

Free energy landscape of Dense Associative Memory

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Using large deviations theory, we solve and obtain a general expression for the free energy functional for a broad class of associative memories, including dense associative memories. We illustrate the method by reproducing classical results for the Hopfield model. For a finite number of patterns, we derive the temperature-dependent free energy functional for dense associative memories featuring polynomial interactions and Log-Sum-Exponential (LSE) activation. We also evaluate the disorder-averaged ground-state energy of these systems in the extensive limit. Our analytical framework reveals how memory retrieval depends on the initial state in higher-order dense networks, and gives the exact full-retrieval threshold for the LSE model. This method provides a systematic procedure for analyzing diverse, complex architectures in associative memory.

Dense Associative Memories (DenseAMs) are energy-based neural architectures that vastly surpass traditional Hopfield networks by using higher-than-quadratic neuron interactions to expand memory storage capacity [1–3]. In these networks, stored patterns represent low-energy configurations, allowing the system to perform powerful error correction by mapping partial or corrupted inputs back to the correct pattern. This process closely mirrors the biological error-correction mechanisms of the human brain, which makes it possible for it to retrieve memories from incomplete information [4]. Because of this high-capacity retrieval and error-correction, DenseAMs have become useful tools in areas like modern deep learning, generative AI, pattern recognition, and transformer architectures [5–8].

In his seminal work, Hopfield mapped neuronal states to Ising spins and synaptic weights to the Hebb rule [9]. This established a spin-glass-like model where specific configurations serve as stored memories. As a result, the model was successfully solved using the replica method [10, 11].

In associative memory (AM) models, gradient descent dynamics drive the retrieval of stored memories by guiding the system toward the nearest local minimum on a free-energy landscape. The recent work in DenseAMs though have primarily concentrated on energy dynamics rather than the full free-energy framework. In this paper, we derive the free energy landscape for a general class of associative memories and use it to study higher DenseAMs [1, 12–14]. We develop a method of solution that makes use of the tilted property of the measure of exponential functions to obtain the rate function. We had earlier used similar methods to study random field quenched disorder for ferromagnets [15–17].

We consider models with neurons as Ising spins. Each neuron exists in two states : ± 1 . A pattern is a configuration of N binary random variables $\{\xi_1^\mu, \dots, \xi_N^\mu\}$, where μ takes P values for P stored patterns. For a configuration

$C : \{s_1, s_2, \dots, s_i, \dots, s_N\}$, a random variable

$$m^\mu = \frac{1}{N} \sum_{i=1}^N \xi_i^\mu s_i \quad (1)$$

captures the overlap between the μ^{th} memory and C . If the configuration C matches with the pattern μ , then $m^\mu = 1$. We define a P dimensional vector $\mathbf{m} = \{m^1, m^2, \dots, m^P\}$ to represent the overlap between a configuration and the patterns.

A general DenseAM [1] is defined by an energy function/Hamiltonian of the form:

$$H(\mathbf{m}) = - \sum_{\mu=1}^P F \left(\sum_{i=1}^N \xi_i^\mu s_i \right) = -Nf(\mathbf{m}) \quad (2)$$

where $F(x)$ is a smooth function. The gradient descent dynamics then works in the direction of lowering the energy and stops when energy can no longer be decreased via single spin flip [5]. Hence H can be considered the cost function of the memory retrieval which is to be minimised. In this work we develop a formalism using large deviations to get the free energy for any function f .

For P patterns represented by the vectors ξ^μ , the joint probability distribution of the order parameter $(\mathbf{m})$, i.e., $Q(\mathbf{m})$ in the absence of cost function is given by

$$Q(\mathbf{m}) = \sum_{\{s_i\}} p(\{s_i\}) \prod_{\mu=1}^P \delta \left(\sum_{i=1}^N \xi_i^\mu s_i = Nm^\mu \right) \quad (3)$$

here $p(\{s_i\})$ is the probability of having a configuration $C = \{s_i\}$. Using the integral representation of the delta function by introducing a set of variables $\lambda = \{\lambda_1, \dots, \lambda_\mu, \dots, \lambda_P\}$ we get

$$Q(\mathbf{m}) = \int \frac{d^P \lambda}{(2\pi)^P} \exp(-N\lambda \cdot \mathbf{m}) \quad (4)$$

$$\left(\sum_{\{s_i\}} p(\{s_i\}) \exp \left(\sum_{\mu} \lambda_{\mu} \sum_i \xi_i^{\mu} s_i \right) \right)$$

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The sum over spins s_i can be done easily to give

$$Q(\mathbf{m}) = \int \frac{d^P \boldsymbol{\lambda}}{(2\pi)^P} \exp(-N \boldsymbol{\lambda} \cdot \mathbf{m}) \prod_{i=1}^N \cosh \left(\sum_{\mu} \lambda_{\mu} \xi_i^{\mu} \right) \quad (5)$$

$$= \int \frac{d^P \boldsymbol{\lambda}}{(2\pi)^P} \exp[-N(\boldsymbol{\lambda} \cdot \mathbf{m} - \Phi(\boldsymbol{\lambda}))]$$

where $\Phi(\boldsymbol{\lambda}, \boldsymbol{\xi}) = \frac{1}{N} \sum_{i=1}^N \log(\cosh(\lambda_{\mu} \xi_i^{\mu}))$. In the limit $N \rightarrow \infty$, via the law of large numbers we get

$$\Phi(\boldsymbol{\lambda}) = \mathbb{E}_{\boldsymbol{\xi}}[\log \cosh(\lambda_{\mu} \xi^{\mu})], \quad (6)$$

where, $\mathbb{E}_{\boldsymbol{\xi}}[\dots]$ is the average over the distribution from which the patterns $\boldsymbol{\xi}$ have been drawn. The contours of integration in Eq. 5 are understood to be analytically deformed, so that they pass the saddle point. The probability $Q(\mathbf{m})$ satisfies large deviation principle with rate function $I_0(\mathbf{m})$, i.e $Q(\mathbf{m}) \asymp \exp[-N I_0(\mathbf{m})]$. The saddle point approximation then gives

$$I_0(\mathbf{m}) = \sup_{\boldsymbol{\lambda} \in \mathbb{R}^P} [\boldsymbol{\lambda} \cdot \mathbf{m} - \Phi(\boldsymbol{\lambda})] \quad (7)$$

Let $\boldsymbol{\lambda}^*$ be the supremum of the function on the right. It is a solution of a set of P equations of the kind

$$m_{\mu}^* = \left. \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_{\mu}} \right|_{\boldsymbol{\lambda}^*} \quad (8)$$

In general, in order to find $\boldsymbol{\lambda}^*$, we need to find the solution of the above P dimensional array of equations. For $P = 1$ it is easy and $I_0(\mathbf{m})$ is the rate function of a set of N non-interacting Ising spins. For higher P we circumvent the step where the hard inversion need to be performed by developing a procedure similar to the one used by us for random field ferro-magnets [16, 17].

The probability of a configuration C of neurons under Gibbs measure is proportional to $\exp(-\beta H(\mathbf{m}))$ with β as the inverse temperature that controls the noise. Since $\mathbf{m}$ is a random variable drawn from a distribution $Q(\mathbf{m})$ that is defined on $\{-1, 1\}^P$, the probability $Q_{H,\beta}(\mathbf{m})$, in the presence of the cost function $f(\mathbf{m})$ is given by

$$Q_{H,\beta}(\mathbf{m}) = \int_A Q(\mathbf{m}) \exp(N\beta f(\mathbf{m})) \quad (9)$$

where A is the subset of all possible configurations compatible with the value $\mathbf{m}$ for P patterns. For $Q_{H,\beta}(\mathbf{m}) \sim \exp(-NI)$, the rate function I can be calculated using the tilted large deviations principal [18, 19] that connects I and I_0 through the relation

$$I_{\beta}(\mathbf{m}) = I_0(\mathbf{m}) - \beta f(\mathbf{m}) \quad (10)$$

where $I_0(\mathbf{m}) = \boldsymbol{\lambda}^* \cdot \mathbf{m} - \Phi(\boldsymbol{\lambda}^*)$. The function $I_{\beta}(\mathbf{m})$ is the free energy landscape for the system whose global minima

gives the value of the free energy at a given β . Let $\mathbf{m}_*$ be a fixed point of I_{β} , then at $\mathbf{m}_*$, since $\left. \frac{\partial I_{\beta}}{\partial m^{\mu}} \right|_{\mathbf{m}_*} = 0$,

$$\left. \frac{\partial I_0}{\partial m^{\mu}} \right|_{\mathbf{m}_*} = \beta \left. \frac{\partial f}{\partial m^{\mu}} \right|_{\mathbf{m}_*} \quad (11)$$

Hence, we have

$$\begin{aligned} \frac{\partial I_0}{\partial m^{\mu}} &= \partial [\boldsymbol{\lambda}^*(\mathbf{m}) \cdot \mathbf{m} - \Phi(\boldsymbol{\lambda}^*(\mathbf{m}))] / \partial m^{\mu} \\ &= \lambda_{\mu}^* + \sum_{\nu} m^{\nu} \left. \frac{\partial \lambda_{\nu}}{\partial m^{\mu}} \right|_{\boldsymbol{\lambda}^*} - \sum_{\nu} \left. \frac{\partial \lambda_{\nu}}{\partial m^{\mu}} \right|_{\boldsymbol{\lambda}^*} \left. \frac{\partial \Phi(\boldsymbol{\lambda})}{\partial \lambda^{\nu}} \right|_{\boldsymbol{\lambda}^*} \end{aligned} \quad (12)$$

Substituting back in Eq. 11, we get:

$$\lambda_{\mu}^* = \beta \left. \frac{\partial f}{\partial m^{\mu}} \right|_{\mathbf{m}_*} \quad (13)$$

For our case of binary spins we use Eq. 6 to arrive at the P self-consistency equations for fixed point $\{\mathbf{m}_*^{\mu}\}$. They are

$$m_{\mu}^* = \mathbb{E}_{\boldsymbol{\xi}} \left[\xi^{\mu} \tanh \left(\sum_{\nu} \beta \left. \frac{\partial f}{\partial m^{\nu}} \right|_{\mathbf{m}_*} \xi^{\nu} \right) \right] \quad (14)$$

The rate function, which is the generalised free energy functional of the DenseAMs comes out to be

$$\begin{aligned} I_{H,\beta}(\mathbf{m}) &= \beta \mathcal{F}(\mathbf{m}) = \beta \sum_{\nu} m_{\nu}^* \left. \frac{\partial f}{\partial m^{\nu}} \right|_{\mathbf{m}_*} \\ &\quad - \mathbb{E}_{\boldsymbol{\xi}} \left[\log \cosh \left(\sum_{\nu} \beta \left. \frac{\partial f}{\partial m^{\nu}} \right|_{\mathbf{m}_*} \xi^{\nu} \right) \right] - \beta f(\mathbf{m}_*). \end{aligned} \quad (15)$$

We have defined $\mathcal{F}(\mathbf{m}) = \beta I_{H,\beta}$, as the free energy functional of the system and the free energy is $\frac{1}{\beta} \inf_{\mathbf{m}} I_{H,\beta}$.

Dense Associative memory Hopfield Model with polynomial interaction: The Hamiltonian is

$$H_N = -\frac{N}{k!} \sum_{\mu=1}^P (m^{\mu})^k \quad (16)$$

The $f(\mathbf{m}) = \frac{1}{k!} \sum_{\mu=1}^P (m^{\mu})^k$ and $\lambda_{\mu}^* = \frac{\beta}{(k-1)!} (m^{\mu})^{k-1}$. Substituting in Eq. 14 and 15 we get

$$m^{\mu} = \mathbb{E}_{\boldsymbol{\xi}} \left[\xi^{\mu} \tanh \left(\sum_{\nu} \frac{\beta}{(k-1)!} (m^{\nu})^{k-1} \xi^{\nu} \right) \right] \quad (17)$$

$$\mathcal{F}(\mathbf{m}) = \frac{k-1}{k!} \sum_{\mu=1}^P (m^{\mu})^k \quad (18)$$

$$- \frac{1}{\beta} \mathbb{E}_{\boldsymbol{\xi}} \left[\log \cosh \left(\sum_{\nu} \frac{\beta}{(k-1)!} (m^{\nu})^{k-1} \xi^{\nu} \right) \right]$$

Note that while for even k , the above equations hold for $-1 < m^\mu < 1$, for odd k they are valid only for positive values $0 < m^\mu < 1$. This is because for odd k , negative values of m^μ results in a higher energy than the mirror state with positive m^μ . The Eq. 17 matches with the stochastic dynamics fixed point equation for DenseAMs obtained recently[20]. For $k = 2$, the expression of $\mathcal{F}(\mathbf{m})$ derived above matches with the classic results in [10].

The $\mathcal{F}(\mathbf{m})$ for $k > 2$ however was not known from the earlier studies. Our method gives the exact expression of the free energy functional for any k . The large deviations approach allowed us to handle higher order interactions with ease, which is not possible with the standard Hubbard-Stratonovich transformation. We can study any finite P using Eqs. 17 and 18. For $P = 1$, the $\mathcal{F}$ and m_μ equation are similar to that of a k -spin Ising model. Hence single pattern DenseAMs has continuous transition for $k = 2$ and first order transition for $k \geq 3$. The transition temperatures for example for $k = 2, 3, 4$ are $\beta_c = 1, 0.23, 0.04$ respectively. The stability analysis can also be performed straightforwardly by finding the eigenvalues of the Hessian for $P = 2$. We would though focus on the extensive limit in the rest of this section.

The capacity of a network is defined as the maximum number of patterns that can be stored such that a randomly chosen pattern is retrieved with small error as $N \rightarrow \infty, P \rightarrow \infty$ at zero temperature. We study the $\mathcal{F}(\mathbf{m})$ by taking $N \rightarrow \infty, P \rightarrow \infty$ along with $\beta \rightarrow \infty$ for the retrieval of a randomly chosen pattern. Each of $(m^\mu)^k$ corresponding to other patterns is taken as a random variable distributed with mean 0 and standard deviation $\frac{e_k}{N^{(k-1)/2}}$ [1, 20], where e_k is a constant dependent on k . Then by central limit theorem, the $z = \sum_{\mu=2}^P (m^\mu)^{k-1} \xi_\mu$ can be taken as a Gaussian random variable with mean 0 and variance $P e_k^2 / N^{k-1}$. For $P = \alpha N^{k-1}$, $\gamma = e_k^2 \alpha$ is the variance and $p(z) = \frac{1}{\sqrt{2\pi\gamma}} e^{-z^2/2\gamma}$. Separating $m^1 = m$ from other patterns and performing the average over ξ^1 and z in the limit of $\beta \rightarrow \infty$, we get the disorder averaged ground state energy of the system as

$$R(\mathbf{m}) = \frac{k-1}{k!} m^k - \frac{1}{(k-1)!} \sqrt{\frac{2\gamma}{\pi}} \exp\left(\frac{-m^{2(k-1)}}{2\gamma}\right) + \frac{m^{k-1}}{(k-1)!} \operatorname{erf}\left(\frac{m^{k-1}}{\sqrt{2\gamma}}\right) \quad (19)$$

We dropped the term $\sum_{\mu=2}^P (m^\mu)^k$ while writing the above expression as its mean is 0 and variance decays as $1/N$. The Eq. 19 gives the landscape of the gradient dynamics [21].

The fixed points of Eq. 19 are given by

$$m = \operatorname{erf}\left(\frac{m^{k-1}}{\sqrt{2\alpha}}\right) \quad (20)$$

For $k = 2$ this equation is the same as the fixed point equation for m obtained in [11], with $\gamma = r\alpha$, where

k	γ_g	γ_l	m_*
2	0.66	0.66	0
3	0.18	0.26	0.85
4	0.1	0.2	0.92
5	0.063	0.17	0.95
10	0.015	0.13	0.98

Table I: For $\gamma < \gamma_l$ a local minima at m_* in $R(m)$ appears which becomes a global minima at γ_g .

r is mean square random overlap obtained via replica calculation.

Let us study the consequence of $R(m)$ for DenseAMs in a bit more detail: for $k = 2$, the function $R(m)$ has a minima at $m = 0$ for high γ , which continuously changes into a double well as γ is lowered as shown in Fig. 1. The system falls into one of the two minima spontaneously with decreasing γ at $\gamma_c = 2/\pi$. The local and global minima are the same and gradient descent always reaches the global minima.

For $k > 2$ the system undergoes a first order transition from $m = 0$ to $m \neq 0$ state. The $m = 0$ stops being a global minima at γ_g but continues to be a local minima as γ is decreased. This can be seen by evaluating the second derivative of $R(m)$ at $m = 0$. $\chi(m) = \partial^2 R(m) / \partial m^2 = 1 - (k-1) \sqrt{2/\pi\gamma} m^{k-2} e^{m^{-2(k-1)}/2\gamma}$. For $k > 2$, $\chi(0) = 1$ and hence $m = 0$ is a minima for all values of γ , though it stops being a global minima at a certain γ_g (see Fig. 1). As a result, the steady state of gradient descent depends on the initial starting state of the system. If the initial state is in the basin of attraction of $m = 0$, the pattern is not retrieved for any γ . At a threshold $\gamma_l < \gamma_g$, the $m \neq 0$ minima shows up for the first time. As k increases the basin of attraction of this non zero fixed point shrinks, though the minima gets closer to $m = 1$. As a result there is less error in retrieval, provided the starting state is in the basin of the retrieval state. Since basin reduces, choices of initial state for successful memory retrieval gets more restricted.

The threshold on α_k for reliable retrieval of memory depends on the percentage of allowed error (given by $(1-m)/2$). This for 1.5% error for Hopfield model is $\alpha \approx 0.14$ [3, 11]. A bound for 0.5% was obtained in [1]. This threshold is different than the transition threshold discussed above as the m at the transition though non zero approaches 1 only in the limit of $k \rightarrow \infty$ (see Table I). We show next for LSE the two thresholds match completely due to full pattern retrieval at and below the transition.

Log-Sum-Exponential (LSE) model: The cost function for binary uncorrelated patterns can be written as

$$H(\mathbf{m}, \lambda) = \frac{N}{2} \sum_{\mu} (m^\mu)^2 - \frac{1}{\lambda} \log \left(\sum_{\mu=1}^P \exp(N\lambda m^\mu) \right) \quad (21)$$

where λ is the interaction strength.

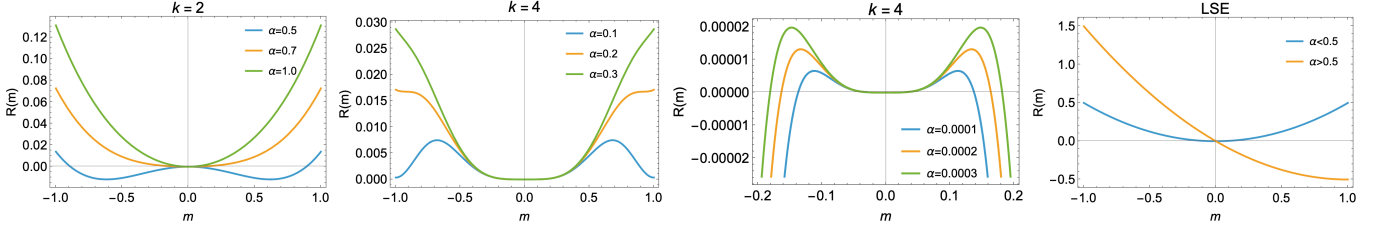


Figure 1: Plot of disorder averaged ground state energy of retrieval ($R(m)$) of a random memory when $N, P \rightarrow \infty$. We have taken $e_k = 1$ and hence $\gamma = \alpha$. a) for Hopfield model, b) $k = 4$ DenseAM. In both cases the function $R(m)$ is plotted near, above and below the transition. In (c) $k = 4$ basin of $m = 0$ state is shown to illustrate that it is always present. In (d) $R(m)$ for LSE model above and below the transition is shown.

We define $\phi = \frac{1}{\lambda N} \ln \sum_{\mu=2}^P \exp(\lambda N m^\mu)$ as was done in [12]. Then we get

$$f = \frac{1}{2} \sum_{\mu=1}^P (m^\mu)^2 - \frac{1}{N\lambda} \log \left(e^{\lambda N m^1} + e^{\lambda N \phi} \right) \quad (22)$$

For small N , substituting the f above in Eqs. 14 and 15 gives the exact $\mathcal{F}(\mathbf{m})$. We instead focus on the large N limit here. Using the law of large numbers we approximate $\phi \sim P \langle \exp(\lambda N m^\mu) \rangle_{m^\mu}$. The m^μ for $\mu \neq 1$ are random variables drawn from a Gaussian distribution with mean 0 and standard deviation $\sigma/\sqrt{N}$. Hence $p(m^\mu) \sim \exp(-N I_0(m^\mu))$ with $I_0 = (m^\mu)^2/(2\sigma^2)$. Using the tilted LDP, we then get, $\langle \exp(\lambda N m^\mu) \rangle_{p(m^\mu)} \sim \exp(-N \lambda^2 \sigma^2/2)$. Assuming $P \sim \exp(\alpha N)$, we get

$$\phi = \frac{1}{\lambda} \left(\alpha + \frac{\lambda^2 \sigma^2}{2} \right) \quad (23)$$

The extrema of ϕ occurs at $\lambda^* = \sqrt{2\alpha}/\sigma$. For $\lambda < \lambda^*$, ϕ increases rapidly with increasing λ and is a slow increasing function for $\lambda > \lambda^*$.

The f hence takes two different forms for $\phi > m^1$ and for $\phi < m^1$ as follows:

$$f_{\phi < m^1} = \frac{1}{2} \sum_{\mu=1}^P (m^\mu)^2 - m^1 \quad (24)$$

$$f_{\phi > m^1} = \frac{1}{2} \sum_{\mu=2}^P (m^\mu)^2 - \phi \quad (25)$$

This gives

$$\mathcal{F}_{\phi < m^1}(\mathbf{m}) = \frac{1}{2} \sum_{\mu=1}^P m^{\mu 2} - m^1 \quad (26)$$

$$- \frac{1}{\beta} \mathbb{E}_\xi \left[\log \cosh \left(\beta (m^1 - 1) \xi^1 + \beta \sum_{\mu=2}^P m^\mu \xi^\mu \right) \right]$$

$$\mathcal{F}_{\phi > m^1}(\mathbf{m}) = \frac{1}{2} \sum_{\mu=1}^P m^{\mu 2} - \frac{1}{\beta} \mathbb{E}_\xi \left[\log \cosh \left(\beta \sum_{\mu=1}^P m^\mu \xi^\mu \right) \right] \quad (27)$$

For $\phi > m^1$ the model behaves just like the standard Hopfield model. Since $P \propto \exp(\alpha N)$ and the capacity of Hopfield model is $O(N)$, there is no retrieval in this case. The memory retrieval is possible only for $\phi < m^1$. For $\beta \rightarrow \infty$, we get

$$R_{\phi < m^1}(m) = \frac{1}{2} (m^1)^2 - m^1 - \sqrt{\frac{2s}{\pi}} \exp \left(-\frac{(m^1 - 1)^2}{2s} \right) - (m^1 - 1) \operatorname{erf} \left(\frac{m^1 - 1}{\sqrt{2s}} \right) \quad (28)$$

with

$$m^1 = 1 + \operatorname{erf} \left(\frac{m^1 - 1}{\sqrt{2s}} \right) \quad (29)$$

where $s \propto \sqrt{P}$. Taking the $s \rightarrow \infty$ limit for $P \propto \exp(\alpha N)$, we get

$$R_{\phi < m^1}(m^1) = \frac{1}{2} (m^1)^2 - m^1 \quad (30)$$

with a fixed point at $m_*^1 = 1$. This is the free energy landscape of the retrieval phase of LSE. As shown in Fig. 1, the landscape has tilted towards the $m = 1$ state and the retrieval of the memory is error free. Equating $\phi = 1$ then gives us the threshold $\alpha_c(\lambda)$ below which the pattern is completely retrieved with no error. We fix $\sigma = 1$. Then $\alpha_c(\lambda) \leq 1/2$ for real λ . Since $\lambda = 1$ at $\alpha_c = 1/2$

$$\alpha_c = \lambda \left(1 - \frac{\lambda}{2} \right) \quad (31)$$

for $\lambda \leq \lambda^* = 1$. We have hence reproduced the threshold for retrieval derived recently using the random energy model [12]. The treatment of LSE here is done assuming exponential number of stored patterns, but the method can be applied to any number of patterns. Unlike polynomial interaction DenseAMs, for LSE the α_c gives the threshold of retrieval as $m_* = 1$ for $\alpha < \alpha_c$.

Discussion: Our method can evaluate any cost function defined by Eq. (1). Although currently demonstrated using binary variables, the framework generalizes effortlessly to other discrete or continuous variables. We validated

our approach by analyzing two prominent AM models. We showed that the disorder-averaged ground state energy, $R(m)$, is useful for decoding gradient dynamics. In DenseAMs, we showed a strong initial-state dependence during memory retrieval, as the basin at $m = 0$ persists

for all α . One would require a mechanism like a stochastic noise [20] to come out of the basin of $m = 0$ state. Finally, for LSE, our results align with the findings by Lucibello et al. [12].

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