcal{Y} := \mathcal{A}^2$. For $y = (a, a') \in \mathcal{Y}$ and $q : \mathcal{A} \rightarrow \mathbb{R}$, define the unilateral difference $\Delta_q(y; i) := q(a'_i, a_{-i}) - q(a_i, a_{-i})$.

Definition 1 (Exact potential function [3]): A function $\phi : \mathcal{A} \rightarrow \mathbb{R}$ is an exact potential function for Γ if $\Delta_\phi(y; i) = \Delta_{u_i}(y; i)$ for all $i \in \mathcal{I}$ and $y \in \mathcal{Y}$. A game admitting such a function is an exact potential game.

General-sum games need not admit an exact potential function. We therefore consider approximate potentials that uniformly bound the discrepancy between unilateral potential and utility differences.

Definition 2 (α -potential function): For $\alpha \geq 0$, a function $\phi : \mathcal{A} \rightarrow \mathbb{R}$ is an α -potential function for Γ if $|\Delta_\phi(y; i) - \Delta_{u_i}(y; i)| \leq \alpha$ for all $i \in \mathcal{I}$ and $y \in \mathcal{Y}$.

The case $\alpha = 0$ recovers an exact potential function. Our objective is to find the smallest α achievable within a prescribed class of candidate potentials. We consider

$$\phi_\theta(a) := \langle \theta, \psi(a) \rangle, \quad \theta \in \Theta \subseteq \mathbb{R}^d, \quad (1)$$

where $\psi : \mathcal{A} \rightarrow \mathbb{R}^d$ is a prescribed Lipschitz continuous feature map and Θ is nonempty, compact, and convex¹.

For each $i \in \mathcal{I}$ and $y \in \mathcal{Y}$, define the signed approximation error $\ell_i(\theta, y) := \Delta_{\phi_\theta}(y; i) - \Delta_{u_i}(y; i)$, and let

$$f(\theta, y) := \max_{i \in \mathcal{I}} \ell_i(\theta, y), \quad \mathcal{E}(\theta) := \max_{y \in \mathcal{Y}} f(\theta, y). \quad (2)$$

Since $\mathcal{Y}$ includes both directions of every unilateral deviation, $\mathcal{E}(\theta) = \max_{i \in \mathcal{I}, y \in \mathcal{Y}} |\ell_i(\theta, y)|$. Thus, $\mathcal{E}(\theta)$ is the smallest α for which ϕ_θ is an α -potential function. The resulting optimization problem is

$$p^* := \min_{\theta \in \Theta} \mathcal{E}(\theta) = \min_{\theta \in \Theta} \max_{y \in \mathcal{Y}} f(\theta, y), \quad (3)$$

with an optimal parameter denoted by $\theta^* \in \arg \min_{\theta \in \Theta} \mathcal{E}(\theta)$.

Computing the optimal approximation in (3) presents two main challenges. First, although the objective is convex in the potential parameters, its evaluation requires solving a generally nonconcave maximization over the continuous space of unilateral deviations. Second, evaluating the objective requires information about all players' utilities, whereas each player has access only to its own utility function.

We address these challenges in the next two sections. Section III derives an equivalent finite-tuple reformulation that separates the global nonconvex search from convex optimization over the potential parameters. Section IV develops a distributed algorithm for solving these convex subproblems using local utility information and combines it with outer optimization methods.

Remark 3 (Choice of parameterization): Linearity in θ makes each $\ell_i(\cdot, y)$ affine and hence $f(\cdot, y)$ and $\mathcal{E}$ convex, while expressiveness is determined by ψ . For a finite action space $\mathcal{A} = \{a^1, \dots, a^M\}$, choosing $d = M$ and $\psi(a^m) = e_m$ recovers any tabular potential permitted by Θ . On compact continuous action spaces, expanding a polynomial basis yields uniform approximation of continuous potentials by the Stone–Weierstrass theorem [14]. The features may instead be fixed neural-network representations or discounted state-action occupancy measures, as in [8, Section 5]. Thus, linearity in the parameters still permits nonlinear dependence on actions or policies. Nevertheless, f remains generally nonconcave in y , making (3) convex–nonconcave minimax problem.

III. FINITE-TUPLE REFORMULATION

We begin by reformulating the convex–nonconcave minimax optimization problem (3) to nonconcave convex maximin optimization problem using results in [10]. The key idea is to replace the maximization over the continuous deviation space with an outer search over finite collections of deviations. For each such collection, the corresponding optimization over the potential parameters is convex.

Specifically, let $\mathcal{W} := \mathcal{Y}^{d+1}$ denote the set of tuples containing $d+1$ deviations. For each $w = (y_1, \dots, y_{d+1}) \in$

¹Although we use linear parametrization of potential function in (1), the results in this paper extend to convex parameterization setting.

$\mathcal{W}$, define

$$F(\theta, w) := \max_{1 \leq j \leq d+1} f(\theta, y_j), \quad G(w) := \min_{\theta \in \Theta} F(\theta, w). \quad (4)$$

For a fixed tuple w , the function $F(\cdot, w)$ measures the maximum approximation error over the deviations contained in w . Since $F(\cdot, w)$ is convex, computing $G(w)$ amounts to solving a convex optimization problem over the potential parameters. Moreover, because $F(\theta, w) \leq \mathcal{E}(\theta)$ for every $\theta \in \Theta$ and $w \in \mathcal{W}$, we have $G(w) \leq p^*$. Thus, every tuple provides a lower bound on the optimal approximation error.

The following result establishes that maximizing these lower bounds over all tuples recovers the optimal value of the original problem. It also provides sufficient conditions under which an optimal parameter of the finite-tuple problem solves the original problem.

Proposition 4 (Reformulation and optimizer recovery): Under the standing assumptions,

$$p^* = \max_{w \in \mathcal{W}} G(w). \quad (5)$$

Moreover, let $w^* = (y_1^*, \dots, y_{d+1}^*) \in \arg \max_{w \in \mathcal{W}} G(w)$, and $\theta_{w^*}^* \in \arg \min_{\theta \in \Theta} F(\theta, w^*)$. Define the set of active player-deviation pairs

$$S^* := \{(i, j) \in \mathcal{I} \times [d+1] : \ell_i(\theta_{w^*}^*, y_j^*) = F(\theta_{w^*}^*, w^*)\}.$$

Then $\theta_{w^*}^*$ is an optimal solution of (3) if either of the following conditions holds:

- (i) The finite-tuple objective is exact at $\theta_{w^*}^*$, i.e., $F(\theta_{w^*}^*, w^*) = \mathcal{E}(\theta_{w^*}^*)$.
- (ii) The active feature-difference vectors positively span $\mathbb{R}^d$, i.e., $0 \in \text{int conv} \{\Delta_\psi(y_j^*; i) : (i, j) \in S^*\}$.

Proof: [Proof Sketch] The optimal-value identity (5) follows from [10, Proposition 2.1]. The required continuity and convexity conditions hold, while compactness of Θ ensures boundedness and attainment.

For optimizer recovery, we use [10, Proposition 2.4]. Condition (i) directly implies that $\mathcal{E}(\theta_{w^*}^*) = p^*$ ([10, Proposition 2.4-(ii)]). Under condition (ii), the active feature-difference vectors ensure that $\theta_{w^*}^*$ is the unique minimizer of $F(\cdot, w^*)$. Since every minimizer of $\mathcal{E}$ also minimizes $F(\cdot, w^*)$, it follows that $\theta_{w^*}^*$ is also the unique minimizer of $\mathcal{E}$ ([10, Proposition 2.4-(i)]). The complete proof is provided in Appendix I. ■

Proposition 4 separates value recovery from parameter recovery. Equality (5) implies that solving the outer problem recovers the optimal approximation value p^* without additional assumptions. For an optimal tuple w^* , an inner minimizer solves the full-domain problem if it satisfies either condition (i) or condition (ii) of Proposition 4. Section IV establishes convergence of the outer objective values to p^* . This value convergence alone does not guarantee recovery of a full-domain optimizer; the optimizer-recovery statement remains conditional on the conditions in Proposition 4.

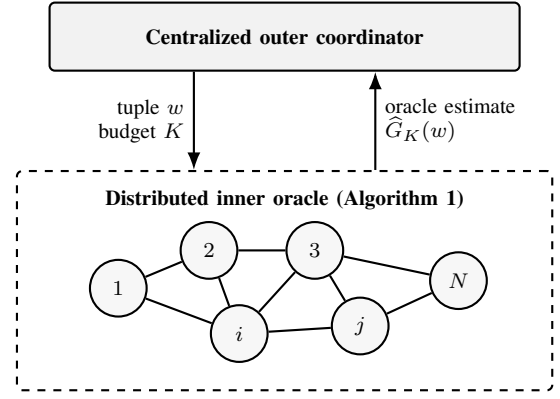


Fig. 1: Centralized outer search coupled with the distributed inner oracle.

IV. DISTRIBUTED ORACLE AND OUTER OPTIMIZATION

Proposition 4 motivates a modular two-level computational framework, illustrated in Figure 1. The framework separates the global maximization over deviation tuples from the convex optimization required to evaluate $G(w)$.

In this paper, we develop a distributed inner oracle tailored to the finite-tuple problem structure. At each outer query, a centralized coordinator broadcasts a candidate tuple w and an iteration budget K . Players then use local utility information and communication with their neighbors to compute an approximate evaluation $\hat{G}_K(w)$. We establish a uniform finite-time accuracy guarantee for the proposed oracle.

For the outer optimization, we consider two methods that offer different computational trade-offs. First, we integrate our distributed oracle with EXOTIC [10], which provides a global optimization guarantee but can be computationally expensive in high-dimensional problems. Second, we develop a projected zeroth-order ascent method as a computationally scalable alternative, without a global optimality guarantee.

We first present the distributed inner oracle and its convergence analysis, followed by the two outer optimization methods. For concision, we give only the main proof arguments in this section; complete proofs are provided in Appendix III.

A. Inner Distributed Optimization Oracle

Fix a candidate tuple $w = (y_1, \dots, y_{d+1}) \in \mathcal{W}$. Using only its local utility information, each player $i \in \mathcal{I}$ defines the finite-tuple objective

$$h_i^w(\theta) := \max_{1 \leq j \leq d+1} \ell_i(\theta, y_j). \quad (6)$$

Then $F(\theta, w) = \max_{i \in \mathcal{I}} h_i^w(\theta)$ and $G(w) = \min_{\theta \in \Theta} \max_{i \in \mathcal{I}} h_i^w(\theta)$. Each h_i^w is convex as a finite maximum of affine functions. Player i can compute its values and subgradients using its own utility u_i , the feature map ψ , and the tuple w .

Introducing an epigraph variable replaces the maximum across players with one convex inequality per player. Choose common finite bounds $t_{\min}$ and $t_{\max}$ such that $t_{\min} \leq h_i^w(\theta) \leq t_{\max}$ for all $i \in \mathcal{I}$, $\theta \in \Theta$, and $w \in \mathcal{W}$, and let $\mathcal{T} := [t_{\min}, t_{\max}]$. Such bounds exist because $h_i^w(\theta)$

is uniformly bounded by continuity and compactness; the shared bounds used in implementation need not be tight. We obtain

$$G(w) = \min_{\theta \in \Theta, t \in \mathcal{T}} t \quad \text{s.t.} \quad h_i^w(\theta) \leq t, \quad \forall i \in \mathcal{I}. \quad (7)$$

Given (θ, t) , player i can evaluate its own constraint $h_i^w(\theta) \leq t$. The optimization requires finding a common pair satisfying every player's constraint. Suppose the players communicate over a connected undirected graph $\mathcal{G} = (\mathcal{I}, \mathcal{E}_{\mathcal{G}})$. Assign each player a local copy $z_i := (\theta_i^\top, t_i)^\top \in \mathcal{Z} := \Theta \times \mathcal{T}$. Enforcing neighbor consensus yields

$$\min_{\{z_i \in \mathcal{Z}\}_{i \in \mathcal{I}}} \frac{1}{N} \sum_{i=1}^N t_i \quad \text{s.t.} \quad h_i^w(\theta_i) \leq t_i, \quad \forall i \in \mathcal{I}, \quad z_i = z_j, \quad \forall (i, j) \in \mathcal{E}_{\mathcal{G}}. \quad (8)$$

At consensus, its objective reduces to the common epigraph value t .

Proposition 5 (Equivalence of the consensus formulation): If $\mathcal{G}$ is connected, the feasible sets of (7) and (8) correspond under replication $z_i = (\theta, t)$ for every $i \in \mathcal{I}$, with identical objective values. Therefore, their optimal values equal $G(w)$, and their optimal solutions correspond to this replication.

1) *Applying existing distributed optimization oracle:* Problem (8) can be solved using existing distributed optimization methods. For fixed w , each constraint $h_i^w(\theta_i) \leq t_i$ is equivalent to the $d+1$ affine inequalities

$$\langle \Delta_\psi(y_j; i), \theta_i \rangle - t_i \leq \Delta_{u_i}(y_j; i), \quad j = 1, \dots, d+1. \quad (9)$$

These inequalities constitute an agent-specific affine conic constraint with cone $\mathbb{R}_+^{d+1}$. Hence, the method of [12] applies directly to each finite-tuple subproblem. More recent methods such as D-APDB [13] accommodate more general private functional constraints.

Remark 6: Our proposed method retains the finite-maximum representation of h_i^w . Applying a general affine-conic method to the expanded formulation in (9) would associate a $(d+1)$ -dimensional constraint-dual variable with each player, while our method uses one scalar multiplier and obtains a subgradient from an active tuple element. Hence, the local dual state is $\mathcal{O}(1)$ rather than $\mathcal{O}(d)$, although evaluating the maximum may still require examining all $d+1$ deviations. Our objective is not to obtain a faster general-purpose convergence rate, but to construct a simpler distributed oracle tailored to the finite-tuple subproblems.

2) *Proposed distributed oracle:* Inspired from [15]–[17], we propose a primal dual update based oracle on regularized Lagrangian function for (8). Let $\mu_i \geq 0$ denote the multiplier associated with player i 's constraint $h_i^w(\theta_i) \leq t_i$, and collect $\mu := (\mu_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^N$. For $\beta_k > 0$, define

$$\mathcal{L}_{\beta_k}(Z, \mu) := \sum_{i \in \mathcal{I}} \left[\frac{t_i}{N} + \mu_i (h_i^w(\theta_i) - t_i) - \frac{\beta_k}{2} \mu_i^2 \right]. \quad (10)$$

Algorithm 1 DRPDM Inner Oracle

Require: $K \geq 1$, $c, \beta_0 > 0$, $\tau \in (0, 1)$, $q \in (0, \tau/2)$, $c\beta_0 \leq 1$, and $\sigma \in (0, \lambda_{\max}^{-1}(L_{\mathcal{G}})]$; set $\eta_k = c(k+1)^{-\tau}$ and $\beta_k = \beta_0(k+1)^{-q}$.

- 1: Initialize $z_i[0] \in \mathcal{Z}$ and $\mu_i[0] = 0$
- 2: **for** $k = 0, \dots, K-1$ **do**
- 3: **for all** $i \in \mathcal{I}$ **in parallel do**
- 4: $j_i^*[k] \in \arg \max_{1 \leq j \leq d+1} \ell_i(\theta_i[k], y_j)$
- 5: $s_i[k] \leftarrow \Delta_\psi(y_{j_i^*[k]}; i)$
- 6: $g_i[k] \leftarrow (\mu_i[k] s_i[k], 1/N - \mu_i[k])$
- 7: $r_i[k] \leftarrow h_i^w(\theta_i[k]) - t_i[k] - \beta_k \mu_i[k]$
- 8: $z_i[k+1] \leftarrow P_{\mathcal{Z}} \left[z_i[k] - \eta_k g_i[k] + \sigma \sum_{j \in \mathcal{N}_i} [A_{\mathcal{G}}]_{ij} (z_j[k] - z_i[k]) \right]$
- 9: $\mu_i[k+1] \leftarrow [\mu_i[k] + \eta_k r_i[k]]_+$
- 10: **end for**
- 11: **end for**
- 12: $S_K \leftarrow \sum_{k=0}^{K-1} \eta_k$
- 13: **for all** $i \in \mathcal{I}$ **in parallel do**
- 14: $\bar{z}_i[K] \leftarrow S_K^{-1} \sum_{k=0}^{K-1} \eta_k z_i[k] =: (\bar{\theta}_i[K], \bar{t}_i[K])$
- 15: **end for**
- 16: $\bar{\theta}_w[K] \leftarrow N^{-1} \sum_i \bar{\theta}_i[K]$; broadcast $\bar{\theta}_w[K]$
- 17: Evaluate $h_i^w(\bar{\theta}_w[K])$ locally; aggregate $\hat{G}_K(w) \leftarrow \max_i h_i^w(\bar{\theta}_w[K])$
- 18: **return** $\bar{\theta}_w[K]$, $\hat{G}_K(w)$, and $\{\bar{z}_i[K]\}_{i \in \mathcal{I}}$

Algorithm 1 implements distributed projected descent-ascent on $\mathcal{L}_{\beta_k}$. At iteration k , let $Z[k]$ have rows $z_i[k]^\top$ and let $\mu[k] := (\mu_i[k])_{i \in \mathcal{I}}$. Specifically, lines 4–6 compute $g_i[k] \in \partial_{z_i} \mathcal{L}_{\beta_k}(Z[k], \mu[k])$, and line 7 computes the dual ascent direction $r_i[k] = \nabla_{\mu_i} \mathcal{L}_{\beta_k}(Z[k], \mu[k])$. Accordingly, line 8 performs projected primal descent with a consensus correction, while line 9 performs projected dual ascent. After the primal–dual iterations, lines 12–15 compute the stepsize-weighted ergodic iterate of each player. Line 16 forms and broadcasts the network-averaged potential parameter, line 17 evaluates the corresponding oracle estimate, and line 18 returns the oracle outputs.

3) *Finite-time analysis:* The finite-budget output of Algorithm 1 is generally not the exact tuple value $G(w)$. Since $\bar{\theta}_w[K] \in \Theta$, the definition of $G(w)$ gives

$$\hat{G}_K(w) := F(\bar{\theta}_w[K], w) \geq G(w). \quad (11)$$

Thus, $\hat{G}_K(w)$ is an upper estimate of $G(w)$ generated by the finite-budget inner solver. The following result controls this overestimation uniformly over $w \in \mathcal{W}$.

Theorem 7 (Uniform inner-oracle accuracy): Under the standing assumptions, run Algorithm 1 with $\tau = 3/4$ and $q = 1/4$. Then, for every $w \in \mathcal{W}$

$$0 \leq \hat{G}_K(w) - G(w) = \tilde{\mathcal{O}}\left(K^{-1/4}\right). \quad (12)$$

Proof: [Proof Sketch] Define $\mathcal{L}(Z, \mu) := \sum_{i \in \mathcal{I}} \left[\frac{t_i}{N} + \mu_i (h_i^w(\theta_i) - t_i) \right]$. Let $\bar{Z}[K]$ have rows $\bar{z}_i[K]^\top$, and define $\bar{\mu}[K] := (S_K)^{-1} \sum_{k=0}^{K-1} \eta_k \mu[k]$.

Fix $w \in \mathcal{W}$, and let $\theta_w^* \in \arg \min_{\theta \in \Theta} F(\theta, w)$, $z_w^* := (\theta_w^*, G(w))$, $Z_c := \mathbf{1}(z_w^*)^\top$. Since z_w^* is feasible and $\bar{\mu}[K] \in \mathbb{R}_+^N$, $\mathcal{L}(Z_c, \bar{\mu}[K]) \leq G(w)$.

To control the disagreement among the local primal iterates, we first use the following bound on the dual iterates.

Lemma 8 (Polynomial bound on dual iterates): Under the standing assumptions, there exists a constant $B_\mu > 0$, independent of w , i , and k , such that

$$0 \leq \mu_i[k] \leq \frac{B_\mu}{\beta_k} = \frac{B_\mu}{\beta_0}(k+1)^q, \quad i \in \mathcal{I}, \quad k \geq 0. \quad (13)$$

Since the primal update direction $g_i[k]$ depends linearly on $\mu_i[k]$, this bound implies $\|g_i[k]\|_2 = \mathcal{O}((k+1)^q)$, which is the key input to the following consensus estimate.

Define

$$z_{\text{av}}[k] := \frac{1}{N} \sum_{i=1}^N z_i[k], \quad e_k := \sqrt{\frac{1}{N} \sum_{i=1}^N \|z_i[k] - z_{\text{av}}[k]\|_2^2}.$$

For the weighted ergodic iterates $\bar{z}_i[K]$ in Algorithm 1, define

$$\bar{z}_w[K] := \frac{1}{N} \sum_{i \in \mathcal{I}} \bar{z}_i[K], \quad D_K := \frac{\sqrt{N}}{S_K} \sum_{k=0}^{K-1} \eta_k e_k.$$

Lemma 9 (Consensus error): Under the standing assumptions, there exist constants $C_e, C_D > 0$, independent of w , K , and the initial primal points, such that

$$e_k \leq C_e(k+1)^{-1/2}, \quad k \geq 0, \quad (14)$$

and for $K \geq 1$,

$$\max_{i \in \mathcal{I}} \|\bar{z}_i[K] - \bar{z}_w[K]\|_2 \leq D_K \leq C_D K^{-1/4}. \quad (15)$$

Let i_K attain $F(\bar{\theta}_w[K], w) = h_{i_K}^w(\bar{\theta}_w[K])$, and let $\mathbf{e}_{i_K}$ be the corresponding coordinate unit vector. Then, by Lemma 9,

$$F(\bar{\theta}_w[K], w) \leq \mathcal{L}(\bar{Z}[K], \mathbf{e}_{i_K}) + (G_\psi + 1)D_K.$$

It remains to control the Lagrangian gap in the preceding bound. For this, we use the following ergodic primal–dual inequality.

Lemma 10 (Ergodic primal–dual inequality): Under the conditions of Theorem 7, let $Z_c := \mathbf{1}z^\top$ be any consensus comparator with $z \in \mathcal{Z}$. Then, for every $\mu \in \mathbb{R}_+^N$,

$$\mathcal{L}(\bar{Z}[K], \mu) - \mathcal{L}(Z_c, \bar{\mu}[K]) \leq R_K + Q_K \|\mu\|_2^2, \quad (16)$$

where, for some constant $C_0 > 0$ independent of w , K , and the initial primal points, $R_K := \frac{C_0}{S_K} \left[1 + \sum_{k=0}^{K-1} \eta_k^2(k+1)^{1/2} + \sum_{k=0}^{K-1} \eta_k(k+1)^{1/4} e_k \right]$, and $Q_K := \frac{1}{S_K} \left[\frac{1}{2} + \frac{1}{4} \sum_{k=0}^{K-1} \eta_k \beta_k \right]$.

Applying Lemma 10 with $\mu = \mathbf{e}_{i_K}$, and using $\|\mathbf{e}_{i_K}\|_2 = 1$, yields $0 \leq \hat{G}_K(w) - G(w) \leq R_K + Q_K + (G_\psi + 1)D_K$.

Finally, under $\tau = 3/4$ and $q = 1/4$, $S_K = \Theta(K^{1/4})$. The standard harmonic-sum bounds applied to the definitions above then give $R_K = \tilde{\mathcal{O}}(K^{-1/4})$, $Q_K = \tilde{\mathcal{O}}(K^{-1/4})$. Together with $D_K = \mathcal{O}(K^{-1/4})$, this proves

$$0 \leq \hat{G}_K(w) - G(w) = \tilde{\mathcal{O}}(K^{-1/4})$$

uniformly over $w \in \mathcal{W}$. ■

B. Outer Optimization

1) *Global Search with EXOTIC:* We first consider EXOTIC [10] as a globally convergent method for solving the outer problem (5). EXOTIC adaptively explores a hierarchical partition of $\mathcal{W}$. At a candidate tuple w , it invokes the distributed inner oracle with an iteration budget K and obtains an approximate value $\hat{G}_K(w)$. It progressively allocates computation between exploring new regions of $\mathcal{W}$ and refining the accuracy of promising oracle evaluations.

Let n denote the total computational budget, measured as the cumulative number of synchronized inner iterations used across all oracle queries, including the final re-evaluations. Given n , EXOTIC determines a search horizon $h_{\max}$ and adaptively allocates inner iteration budgets across the nodes of the partition tree. After the tree-search phase, selected candidates are re-evaluated with a larger budget K_{final} , and the tuple with the largest refined estimate is returned. We defer the complete EXOTIC procedure and its budget allocation rule to the Appendix IV.

To invoke the convergence guarantee of EXOTIC from [10], we verify two properties of the outer problem. First, the objective G must vary regularly within cells of the hierarchical partition. Second, the number of cells that remain near-optimal at increasingly fine resolutions must be controlled. We establish these properties below.

Equip $\mathcal{W}$ with the metric $d_{\mathcal{W}}(w, \tilde{w}) := \max_{1 \leq j \leq d+1} \|y_j - \tilde{y}_j\|_2$, and let L_u and L_ψ denote common Lipschitz constants of the utilities and the feature map, respectively.

Lemma 11 (Regularity of the outer objective): Let $C_\theta := \max_{\theta \in \Theta} \|\theta\|_2$. Then, for all $w, \tilde{w} \in \mathcal{W}$,

$$|G(w) - G(\tilde{w})| \leq L_{\text{game}} d_{\mathcal{W}}(w, \tilde{w}), \quad (17)$$

where $L_{\text{game}} := 2(C_\theta L_\psi + L_u)$.

EXOTIC uses a B -ary hierarchical partition $\{\mathcal{W}_{h,r}\}_{r=1}^{B^h}$ of $\mathcal{W}$, where h denotes the depth. Assume that the cell diameters decrease geometrically:

$$\text{diam}_{d_{\mathcal{W}}}(\mathcal{W}_{h,r}) \leq \alpha_{\mathcal{W}} \rho^h, \quad \alpha_{\mathcal{W}} > 0, \quad 0 < \rho < 1. \quad (18)$$

By Lemma 11, if $\mathcal{W}_{h,r}$ contains an outer maximizer w^* , then

$$G(w^*) - G(w) \leq \nu_{\text{game}} \rho^h, \quad w \in \mathcal{W}_{h,r},$$

where $\nu_{\text{game}} := \alpha_{\mathcal{W}} L_{\text{game}}$. Thus, the variation of G within an optimal cell decreases geometrically with the partition depth.

It remains to characterize how many cells can remain near-optimal at a given depth. Define $\mathcal{N}_h(\varepsilon) := \{r : \max_{w \in \mathcal{W}_{h,r}} G(w) \geq p^* - \varepsilon\}$. Following [10, Definition 3.2], let $d_G \geq 0$ be an admissible near-optimality exponent if, for some $C_G > 1$,

$$|\mathcal{N}_h(2\nu_{\text{game}} \rho^h)| \leq C_G \rho^{-d_G h}, \quad h \geq 0. \quad (19)$$

The exponent d_G captures the effective difficulty of the outer search: larger d_G allows more cells to remain competitive as the partition is refined. We can now combine these

outer regularity properties with the uniform accuracy of the distributed inner oracle.

Theorem 12 (Global outer-value guarantee): Suppose that the assumptions of Theorem 7 hold and that the hierarchical partition satisfies (18) and (19). Run EXOTIC with Algorithm 1 as its inner oracle and total inner-iteration budget n . Then, for the returned tuple $\hat{w}_n$,

$$0 \leq p^* - G(\hat{w}_n) = \tilde{\mathcal{O}}\left(n^{-1/(d_G+4)}\right),$$

with all problem and algorithm constants fixed.

Proof: [Proof sketch] Theorem 7 gives the uniform one-sided oracle bound $0 \leq \hat{G}_K(w) - G(w) = \tilde{\mathcal{O}}(K^{-1/4})$, for every $w \in \mathcal{W}$. Lemma 11 together with (18) verifies the geometric variation condition required by EXOTIC, while (19) controls the number of near-optimal cells at each depth. Substituting the inner-oracle exponent $1/4$ into [10, Theorem 3.3 and Corollary 3.5] gives the stated rate, with logarithmic factors absorbed into $\tilde{\mathcal{O}}$. ■

Theorem 12 guarantees convergence of the *outer value* $G(\hat{w}_n)$ to p^* . The associated potential parameter is obtained from the final inner solve, $\hat{\theta}_n := \bar{\theta}_{\hat{w}_n}[K_{\text{final}}]$. The theorem does not, by itself, provide a finite-budget bound on the full-domain error $\mathcal{E}(\hat{\theta}_n)$. Exact optimizer recovery is characterized by Proposition 4; obtaining a quantitative finite-budget recovery guarantee would require additional stability or growth conditions.

2) *Projected Zeroth-Order Ascent:* Although EXOTIC provides a global outer-search guarantee, its hierarchical partition can become computationally expensive when $\mathcal{W}$ is high-dimensional. We therefore also consider a projected zeroth-order ascent method that uses only evaluations of the distributed oracle and avoids maintaining a partition tree.

Assume that $\mathcal{W} \subseteq \mathbb{R}^{D_{\mathcal{W}}}$ is convex with nonempty interior, and define the δ -interior $\mathcal{W}_{\delta} := \{w \in \mathcal{W} : w + \delta v \in \mathcal{W} \text{ for all } \|v\|_2 \leq 1\}$. At iteration t , we sample $v_t \sim \text{Unif}(\mathbb{S}^{D_{\mathcal{W}}-1})$ and query the distributed oracle at w_t and $w_t^+ := w_t + \delta_t v_t$. Let $\hat{G}_t^0$ and $\hat{G}_t^+$ denote the corresponding oracle estimates. We construct the zeroth-order ascent direction $\hat{g}_t := \frac{D_{\mathcal{W}}}{\delta_t}(\hat{G}_t^+ - \hat{G}_t^0)v_t$, and update $w_{t+1} := P_{\mathcal{W}_{\delta_{t+1}}}(w_t + \alpha_t \hat{g}_t)$, where δ_t is the perturbation radius and α_t is the outer stepsize.

Unlike EXOTIC, this procedure does not perform a global search over $\mathcal{W}$ and requires only two oracle evaluations per outer iteration. It therefore provides a more scalable alternative for high-dimensional problems. In this work, however, we use the method as a heuristic and do not establish global-optimality or stationarity guarantees for the resulting inexact projected updates. The complete procedure, including the decay schedules for δ_t and α_t and the final re-evaluation step, is given in Appendix V.

V. NUMERICAL EXPERIMENTS

We evaluate the framework in three complementary settings. A two-player benchmark with known optimum compares EXOTIC and zeroth-order (ZO) search and assesses parameter recovery; a three-player linear-quadratic game tests improvement over an analytical α -potential construction; and

a 50-player game examines the scalability of ZO search in a high-dimensional potential class.

A. Experimental Protocol

All experiments use Algorithm 1 as the distributed inner solver with $c = 1$, $\tau = \frac{1}{2}$, $\beta_0 = 0.125$, and $q = \frac{1}{6}$, unless stated otherwise; the consensus stepsize σ is chosen separately for each communication graph. Numerically, we use the uniform ergodic average $\bar{z}_i^{\text{num}}[K] := K^{-1} \sum_{k=0}^{K-1} z_i[k]$, instead of the stepsize-weighted average used in the analysis.

For EXOTIC, we use a deterministic binary midpoint partition with $h_{\text{max}} \in \{500, 1000, 2000\}$ and $s \in \{10^3, 10^4\}$, where a nominal inner budget K corresponds to sK DRPDM iterations. For ZO search, we set $\delta_t = \delta_0(t+1)^{-1/4}$ and $\alpha_t = \alpha_0(t+1)^{-4/5}$, and cold-start each inner solve. Experiment-specific values of α_0 , δ_0 , and the iteration budgets are given below.

B. Two-Player Benchmark with Known Optimum

Our first experiment considers a controlled benchmark for which the optimal full-domain approximation and corresponding potential parameter can be computed exactly. This allows us to evaluate both objective and parameter recovery and to compare the two outer-search methods.

Consider a two-player, two-action game, where $x_i \in [0, 1]$ denotes the probability that player i selects its first action. The mixed-strategy utilities are $u_1(x_1, x_2) = 12 - 4x_1 - 10x_2 + 10x_1x_2$ and $u_2(x_1, x_2) = 10 - 2x_1 - 8x_2 + 2x_1x_2$. We use $\psi(x_1, x_2) = (x_1, x_2, x_1x_2)^\top$ and $\phi_\theta(x_1, x_2) = \langle \theta, \psi(x_1, x_2) \rangle$. Because these utilities are multilinear extensions of a finite 2×2 game, the optimal approximation can be computed exactly using the finite-game linear program of [4]. For the feature class above, this yields $\theta^* = (-2, -10, 6)^\top$ and $p^* = 2$. Consequently, we can evaluate both the full-domain optimality gap $\mathcal{E}(\hat{\theta}) - p^*$ and the parameter-recovery error $\|\hat{\theta} - \theta^*\|_2$.

The players communicate over a single undirected edge. We set $\sigma = 0.45$ and $\Theta = [-25, 25]^3$. Since $x_i, x'_i \in [0, 1]$, the resulting outer search space is $\mathcal{W} = \mathcal{Y}^{d+1} = [0, 1]^{16}$. For ZO search, we use $T = 2000$, $K_{\text{inner}} = 10^4$, and $K_{\text{reeval}} = 10^5$, and test $\alpha_0 \in \{0.01, 0.03, 0.1\}$ and $\delta_0 \in \{0.0125, 0.025, 0.05\}$. For EXOTIC, we use the parameter grid specified in Section V-A.

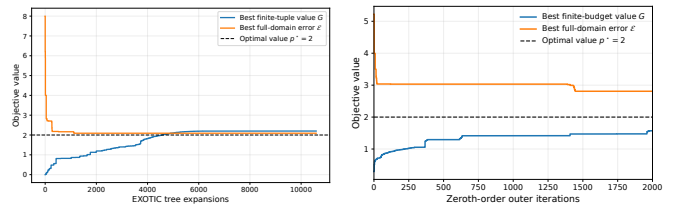


Fig. 2: Representative best-so-far finite-tuple values and exact full-domain errors for the two-player game: DRPDM-based EXOTIC ($h_{\text{max}} = 2000$, $s = 10^4$; left) and ZO ($\alpha_0 = 0.1$, $\delta_0 = 0.025$; right). The dashed lines mark $p^* = 2$.

Table I compares the two methods. For EXOTIC, increasing h_{max} consistently reduces the recovered full-domain er-

TABLE I: Performance on the two-player continuous game. For ZO, the configuration denotes (α_0, δ_0) ; for EXOTIC, it denotes $(h_{\max}, s)$.

Method	Configuration	$\hat{G}$	$\mathcal{E}(\hat{\theta})$	$\mathcal{E}(\hat{\theta}) - p^*$	$\ \hat{\theta} - \theta^*\ _2$	Time (s)
ZO	$(10^{-2}, 0.0125)$	0.3063	2.3436	0.3436	0.4650	12.68
	$(3 \times 10^{-2}, 0.0125)$	0.4363	2.4676	0.4676	1.0612	12.43
	$(10^{-1}, 0.025)$	0.6550	2.4251	0.4251	0.8502	12.30
EXOTIC	$(500, 10^4)$	1.6265	2.4512	0.4512	0.5217	6.98
	$(1000, 10^4)$	1.9006	2.1555	0.1555	0.1166	16.39
	$(2000, 10^4)$	1.9214	2.0655	0.0655	0.1088	38.37

ror. On this low-dimensional benchmark, EXOTIC achieves more accurate objective and parameter recovery, at the cost of greater computation at its highest-accuracy setting. Figure 2 shows representative trajectories for EXOTIC with $(h_{\max}, s) = (2000, 10^4)$ and ZO with $(\alpha_0, \delta_0) = (0.1, 0.025)$; the latter is shown for illustration and is not the best-performing ZO configuration in Table I.

C. Linear-Quadratic Network Games

We next consider network games for which an analytical α -potential construction is available from [6]. Rather than recovering a known optimum, these experiments test whether our optimization-based approach can improve upon this analytical construction. We consider a small instance amenable to EXOTIC and a larger instance for which we use the ZO outer method. Consider the N -player linear-quadratic network game

$$u_i(a) = -\frac{1}{2}a_i^2 + \zeta_i a_i + \gamma a_i \sum_{j=1}^N V_{ij} a_j, \quad a_i \in [-1, 1], \quad (20)$$

where $V_{ii} = 0$ and the off-diagonal entries are sampled independently as $V_{ij} \sim \text{Unif}[1 - \varepsilon_N, 1 + \varepsilon_N]$. The players communicate over an undirected cycle. We optimize over the quadratic potential class

$$\phi_\theta(a) = \sum_{i=1}^N p_i a_i^2 + \sum_{i=1}^N q_i a_i + \sum_{1 \leq i < j \leq N} r_{ij} a_i a_j, \quad (21)$$

where $\theta = (\{p_i\}_{i=1}^N, \{q_i\}_{i=1}^N, \{r_{ij}\}_{1 \leq i < j \leq N}) \in \Theta \subseteq \mathbb{R}^d$ and $d = 2N + \frac{N(N-1)}{2}$. As the analytical baseline, we use

$$p_i^{(0)} = -\frac{1}{2}, \quad q_i^{(0)} = \zeta_i, \quad r_{ij}^{(0)} = \frac{\gamma}{2}(V_{ij} + V_{ji}), \quad i < j. \quad (22)$$

By [6], its full-domain approximation error is $\mathcal{E}(\theta_0) = |\gamma| \|V - V^\top\|_\infty$. For this class of games, $\mathcal{E}(\theta)$ can be evaluated exactly over the continuous action domain using a closed-form expression provided in Appendix VI. We measure improvement by

$$\text{Imp}(\hat{\theta}) := \mathcal{E}(\theta_0) - \mathcal{E}(\hat{\theta}), \quad \text{Imp}_{\%}(\hat{\theta}) := 100 \frac{\text{Imp}(\hat{\theta})}{\mathcal{E}(\theta_0)}. \quad (23)$$

Positive values therefore indicate improvement over the analytical construction in the original full-domain objective.

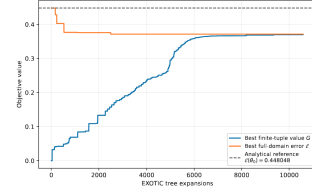


Fig. 3: Representative EXOTIC search ($h_{\max} = 2000$, $s = 10^4$). The dashed line marks $\mathcal{E}(\theta_0) = \|V - V^\top\|_\infty$.

1) *Three-Player Game: Improving the Analytical Construction:* We first consider a low-dimensional instance for which EXOTIC can be used for global search over the finite-tuple space. The goal is to test whether optimizing the finite-tuple objective can improve upon the analytical baseline in (22). We consider a three-player instance with $\gamma = \zeta_i = 1$, $\varepsilon_N = 0.5$, and communication over the three-node cycle. We consider the following parameterization of potential

$$\phi_\theta(a) = -\frac{1}{2} \sum_{i=1}^3 a_i^2 + \sum_{i=1}^3 a_i + \sum_{1 \leq i < j \leq 3} \theta_{ij} a_i a_j, \quad \theta \in [-2, 2]^3.$$

Thus, $d_\theta = 3$, $d_y = 6$, and $\mathcal{W} = \mathcal{Y}^4 \subseteq \mathbb{R}^{24}$. The analytical baseline is $\theta_{ij}^{(0)} = (V_{ij} + V_{ji})/2$, for which $\mathcal{E}(\theta_0) = \|V - V^\top\|_\infty = 0.4480$. We use the EXOTIC parameter grid from Section V-A and set $\sigma = 0.3$.

Across the tested configurations, the smallest observed exact full-domain error is 0.3693, corresponding to a 17.58% improvement over the analytical baseline. Figure 3 shows that the best finite-tuple value increases during the search, while the diagnostic full-domain error initially decreases and then stabilizes. Thus, in this instance, finite-tuple optimization identifies a potential that improves upon the analytical construction over the full action domain.

2) *Fifty-Player Game: Scalable Zeroth-Order Search:* We finally consider a larger instance to examine the scalability of the ZO outer method. The full quadratic potential class has $d_\theta = 1325$, making hierarchical global search over $\mathcal{W} = \mathcal{Y}^{d_\theta+1}$ computationally impractical. We therefore combine the ZO outer search with the distributed DRPDM oracle.

We consider a 50-player instance with $\gamma = \zeta_i = 1$ and $\varepsilon_N = 0.8$. The interaction matrix V is generated with seed 42 and held fixed throughout the experiment. The players communicate over a 50-node cycle, with $\sigma = 0.225$. We use the full quadratic potential class in (21), with $d_\theta = 1325$ and $\Theta = [-2, 2]^{1325}$. The analytical baseline in (22) has exact full-domain error $\mathcal{E}(\theta_0) = \|V - V^\top\|_\infty = 33.5072$.

We run Algorithm 3 for $T = 200$ iterations and test $\alpha_0 \in \{10^{-2}, 10^{-3}, 10^{-4}\}$ and $\delta_0 \in \{0.00625, 0.0125, 0.025\}$. Each base and perturbed query uses $K_{\text{inner}} = 10,000$ DRPDM iterations, with the two queries evaluated in parallel from the same cold start θ_0 . All nine configurations share the same interaction matrix, initial outer iterate, and zeroth-order directions.

Because the outer method optimizes the finite-tuple objective rather than the full-domain error, we use the closed-form expression for $\mathcal{E}$ available in this example only for diagnostic

post-selection. Candidates whose initial full-domain error improves upon $\mathcal{E}(\theta_0)$ are independently re-solved using $K_{\text{reeval}} = 50,000$ DRPDM iterations, and we report the candidate with the smallest resulting exact error. If no candidate passes this screening step, we instead re-evaluate the candidate with the smallest initial error.

Figure 4 shows the best-so-far trajectories, and Table II summarizes the post-selected outputs. Four of the nine configurations improve upon the analytical baseline. The best result is obtained with $(\alpha_0, \delta_0) = (10^{-2}, 0.025)$, for which the iterate selected at $t = 163$ achieves $\mathcal{E}(\hat{\theta}_{\text{out}}) = 31.7768$, corresponding to a 5.16% improvement. In contrast, all configurations with $\alpha_0 = 10^{-4}$ perform worse than the baseline, suggesting that this outer stepsize is too small for the available iteration budget.

The experiment also illustrates the distinction between outer-value recovery and parameter recovery. For an arbitrary nonoptimal tuple w , increasing $G(w)$ need not monotonically reduce $\mathcal{E}(\theta_w)$, since θ_w is optimized only against the deviations contained in w . Theorem 12 therefore controls the outer value, while recovery of the corresponding full-domain optimal potential is governed asymptotically by the structural conditions of Proposition 4.

TABLE II: Post-selected outputs for the 50-player game with $c = 1$, after independent DRPDM re-evaluation with $K_{\text{reeval}} = 50,000$.

α_0	δ_0	Selected t	$\mathcal{E}(\theta_{\text{out}})$	Improvement
10^{-2}	0.0125	140	32.730435	0.776793 (2.32%)
10^{-2}	0.025	163	31.776847	1.730381 (5.16%)
10^{-3}	0.0125	67	33.584310	-0.077082 (-0.23%)
10^{-3}	0.025	160	32.786437	0.720791 (2.15%)

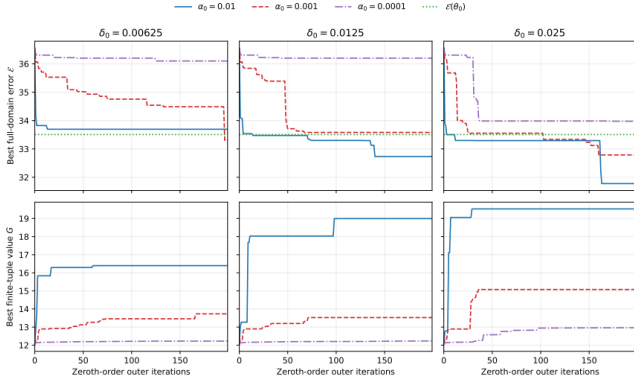


Fig. 4: Best-so-far exact errors (top) and finite-tuple values (bottom) for the 50-player ZO experiment. Columns correspond to $\delta_0 \in \{0.00625, 0.0125, 0.025\}$, and curves to $\alpha_0 \in \{10^{-2}, 10^{-3}, 10^{-4}\}$. The dashed lines mark $\mathcal{E}(\theta_0) = 33.5072$.

VI. CONCLUSION

We studied distributed computation of near-potential approximations for continuous-action general-sum games. An

exact finite-tuple reformulation separates the problem into a global search over deviation tuples and distributed convex inner problem. For the inner problem, we developed a regularized primal-dual oracle with a uniform one-sided accuracy guarantee. Combined with EXOTIC, this yields an end-to-end global-search guarantee, while projected zeroth-order ascent provides a more scalable alternative. We also established sufficient conditions for recovering an optimizer of the original full-domain problem. Numerical experiments illustrate the tradeoff between global search accuracy and computational scalability.

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APPENDIX I
PROOF OF PROPOSITION 4

The value identity follows from the reformulation argument of [10, Proposition 2.1]. Indeed, $\ell_i(\theta, y)$ is continuous in (θ, y) and affine, hence convex, in θ , while Θ and $\mathcal{Y}$ are compact. Compactness of Θ replaces the coercivity assumption used in [10] to guarantee boundedness and attainment. Therefore,

$$p^* = \max_{w \in \mathcal{W}} G(w).$$

Let $w^* \in \arg \max_{w \in \mathcal{W}} G(w)$ and $\theta_{w^*} \in \arg \min_{\theta \in \Theta} F(\theta, w^*)$. Then

$$F(\theta_{w^*}, w^*) = G(w^*) = p^*.$$

For optimizer recovery, we verify the two alternative conditions of [10, Proposition 2.4]. Under condition (i),

$$\mathcal{E}(\theta_{w^*}) = F(\theta_{w^*}, w^*) = p^*,$$

so θ_{w^*} solves the full-domain problem.

It remains to consider condition (ii). For every $(i, j) \in S^*$, define the active loss gradient

$$g_{ij}^* := \Delta_\psi(y_j^*; i).$$

Fix any $\theta \in \Theta$ with $\theta \neq \theta_{w^*}$, and let $v := \theta - \theta_{w^*} \neq 0$. For every $(i, j) \in S^*$, the affinity gives

$$\ell_i(\theta, y_j^*) = \ell_i(\theta_{w^*}, y_j^*) + \langle g_{ij}^*, \theta - \theta_{w^*} \rangle.$$

Since every $(i, j) \in S^*$ is active at θ_{w^*} ,

$$\begin{aligned} F(\theta, w^*) &\geq \max_{(i,j) \in S^*} \ell_i(\theta, y_j^*) \\ &= F(\theta_{w^*}, w^*) + \max_{(i,j) \in S^*} \langle g_{ij}^*, v \rangle. \end{aligned}$$

We claim that

$$\max_{(i,j) \in S^*} \langle g_{ij}^*, v \rangle > 0.$$

Suppose instead that $\langle g_{ij}^*, v \rangle \leq 0$ for every $(i, j) \in S^*$. Then the convex hull of the active gradients is contained in the closed halfspace

$$\{g \in \mathbb{R}^d : \langle g, v \rangle \leq 0\}.$$

This contradicts condition (ii), because

$$0 \in \text{int conv}\{g_{ij}^* : (i, j) \in S^*\}$$

implies that this convex hull contains a neighborhood of the origin, including a point in the direction v with strictly positive inner product with v . Hence,

$$F(\theta, w^*) > F(\theta_{w^*}, w^*) \quad \text{for every } \theta \in \Theta \setminus \{\theta_{w^*}\}.$$

Thus, θ_{w^*} is the unique minimizer of $F(\cdot, w^*)$.

Finally, let $\theta^* \in \arg \min_{\theta \in \Theta} \mathcal{E}(\theta)$. Since w^* maximizes G ,

$$p^* = G(w^*) = \min_{\theta \in \Theta} F(\theta, w^*) \leq F(\theta^*, w^*).$$

On the other hand, the tuple elements $y_1^*, \dots, y_{d+1}^*$ belong to $\mathcal{Y}$, so

$$F(\theta^*, w^*) = \max_{i,j} \ell_i(\theta^*, y_j^*) \leq \max_{i,y \in \mathcal{Y}} \ell_i(\theta^*, y) = \mathcal{E}(\theta^*) = p^*.$$

Hence $F(\theta^*, w^*) = G(w^*)$, and therefore θ^* also minimizes $F(\cdot, w^*)$. Its uniqueness gives $\theta^* = \theta_{w^*}$, and hence

$$\theta_{w^*} \in \arg \min_{\theta \in \Theta} \mathcal{E}(\theta).$$

This proves optimizer recovery under condition (ii).

APPENDIX II
AUXILIARY RESULTS

Lemma 13: Let $\rho \in (0, 1)$ and $\xi \in (0, 1)$. Then, for every $k \geq 1$,

$$\rho^k \leq \left(\frac{\xi}{e|\log \rho|} \right)^\xi k^{-\xi}, \quad (24)$$

and

$$\sum_{s=0}^{k-1} \rho^{k-1-s} (s+1)^{-\xi} \leq C_{\rho,\xi} k^{-\xi}, \quad (25)$$

where

$$C_{\rho,\xi} := \frac{2^\xi}{1-\rho} + \frac{2}{e(1-\xi)|\log \rho|}.$$

Proof: Let $b := |\log \rho| > 0$ and define $f(x) := x^\xi e^{-bx}$ for $x > 0$. Since

$$f'(x) = x^{\xi-1} e^{-bx} (\xi - bx),$$

f is increasing on $(0, \xi/b)$ and decreasing on $(\xi/b, \infty)$. Hence,

$$\sup_{x>0} x^\xi \rho^x = f\left(\frac{\xi}{b}\right) = \left(\frac{\xi}{e|\log \rho|} \right)^\xi.$$

Taking $x = k$ proves (24).

To prove (25), first rewrite the convolution as

$$\sum_{s=0}^{k-1} \rho^{k-1-s} (s+1)^{-\xi} = \sum_{t=1}^k \rho^{k-t} t^{-\xi}.$$

We split the index set into

$$\mathcal{I}_1 := \{t \in [k] : t < k/2\}, \quad \mathcal{I}_2 := \{t \in [k] : t \geq k/2\}.$$

For $t \in \mathcal{I}_2$, we have $t^{-\xi} \leq 2^\xi k^{-\xi}$. Therefore,

$$\begin{aligned} \sum_{t \in \mathcal{I}_2} \rho^{k-t} t^{-\xi} &\leq 2^\xi k^{-\xi} \sum_{t \in \mathcal{I}_2} \rho^{k-t} \\ &\leq \frac{2^\xi}{1-\rho} k^{-\xi}. \end{aligned}$$

For $t \in \mathcal{I}_1$, we have $k-t > k/2$ and hence $\rho^{k-t} \leq \rho^{k/2}$. Since $\xi \in (0, 1)$,

$$\sum_{t \in \mathcal{I}_1} t^{-\xi} \leq \sum_{t=1}^k t^{-\xi} \leq \int_0^k x^{-\xi} dx = \frac{k^{1-\xi}}{1-\xi}.$$

It follows that

$$\begin{aligned} \sum_{t \in \mathcal{I}_1} \rho^{k-t} t^{-\xi} &\leq \frac{\rho^{k/2} k^{1-\xi}}{1-\xi} \\ &= \frac{k \rho^{k/2}}{1-\xi} k^{-\xi}. \end{aligned}$$

To bound the remaining factor, define $g(x) := x\rho^{x/2}$ for $x > 0$. Since

$$g'(x) = \rho^{x/2} \left(1 - \frac{|\log \rho|}{2} x \right),$$

the maximum of g is attained at $x = 2/|\log \rho|$. Consequently,

$$k\rho^{k/2} \leq \sup_{x>0} x\rho^{x/2} = \frac{2}{e|\log \rho|}.$$

Thus,

$$\sum_{t \in \mathcal{I}_1} \rho^{k-t} t^{-\xi} \leq \frac{2}{e(1-\xi)|\log \rho|} k^{-\xi}.$$

Combining the bounds over $\mathcal{I}_1$ and $\mathcal{I}_2$ gives

$$\begin{aligned} \sum_{s=0}^{k-1} \rho^{k-1-s} (s+1)^{-\xi} &\leq \left[\frac{2^\xi}{1-\rho} + \frac{2}{e(1-\xi)|\log \rho|} \right] k^{-\xi} \\ &= C_{\rho, \xi} k^{-\xi}. \end{aligned}$$

This proves (25). $\blacksquare$

Lemma 14: For $r > 0$ and $K \geq 1$, define $H_r(K) := \sum_{k=1}^K k^{-r}$. Then,

$$H_r(K) \leq \begin{cases} 1 + \frac{K^{1-r} - 1}{1-r}, & 0 < r < 1, \\ 1 + \log K, & r = 1, \\ 1 + \frac{1}{r-1}, & r > 1. \end{cases}$$

Consequently, for every fixed $r > 0$, there exists a constant $C_r > 0$, independent of K , such that

$$H_r(K) \leq C_r(1 + \log K)K^{[1-r]_+}.$$

More generally, for every $b \in \mathbb{R}$,

$$K^{-b} H_r(K) \leq C_r(1 + \log K)K^{-\min\{b, b+r-1\}}.$$

Proof: If $0 < r < 1$, the integral comparison gives

$$H_r(K) \leq 1 + \int_1^K x^{-r} dx = 1 + \frac{K^{1-r} - 1}{1-r}.$$

If $r = 1$, then

$$H_1(K) \leq 1 + \int_1^K \frac{1}{x} dx = 1 + \log K.$$

Finally, if $r > 1$,

$$H_r(K) \leq 1 + \int_1^\infty x^{-r} dx = 1 + \frac{1}{r-1}.$$

These three estimates imply

$$H_r(K) \leq C_r(1 + \log K)K^{[1-r]_+}$$

for some constant $C_r > 0$.

Multiplying both sides by K^{-b} gives

$$K^{-b} H_r(K) \leq C_r(1 + \log K)K^{-b+[1-r]_+}.$$

Since

$$b - [1-r]_+ = \begin{cases} b + r - 1, & 0 < r < 1, \\ b, & r \geq 1, \end{cases} = \min\{b, b+r-1\},$$

we obtain

$$K^{-b} H_r(K) \leq C_r(1 + \log K)K^{-\min\{b, b+r-1\}}.$$

$\blacksquare$ for all i, k , and w .

APPENDIX III COMPLETE PROOFS FOR SECTION IV

This appendix supplies the complete proofs of the results used in the finite-time analysis of the distributed oracle and in the outer-search guarantee. Throughout, constants may depend on the fixed game, compact sets, communication graph, and algorithm parameters, but not on the queried tuple w , the iteration budget K , or the initial primal points in $\mathcal{Z}$.

A. Proof of Proposition 5

Given any feasible point (θ, t) of (7), for $i \in \mathcal{I}$, set $z_i = (\theta, t)$. The resulting collection satisfies all the local and consensus constraints of the consensus formulation, and its objective value is $(N)^{-1} \sum_{i=1}^N t_i = t$. Conversely, consider any feasible collection $\{z_i\}_{i \in \mathcal{I}}$ of the consensus formulation. Because $\mathcal{G}$ is connected, the consensus constraints imply that $z_1 = \dots = z_N = (\theta, t)$ for some (θ, t) . The local constraints then imply $h_i^w(\theta) \leq t$ for every $i \in \mathcal{I}$. Therefore, (θ, t) is feasible for (7), and the two corresponding points have the same objective value.

B. Proof of Lemma 8

Let

$$C_\theta := \max_{\theta \in \Theta} \|\theta\|_2, \quad C_t := \max\{|t_{\min}|, |t_{\max}|\},$$

and recall the uniform bounds

$$G_\psi := \max_{i \in \mathcal{I}, y \in \mathcal{Y}} \|\Delta\psi(y; i)\|_2,$$

and

$$U := \max_{i \in \mathcal{I}, y \in \mathcal{Y}} |\Delta u_i(y; i)|.$$

These constants are finite by compactness and continuity. Since $|\ell_i(\theta, y)| \leq C_\theta G_\psi + U$, compactness of $\mathcal{T} = [t_{\min}, t_{\max}]$ gives

$$|h_i^w(\theta_i[k]) - t_i[k]| \leq B_\mu, \quad B_\mu := C_\theta G_\psi + U + C_t,$$

uniformly over w, i , and k .

The dual update and $\eta_k \beta_k \leq c\beta_0 \leq 1$ yield

$$\begin{aligned} \mu_i[k+1] &= \left[(1 - \eta_k \beta_k) \mu_i[k] + \eta_k (h_i^w(\theta_i[k]) - t_i[k]) \right]_+ \\ &\leq (1 - \eta_k \beta_k) \mu_i[k] + \eta_k B_\mu. \end{aligned}$$

We prove by induction that $\mu_i[k] \leq B_\mu / \beta_k$. The claim holds at $k = 0$ because $\mu_i[0] = 0$. If it holds at iteration k , then

$$\begin{aligned} \mu_i[k+1] &\leq (1 - \eta_k \beta_k) \frac{B_\mu}{\beta_k} + \eta_k B_\mu \\ &= \frac{B_\mu}{\beta_k} \leq \frac{B_\mu}{\beta_{k+1}}, \end{aligned}$$

where the last inequality follows from $\beta_{k+1} \leq \beta_k$. Therefore,

$$0 \leq \mu_i[k] \leq \frac{B_\mu}{\beta_k} = \frac{B_\mu}{\beta_0} (k+1)^q$$

For later use, Algorithm 1 and $\|s_i[k]\|_2 \leq G_\psi$ also give

$$\|g_i[k]\|_2 \leq \mu_i[k](G_\psi + 1) + \frac{1}{N} \leq C_g(k+1)^q, \quad (26)$$

$$|r_i[k]| \leq B_\mu + \beta_k \mu_i[k] \leq 2B_\mu, \quad (27)$$

where

$$C_g := \frac{B_\mu}{\beta_0}(G_\psi + 1) + \frac{1}{N}.$$

All these bounds are uniform over w .

C. Proof of Lemma 9

Let

$$P := I_N - \sigma L_G, \quad J := \frac{1}{N} \mathbf{1}\mathbf{1}^\top.$$

Because the graph is undirected, $L_G \mathbf{1} = 0$ and $\mathbf{1}^\top L_G = 0^\top$, so

$$P\mathbf{1} = \mathbf{1}, \quad \mathbf{1}^\top P = \mathbf{1}^\top.$$

Moreover, $d_i = e_i^\top L_G e_i \leq \lambda_{\max}(L_G)$. The condition $\sigma \leq 1/\lambda_{\max}(L_G)$ therefore implies

$$P_{ii} = 1 - \sigma d_i \geq 0, \quad P_{ij} = \sigma [A_G]_{ij} \geq 0 \quad (i \neq j).$$

Hence, P is nonnegative and doubly stochastic.

Let the eigenvalues of L_G be ordered as

$$0 = \lambda_1(L_G) < \lambda_2(L_G) \leq \dots \leq \lambda_N(L_G).$$

Connectivity gives $\lambda_2(L_G) > 0$. Since $0 < \sigma \leq 1/\lambda_{\max}(L_G)$, spectral decomposition gives

$$\rho_g := \|P - J\|_2 = 1 - \sigma \lambda_2(L_G) < 1. \quad (28)$$

Let $Z[k]$ and $G[k]$ have rows $z_i[k]^\top$ and $g_i[k]^\top$, respectively. Since

$$\sum_{j \in \mathcal{N}_i} [A_G]_{ij} (z_j[k] - z_i[k]) = -(L_G Z[k])_i^\top,$$

the stacked unprojected primal update is

$$PZ[k] - \eta_k G[k].$$

Also,

$$e_k = \frac{1}{\sqrt{N}} \|(I_N - J)Z[k]\|_F.$$

The pairwise-variance identity and nonexpansiveness of the common row-wise projection onto $\mathcal{Z}$ give

$$\begin{aligned} e_{k+1} &\leq \frac{1}{\sqrt{N}} \|(I_N - J)(PZ[k] - \eta_k G[k])\|_F \\ &\leq \frac{1}{\sqrt{N}} \|P - J\|_2 \|(I_N - J)Z[k]\|_F \\ &\quad + \frac{\eta_k}{\sqrt{N}} \|I_N - J\|_2 \|G[k]\|_F \\ &\leq \rho_g e_k + \eta_k C_g (k+1)^q, \end{aligned}$$

where the last inequality uses (26). Substituting $\eta_k = c(k+1)^{-\tau}$ yields

$$e_{k+1} \leq \rho_g e_k + cC_g (k+1)^{-(\tau-q)}. \quad (29)$$

Put $\xi := \tau - q > 0$. Iterating (29) gives, for every $k \geq 1$,

$$e_k \leq \rho_g^k e_0 + cC_g \sum_{s=0}^{k-1} \rho_g^{k-1-s} (s+1)^{-\xi}.$$

Let

$$D_{\mathcal{Z}} := \max_{z, z' \in \mathcal{Z}} \|z - z'\|_2.$$

Convexity of $\mathcal{Z}$ implies $z_{\text{av}}[0] \in \mathcal{Z}$ and hence $e_0 \leq D_{\mathcal{Z}}$. If $0 < \rho_g < 1$, Lemma 14 gives

$$\rho_g^k \leq C_{\rho,1} k^{-\xi}, \quad \sum_{s=0}^{k-1} \rho_g^{k-1-s} (s+1)^{-\xi} \leq C_{\rho,2} k^{-\xi}.$$

Thus, for a constant $C_e > 0$ independent of w and k ,

$$e_k \leq C_e k^{q-\tau}.$$

When $\rho_g = 0$, the same conclusion follows directly from (29). Enlarging C_e to cover $k = 0$ gives

$$e_k \leq C_e (k+1)^{q-\tau}, \quad k \geq 0. \quad (30)$$

We next pass to the weighted ergodic averages. Since

$$\bar{z}_w[K] = \frac{1}{S_K} \sum_{k=0}^{K-1} \eta_k z_{\text{av}}[k],$$

for every player i ,

$$\bar{z}_i[K] - \bar{z}_w[K] = \frac{1}{S_K} \sum_{k=0}^{K-1} \eta_k (z_i[k] - z_{\text{av}}[k]).$$

The definition of e_k implies $\|z_i[k] - z_{\text{av}}[k]\|_2 \leq \sqrt{N} e_k$, and hence

$$\max_i \|\bar{z}_i[K] - \bar{z}_w[K]\|_2 \leq \frac{\sqrt{N}}{S_K} \sum_{k=0}^{K-1} \eta_k e_k = D_K. \quad (31)$$

Furthermore, (30) gives

$$\begin{aligned} \sum_{k=0}^{K-1} \eta_k e_k &\leq cD_{\mathcal{Z}} + cC_e \sum_{k=1}^K k^{-(2\tau-q)} \\ &\leq c[D_{\mathcal{Z}} + C_e H_{2\tau-q}(K)]. \end{aligned}$$

Since $S_K = c \sum_{k=1}^K k^{-\tau} \geq cK^{1-\tau}$,

$$D_K \leq \sqrt{N} K^{\tau-1} [D_{\mathcal{Z}} + C_e H_{2\tau-q}(K)].$$

Applying Lemma 14 in Appendix II with $b = 1 - \tau$ and $r = 2\tau - q$ yields

$$D_K \leq C_D (1 + \log K) K^{-\min\{\tau-q, 1-\tau\}},$$

which proves both claims.

D. Proof of Lemma 10

For compactness, write

$$c_i[k] := h_i^w(\theta_i[k]) - t_i[k]$$

and define the local unregularized Lagrangian

$$\mathcal{L}_i(z_i, \mu_i) := \frac{t_i}{N} + \mu_i(h_i^w(\theta_i) - t_i).$$

Thus,

$$\mathcal{L}(Z, \mu) := \sum_{i=1}^N \mathcal{L}_i(z_i, \mu_i).$$

Fix a consensus comparator $Z_c = \mathbf{1}z^\top$, with $z \in \mathcal{Z}$, and any $\mu \in \mathbb{R}_+^N$. Let

$$v_i[k] := \sum_{j=1}^N P_{ij} z_j[k].$$

Then the primal update is

$$z_i[k+1] = P_Z(v_i[k] - \eta_k g_i[k]).$$

Because $z = P_Z(z)$, nonexpansiveness gives

$$\begin{aligned} \|z_i[k+1] - z\|_2^2 &\leq \|v_i[k] - \eta_k g_i[k] - z\|_2^2 \\ &= \|v_i[k] - z\|_2^2 - 2\eta_k \langle g_i[k], z_i[k] - z \rangle \\ &\quad + 2\eta_k \langle g_i[k], z_i[k] - v_i[k] \rangle + \eta_k^2 \|g_i[k]\|_2^2. \end{aligned}$$

Since P is doubly stochastic, Jensen's inequality yields

$$\sum_i \|v_i[k] - z\|_2^2 \leq \sum_i \|z_i[k] - z\|_2^2.$$

Moreover, $\mu_i[k] \geq 0$ makes $\mathcal{L}_i(\cdot, \mu_i[k])$ convex, and $g_i[k] \in \partial_z \mathcal{L}_i(z_i[k], \mu_i[k])$. Therefore,

$$\langle g_i[k], z_i[k] - z \rangle \geq \mathcal{L}_i(z_i[k], \mu_i[k]) - \mathcal{L}_i(z, \mu_i[k]).$$

Because $P = I_N - \sigma L_G$,

$$\sum_i \langle g_i[k], z_i[k] - v_i[k] \rangle = \sigma \langle G[k], L_G Z[k] \rangle_F.$$

Summing the preceding projection inequality over i gives

$$\begin{aligned} &\eta_k \sum_i [\mathcal{L}_i(z_i[k], \mu_i[k]) - \mathcal{L}_i(z, \mu_i[k])] \\ &\leq \frac{1}{2} \sum_i (\|z_i[k] - z\|_2^2 - \|z_i[k+1] - z\|_2^2) \\ &\quad + \eta_k \sigma \langle G[k], L_G Z[k] \rangle_F + \frac{\eta_k^2}{2} \sum_i \|g_i[k]\|_2^2. \end{aligned} \quad (32)$$

For the dual update, projection onto $\mathbb{R}_+$ gives, for every comparator $\mu_i \geq 0$,

$$\begin{aligned} |\mu_i[k+1] - \mu_i|^2 &\leq |\mu_i[k] - \mu_i + \eta_k r_i[k]|^2 \\ &= |\mu_i[k] - \mu_i|^2 + 2\eta_k (\mu_i[k] - \mu_i) c_i[k] \\ &\quad - 2\eta_k \beta_k \mu_i[k] (\mu_i[k] - \mu_i) \\ &\quad + \eta_k^2 |r_i[k]|^2. \end{aligned}$$

Completing the square gives

$$(\mu_i - \mu_i[k])\mu_i[k] = \frac{\mu_i^2}{4} - \left(\mu_i[k] - \frac{\mu_i}{2}\right)^2 \leq \frac{\mu_i^2}{4}.$$

Since $\mathcal{L}_i(z_i[k], \cdot)$ is affine,

$$\mathcal{L}_i(z_i[k], \mu_i) - \mathcal{L}_i(z_i[k], \mu_i[k]) = (\mu_i - \mu_i[k])c_i[k].$$

Rearranging the dual projection inequality therefore gives

$$\begin{aligned} &\eta_k [\mathcal{L}_i(z_i[k], \mu_i) - \mathcal{L}_i(z_i[k], \mu_i[k])] \\ &\leq \frac{1}{2} (|\mu_i[k] - \mu_i|^2 - |\mu_i[k+1] - \mu_i|^2) \\ &\quad + \frac{\eta_k^2}{2} |r_i[k]|^2 + \frac{\eta_k \beta_k}{4} \mu_i^2. \end{aligned} \quad (33)$$

Let $\bar{Z}[K]$ have rows $\bar{z}_i[K]^\top$ and define

$$\bar{\mu}[K] := \frac{1}{S_K} \sum_{k=0}^{K-1} \eta_k \mu[k].$$

Convexity of $\mathcal{L}(\cdot, \mu)$ and affinity of $\mathcal{L}(Z_c, \cdot)$ imply

$$\begin{aligned} &S_K [\mathcal{L}(\bar{Z}[K], \mu) - \mathcal{L}(Z_c, \bar{\mu}[K])] \\ &\leq \sum_{k=0}^{K-1} \eta_k [\mathcal{L}(Z[k], \mu) - \mathcal{L}(Z_c, \mu[k])]. \end{aligned}$$

Adding (33), summed over i , to (32), and then summing over k , makes all distance terms telescope. Since $z_i[0], z \in \mathcal{Z}$, $\mu_i[0] = 0$, and the dropped terminal squared distances are nonnegative, we obtain

$$\begin{aligned} &S_K [\mathcal{L}(\bar{Z}[K], \mu) - \mathcal{L}(Z_c, \bar{\mu}[K])] \\ &\leq \frac{ND_Z^2}{2} + \sigma \sum_{k=0}^{K-1} \eta_k |\langle G[k], L_G Z[k] \rangle_F| \\ &\quad + \frac{1}{2} \sum_{k=0}^{K-1} \eta_k^2 \sum_i (\|g_i[k]\|_2^2 + |r_i[k]|^2) \\ &\quad + \left(\frac{1}{2} + \frac{1}{4} \sum_{k=0}^{K-1} \eta_k \beta_k\right) \|\mu\|_2^2. \end{aligned} \quad (34)$$

It remains to control the mixing and squared-direction terms. Since $L_G J = 0$, Cauchy-Schwarz, (26), and the definition of e_k yield

$$\begin{aligned} |\langle G[k], L_G Z[k] \rangle_F| &\leq \|G[k]\|_F \|L_G Z[k]\|_F \\ &\leq N \|L_G\|_2 C_g (k+1)^q e_k. \end{aligned}$$

Furthermore, (26)–(27) imply

$$\sum_i (\|g_i[k]\|_2^2 + |r_i[k]|^2) \leq C_1 (k+1)^{2q}$$

for a uniform constant $C_1 > 0$. Substituting these two bounds into (34), absorbing all fixed constants into $C_0 > 0$, and dividing by S_K gives

$$\mathcal{L}(\bar{Z}[K], \mu) - \mathcal{L}(Z_c, \bar{\mu}[K]) \leq R_K + Q_K \|\mu\|_2^2,$$

where R_K and Q_K are exactly those defined in Lemma 10. The constant C_0 is independent of w , K , and the initial primal points because D_Z and all constants above are uniform.

E. Proof of Theorem 7

Fix $w \in \mathcal{W}$ and choose

$$\begin{aligned} \theta_w^* &\in \arg \min_{\theta \in \Theta} F(\theta, w), \\ z_w^* &:= (\theta_w^*, G(w)), \\ Z_c &:= \mathbf{1}(z_w^*)^\top. \end{aligned}$$

The point z_w^* is feasible for the epigraph problem, so

$$\mathcal{L}(Z_c, \bar{\mu}[K]) \leq G(w). \quad (35)$$

Let i_K attain

$$F(\bar{\theta}_w[K], w) = h_{i_K}^w(\bar{\theta}_w[K]),$$

let e_{i_K} be the corresponding coordinate unit vector, and set $\bar{t}_w[K] := N^{-1} \sum_i \bar{t}_i[K]$. Then

$$\mathcal{L}(\bar{Z}[K], e_{i_K}) = \bar{t}_w[K] + h_{i_K}^w(\bar{\theta}_{i_K}[K]) - \bar{t}_{i_K}[K].$$

Since $h_{i_K}^w$ is G_ψ -Lipschitz and $\|\bar{z}_{i_K}[K] - \bar{z}_w[K]\|_2 \leq D_K$,

$$\begin{aligned} F(\bar{\theta}_w[K], w) &\leq h_{i_K}^w(\bar{\theta}_{i_K}[K]) + G_\psi D_K \\ &\leq \mathcal{L}(\bar{Z}[K], e_{i_K}) + (G_\psi + 1)D_K. \end{aligned}$$

Applying Lemma 10 with $\mu = e_{i_K}$, using $\|e_{i_K}\|_2 = 1$, and invoking (35) gives

$$0 \leq \hat{G}_K(w) - G(w) \leq R_K + Q_K + (G_\psi + 1)D_K. \quad (36)$$

The lower bound follows directly from $G(w) = \min_{\theta \in \Theta} F(\theta, w)$.

It remains to bound the three terms uniformly. Let $H_r(K) := \sum_{k=1}^K k^{-r}$. From the definitions of R_K and Q_K , the pointwise consensus estimate (30), and $S_K \geq cK^{1-\tau}$, the right-hand side of (36) is bounded by a uniform constant multiple of

$$K^{\tau-1} [1 + H_{2\tau-2q}(K) + H_{\tau+q}(K) + H_{2\tau-q}(K)].$$

Indeed,

$$\begin{aligned} \sum_{k=0}^{K-1} \eta_k^2 (k+1)^{2q} &= c^2 H_{2\tau-2q}(K), \\ \sum_{k=0}^{K-1} \eta_k (k+1)^q e_k &\leq cC_e H_{2\tau-2q}(K), \\ \sum_{k=0}^{K-1} \eta_k \beta_k &= c\beta_0 H_{\tau+q}(K), \end{aligned}$$

while Lemma 9 supplies the $H_{2\tau-q}(K)$ term through D_K .

Applying Lemma 14 gives the respective decay exponents

$$\begin{aligned} 1 - \tau, & \quad \min\{1 - \tau, \tau - 2q\}, \\ \min\{1 - \tau, q\}, & \quad \min\{1 - \tau, \tau - q\}, \end{aligned}$$

up to a factor $1 + \log K$. Their minimum is

$$\gamma_c(\tau, q) = \min\{q, \tau - 2q, 1 - \tau\} > 0.$$

Therefore, for a constant $C > 0$ independent of w , K , and the initial primal points,

$$0 \leq \hat{G}_K(w) - G(w) \leq C(1 + \log K)K^{-\gamma_c(\tau, q)}.$$

Setting $\tau = 3/4$ and $q = 1/4$ gives

$$\gamma_c(\tau, q) = \min\{q, \tau - 2q, 1 - \tau\} = \frac{1}{4}.$$

Therefore,

$$0 \leq \hat{G}_K(w) - G(w) \leq C(1 + \log K)K^{-1/4} = \tilde{\mathcal{O}}(K^{-1/4})$$

uniformly over $w \in \mathcal{W}$, which proves Theorem 7.

F. Proof of Lemma 11

Let $w = (y_1, \dots, y_{d+1})$ and $\tilde{w} = (\tilde{y}_1, \dots, \tilde{y}_{d+1})$. For any i , θ , and $y, \tilde{y} \in \mathcal{Y}$, the definitions of ℓ_i , C_θ , L_ψ , and L_u give

$$\begin{aligned} |\ell_i(\theta, y) - \ell_i(\theta, \tilde{y})| &\leq |\langle \theta, \Delta\psi(y; i) - \Delta\psi(\tilde{y}; i) \rangle| \\ &\quad + |\Delta u_i(y; i) - \Delta u_i(\tilde{y}; i)| \\ &\leq 2(C_\theta L_\psi + L_u)\|y - \tilde{y}\|_2 \\ &= L_{\text{game}}\|y - \tilde{y}\|_2. \end{aligned}$$

Taking finite maxima over players and tuple components preserves this common Lipschitz bound. Thus, for every $\theta \in \Theta$,

$$|F(\theta, w) - F(\theta, \tilde{w})| \leq L_{\text{game}} d_{\mathcal{W}}(w, \tilde{w}).$$

Let $\theta_{\tilde{w}}$ minimize $F(\cdot, \tilde{w})$. Then

$$\begin{aligned} G(w) - G(\tilde{w}) &\leq F(\theta_{\tilde{w}}, w) - F(\theta_{\tilde{w}}, \tilde{w}) \\ &\leq L_{\text{game}} d_{\mathcal{W}}(w, \tilde{w}). \end{aligned}$$

Interchanging w and $\tilde{w}$ proves

$$|G(w) - G(\tilde{w})| \leq L_{\text{game}} d_{\mathcal{W}}(w, \tilde{w}).$$

If w^* and w lie in the same cell $\mathcal{W}_{h,r}$, the partition-diameter condition consequently gives

$$G(w^*) - G(w) \leq L_{\text{game}} \alpha_{\mathcal{W}} \rho^h = \nu_{\text{game}} \rho^h.$$

G. Proof of Theorem 12

Theorem 7 gives the uniform one-sided oracle bound for all $w \in \mathcal{W}$, $K \geq 1$,

$$0 \leq \hat{G}_K(w) - G(w) \leq C(1 + \log K)K^{-1/4}, \quad (37)$$

for some constant $C > 0$ independent of w and K .

Lemma 11 and the geometric partition-diameter condition in (18) imply that every cell $\mathcal{W}_{h,r}$ containing an optimizer w^* satisfies

$$\sup_{w \in \mathcal{W}_{h,r}} (G(w^*) - G(w)) \leq \nu_{\text{game}} \rho^h.$$

Moreover, (19) supplies the admissible near-optimality exponent d_G .

To make the oracle error in (37) comparable to the depth- h resolution ρ^h , an inner budget of order

$$K_h = \tilde{\mathcal{O}}(\rho^{-4h})$$

is sufficient. At the same depth, the number of near-optimal cells is at most of order $\rho^{-d_G h}$. Consequently, the total inner work required by the EXOTIC search to resolve depth h is

$$\tilde{\mathcal{O}}(\rho^{-(d_G+4)h}).$$

Equivalently, under total inner-iteration budget n , the finest achievable resolution satisfies

$$\rho^h = \tilde{\mathcal{O}}\left(n^{-1/(d_G+4)}\right).$$

This is the budget argument in [10, Theorem 3.3 and Corollary 3.5], with the logarithmic oracle factor absorbed into $\tilde{\mathcal{O}}$. Hence, for the refined estimate returned by EXOTIC,

$$p^* - \hat{G}_n = \tilde{\mathcal{O}}\left(n^{-1/(d_G+4)}\right), \quad (38)$$

where $\hat{G}_n := \hat{G}_{K_{\text{final}}}(\hat{w}_n)$.

Under the EXOTIC budget rule,

$$h_{\max} = \left\lfloor \frac{2n}{5B(1 + \log n)^2} \right\rfloor, \quad K_{\text{final}} = \lfloor h_{\max}/2 \rfloor = \tilde{\Theta}(n).$$

Therefore, the final re-evaluation satisfies

$$0 \leq \hat{G}_n - G(\hat{w}_n) = \tilde{\mathcal{O}}(n^{-1/4}).$$

Combining this bound with (38) gives

$$\begin{aligned} p^* - G(\hat{w}_n) &= p^* - \hat{G}_n + \hat{G}_n - G(\hat{w}_n) \\ &= \tilde{\mathcal{O}}\left(n^{-1/(d_G+4)}\right) + \tilde{\mathcal{O}}(n^{-1/4}). \end{aligned}$$

Since $d_G \geq 0$,

$$\frac{1}{d_G + 4} \leq \frac{1}{4},$$

so the final re-evaluation error is no slower than the outer-search rate. Thus,

$$0 \leq p^* - G(\hat{w}_n) = \tilde{\mathcal{O}}\left(n^{-1/(d_G+4)}\right).$$

APPENDIX IV

THE EXOTIC TREE-SEARCH PROCEDURE

For completeness, Algorithm 2 reproduces the EXOTIC procedure of [10], with minor notational changes and corrections of typographical errors. We use B for the branching factor, reserving K for the DRPDM iteration budget. For $w \in \mathcal{W}$, $z \in \varphi(w)$, and $H \geq 1$, define

$$\text{Eval}_H(w, z) := F(\text{OPT}(w, z, H), w).$$

In our implementation, OPT is instantiated by the distributed oracle in Algorithm 1.

APPENDIX V

PROJECTED ZERO-TH-ORDER ASCENT

Algorithm 3 gives the complete projected zeroth-order procedure. Each outer iteration uses two DRPDM evaluations, and the final tuple is re-evaluated using the larger budget K_{final} . The exponents r_δ and r_α control the decay of the perturbation radius and outer stepsize, respectively; their numerical values are specified in Section V.

Algorithm 2 EXOTIC tree search [10]

Require: Depth $h_{\max}$, branching factor B , representatives $\{(w_{h,r}, z_{h,r})\}_{h,r}$

- 1: Initialize $\mathcal{T}^{(B)}$ with root $(w_{0,1}, G_{0,1} = 0, z_{0,1}, \#_{0,1} = 0)$
- 2: **for** $r = 1, \dots, B$ **do**
- 3: $G_{1,r} \leftarrow \text{Eval}_{h_{\max}}(w_{1,r}, z_{1,r})$
- 4: $\#_{1,r} \leftarrow h_{\max}$; append $(1, r)$ to the root
- 5: **end for**
- Tree-expansion phase**
- 6: **for** $h = 1, \dots, h_{\max}$ **do**
- 7: **for** $m = 1, \dots, \lfloor h_{\max}/h \rfloor$ **do**
- 8: $H \leftarrow \lfloor h_{\max}/(hm) \rfloor$
- 9: $(h, \bar{r}) \leftarrow \arg \max_{\substack{(h,r) \in \mathcal{T}_{\text{leaf}}^{(B)} \\ \#_{h,r} \geq H}} G_{h,r}$
- 10: Append the B children of $(h, \bar{r})$
- 11: **for** $r = 1, \dots, B$ **do**
- 12: $G_{h+1,r} \leftarrow \text{Eval}_H(w_{h+1,r}, z_{h+1,r})$
- 13: $\#_{h+1,r} \leftarrow H$
- 14: **end for**
- 15: **end for**
- 16: **end for**
- Re-evaluation phase**
- 17: **for** $p = 0, \dots, \lfloor \log_2 h_{\max} \rfloor$ **do**
- 18: $(h_p, r_p) \leftarrow \arg \max_{\substack{(h,r) \in \mathcal{T}^{(B)} \\ \#_{h,r} \geq 2^p}} G_{h,r}$
- 19: $G_{h_p, r_p} \leftarrow \text{Eval}_{\lfloor h_{\max}/2 \rfloor}(w_{h_p, r_p}, z_{h_p, r_p})$
- 20: **end for**
- 21: $\hat{p} \leftarrow \arg \max_{0 \leq p \leq \lfloor \log_2 h_{\max} \rfloor} G_{h_p, r_p}$
- 22: $\hat{w}_n \leftarrow w_{h_{\hat{p}}, r_{\hat{p}}}$
- 23: $\hat{G}_n \leftarrow G_{h_{\hat{p}}, r_{\hat{p}}}$
- 24: **return** $\hat{G}_n, \hat{w}_n$

APPENDIX VI

EXACT FULL-DOMAIN ERROR FOR LINEAR-QUADRATIC NETWORK GAMES

This appendix derives the closed-form expression used to evaluate the full-domain error in the linear-quadratic network-game experiments. Recall the utility functions and quadratic potential parameterization introduced in Section V-C. For a unilateral deviation $y = (a, a'_i)$, the corresponding error is

$$\ell_i(\theta, y) = (a'_i - a_i) \left[A_i(\theta)(a'_i + a_i) + B_i(\theta) + \sum_{j \neq i} C_{ij}(\theta) a_j \right], \quad (39)$$

where $A_i(\theta) := p_i + \frac{1}{2}$, $B_i(\theta) := q_i - \zeta_i$, and for all $j \neq i$, $C_{ij}(\theta) := r_{\min\{i,j\}, \max\{i,j\}} - \gamma V_{ij}$. Define

$$K_i(\theta) := |B_i(\theta)| + \sum_{j \neq i} |C_{ij}(\theta)|. \quad (40)$$

Maximizing (39) over the full action domain gives

$$\mathcal{E}(\theta) = \max_{i \in [N]} H_i(\theta), \quad (41)$$

where

$$H_i(\theta) = \begin{cases} 2K_i(\theta), & K_i(\theta) \geq 2|A_i(\theta)|, \\ |A_i(\theta)| + K_i(\theta) + \frac{K_i(\theta)^2}{4|A_i(\theta)|}, & 0 \leq K_i(\theta) < 2|A_i(\theta)|. \end{cases} \quad (42)$$

Algorithm 3 Projected Zeroth-Order Ascent with DRPDM Evaluations

Require: Outer budget T ; inner budgets $\{K_t\}_{t=0}^{T-1}$; final budget K_{final} ; $\delta_0, \alpha_0 > 0$ with $\mathcal{W}_{\delta_0} \neq \emptyset$; $w_0 \in \mathcal{W}_{\delta_0}$; set $\delta_t = \delta_0(t+1)^{-r_\delta}$ and $\alpha_t = \alpha_0(t+1)^{-r_\alpha}$.

- 1: **for** $t = 0, \dots, T-1$ **do**
- 2: Sample $v_t \sim \text{Unif}(\mathbb{S}^{D_{\mathcal{W}}-1})$
- 3: $w_t^+ \leftarrow w_t + \delta_t v_t$
- 4: Run Algorithm 1 on w_t and w_t^+ , each for K_t iterations with multipliers initialized to zero
- 5: Obtain $\hat{G}_t^0$ and $\hat{G}_t^+$
- 6: $\hat{g}_t \leftarrow (D_{\mathcal{W}}/\delta_t)(\hat{G}_t^+ - \hat{G}_t^0)v_t$
- 7: $w_{t+1} \leftarrow P_{\mathcal{W}_{\delta_{t+1}}}(w_t + \alpha_t \hat{g}_t)$
- 8: **end for**
- 9: Run Algorithm 1 on w_T for K_{final} iterations
- 10: **return** w_T , $\hat{\theta}_T := \hat{\theta}_{w_T}[K_{\text{final}}]$, and $\hat{G}_T := F(\hat{\theta}_T, w_T)$

When $A_i(\theta) = 0$, the first case applies. Hence, all reported full-domain errors are evaluated exactly, without discretizing or sampling the action space.