

AgentCue: Statistical Summaries for Guiding Decentralized Multi-Agent Self-Organization

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Abstract

Multi-agent systems (MAS) combine multiple agents to solve complex tasks, and recent MAS scale to more than a thousand agents. At this scale, decentralized MAS remove the central orchestrator that assigns sub-tasks, and the agents, which we call workers, self-organize through their own choices of sub-tasks and collaborators on a shared board. As workers keep adding participation and collaboration records, however, the information explodes: workers spend more and more effort on reading the records of other workers, and self-organization loses efficiency. In the statistical view, we represent the execution history of an MAS by two continually updated networks, a collaboration network among workers and a participation network between workers and task types. We then propose AgentCue, a method built on a new latent space model that places workers and task types in a shared latent space. Each time the networks are updated, the estimated parameters of the latent space model form a compact and interpretable statistical summary, and workers use the statistical summary to choose sub-tasks and collaborators without repeated searches of the shared board. We establish the identifiability of the latent space model and non-asymptotic error bounds for the estimates under network increments, and we show that a fixed number of gradient updates after each network update suffices for the same error rate. Experiments on LongBench v2 Multi-Doc QA show that AgentCue achieves competitive effectiveness, efficiency, and interpretability.

Keywords: Multi-agent systems; Latent space model; Organization learning; Large language model; Self-organization.

1 Introduction

Over the past few years, foundation models, that is, large pretrained models such as large language models, have advanced rapidly, and so has the software around these models, which agent developers call a harness. A harness runs a foundation model in a loop: the harness maintains the context of the foundation model, executes the tools that the foundation model calls, and feeds the results back to the foundation model. A foundation model together with a harness forms an agent, and agents are now in wide use (Yao et al. 2023). On complex tasks, however, a single agent can only explore step by step: the agent must first decompose the task and then spend a long time on sub-tasks such as coding, inference, and review, one after another (Zhan et al. 2026). Moreover, on a complex task, a single agent may even lose focus on local details and become highly prone to errors, and the decomposition of the task may itself be flawed (Yang et al. 2026, Huang, Lee, Tng & Ge 2026). Therefore, agent developers equip multiple agents with an auxiliary system for collaboration, and the result is a multi-agent system (MAS): several agents work on different sub-tasks at the same time, and one or more agents then integrate their results. Such collaboration greatly improves performance and saves time (Zhan et al. 2026), and MAS have become a mainstream research paradigm for complex tasks (Wu et al. 2024, Li et al. 2026).

To meet increasingly complex real-world tasks, MAS continue to grow in scale. Well-known MAS such as CAMEL, ChatDev, and LAMBDA use only two to seven agents for role-playing task solving, software development, and data analysis, respectively (Li et al. 2023, Qian et al. 2024, Sun et al. 2026), whereas the reconstruction of a complex codebase within a limited time can now call for as many as 1,024 agents (Zhan et al. 2026). However, most current MAS rely on a central orchestrator, that is, a main agent that decomposes the task and assigns sub-tasks to the other agents, which we call workers. The capacity of the central orchestrator to allocate sub-tasks and coordinate workers often limits the scalability

of the whole MAS (Zhan et al. 2026). Therefore, decentralized MAS have emerged as a core research paradigm for large-scale MAS (Yang et al. 2025, Li et al. 2026). Decentralized MAS replace the central orchestrator with environment components such as a shared workspace for sub-tasks in progress and finished sub-tasks, shared context for reusable findings, a message interface for communication, and a task queue for open sub-tasks (Mao & Mirhoseini 2026, Zhan et al. 2026), and we refer to these components collectively as the shared board. Decentralized MAS raise many research questions, and the most important question asks how a decentralized MAS achieves self-organization through the shared board. Here, self-organization means that the division of labor and collaboration arise from the local choices of workers instead of central assignment, where a local choice is a choice of a sub-task or a collaborator that a worker makes on its own. Studies such as DeLM (Mao & Mirhoseini 2026), Proxifield (Tambwekar et al. 2026), and recent analyses of agent systems (Kim et al. 2025, Li et al. 2026) agree that self-organization without any external design suffers from low efficiency, redundant work, and error propagation, where an external design refers to a mechanism that agent developers add to shape the interactions among workers. Figure 1 illustrates how the difficulties of self-organization grow with the number of workers. There is therefore an urgent need for a suitable external design that helps workers choose their next sub-tasks and collaborators efficiently.

Several recent studies introduce such external designs. AgentNet (Yang et al. 2025) equips each agent with a router, which decides whether the agent executes a task, splits the task, or forwards the task to the agent whose capability best matches the task. Decentralized Expertise-Aware Load Serving (DEALS) (Huang, Wang, Yin, Velasquez & Liu 2026) lets each agent forward a task to a neighbor with fewer unfinished tasks or a higher success rate on tasks of the same category. Economy of Minds (EoM) (Qi et al. 2026) lets agents bid in auctions for the right to act and pay one another, and the wealth of each agent then

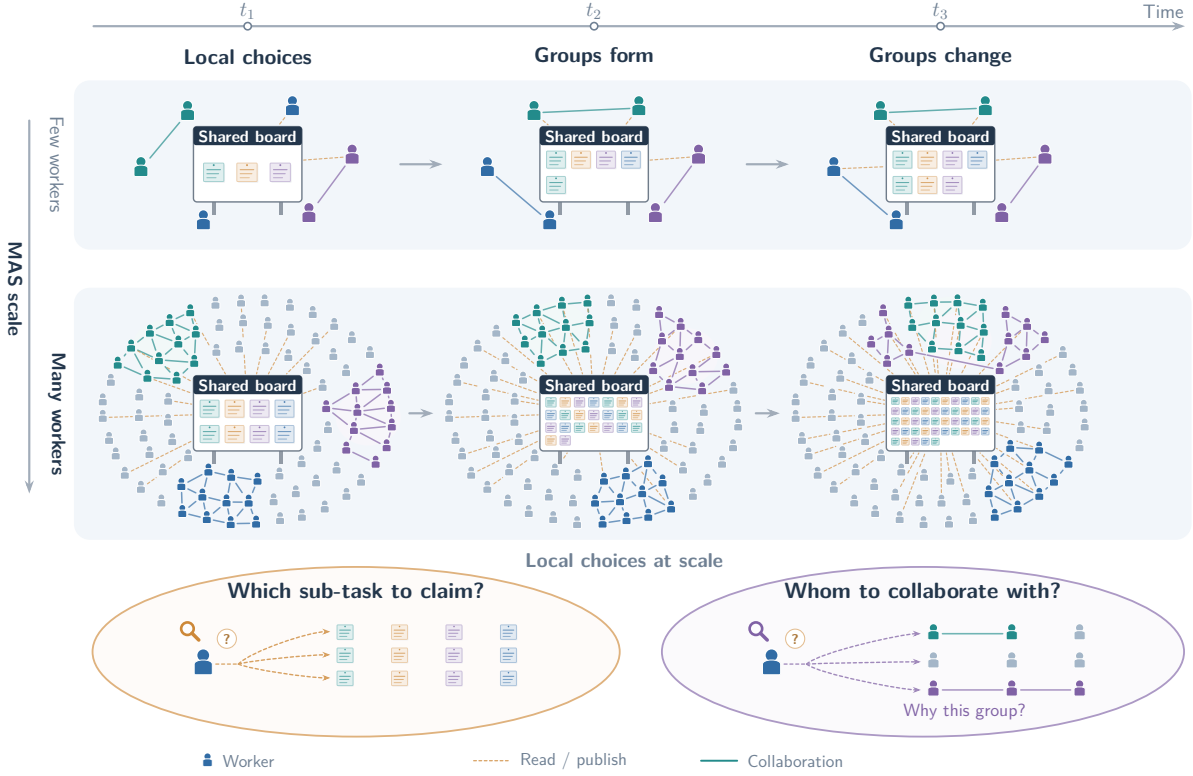


Figure 1: Self-organization in a decentralized MAS. Workers read from and publish to the shared board (dashed lines) and collaborate with one another (solid lines), and groups of collaborators form and change over time through local choices. With few workers (upper panel), the shared board holds few records. With many workers (middle panel), records accumulate quickly, and each local choice (lower panel), which sub-task to claim and whom to collaborate with, requires a search through many records.

decides whether the agent persists and mutates or goes bankrupt. Proxifield (Tambwekar et al. 2026) builds a sparse communication graph in every round through deterministic routing rules. Self-Organizing Agent Teams (SAT) (Pappu et al. 2026) learns a bank of collaboration strategies offline, which specify roles, conversation phases, participation, and information flow, and applies the bank unchanged to new problems. Each design presets a mechanism of interaction, within which the agents still make their own local choices.

However, AgentNet, DEALS, EoM, Proxifield, and SAT were developed and evaluated in

small-scale MAS, and a new problem takes the stage in large-scale MAS. The new problem lies in the participation and collaboration records: participation records state the task types, that is, pre-specified categories of sub-tasks, in which each worker works, and collaboration records state whose published results, that is, results that workers publish on the shared board, each worker uses or reviews. In a large-scale MAS, many workers keep adding participation and collaboration records, and the information explodes: workers spend more and more effort on reading the records of other workers and choose their sub-tasks more and more slowly, so the efficiency of self-organization declines. Agensh (Zhan et al. 2026) shows the burden at scale: with 1,024 workers, every new entry in the shared context goes to all other workers, while a direct read of the shared context returns only the 2,000 latest entries. The more entries the workers write, the more time a worker must spend on keyword searches to trace entries beyond the 2,000 latest entries. The gain from more workers also shrinks: in the codebase reconstruction of Agensh, the final test-pass rate rises by 17.05% from 1 to 128 workers but by only 4.12% from 128 to 1,024 workers. The resulting dilemma, an information explosion without a fast and interpretable summary, is exactly where statistics excels. Picture the workers as a large crowd of “cowboys”: each cowboy rides a foundation model as a horse and steers the horse through a harness, while a vast list of sub-tasks and potential collaborators awaits selection. What the cowboys need is a statistical estimation procedure¹ that summarizes the information. With such a procedure, the cowboys no longer depend on repeated searches, that is, retrievals of the same unchanged records of the shared board by many workers: the cowboys quickly screen the sub-tasks and potential collaborators, select the sub-tasks and collaborators that fit them best, and reach an efficient self-organization.

In the statistical view, we first characterize the data. A preprocessing step condenses the

¹Coincidentally, one of the most influential estimation procedures in statistics shares its name with the cowboy’s rope: the *Lasso* (Tibshirani 1996), which shrinks and selects among many candidate variables.

massive execution history of an MAS into two networks. The first network is a collaboration network, whose nodes are workers and whose edges count the collaborations between two workers, that is, the times one worker uses or reviews a published result of the other worker. The second network is a participation network, a worker–task-type bipartite network whose nodes are the workers and the task types and whose edges count the times each worker performs each task type. Latent space models offer a powerful tool for such network data (Hoff et al. 2002, Ma et al. 2020). A latent space model represents each node by a latent position, a low-dimensional vector to be estimated, and nodes with closer latent positions tend to have closer relations. A rich literature has extended latent space models to various network data. For example, He et al. (2025) analyze longitudinal networks of interaction counts with timestamps, such as bike trips between stations, and Fan et al. (2024) jointly analyze a one-mode network, such as friendships among users, and a two-mode network, such as reviews from users to businesses. For a new form of data, which we call continually updated networks because the edge counts accumulate as the MAS runs, we propose a new latent space model, to our knowledge the first latent space model for such networks, and build our method **AgentCue** on the latent space model. In AgentCue, at each checkpoint, that is, each time the networks are updated, the MAS estimates the parameters of the latent space model and publishes the estimates as a statistical summary. Workers then locate suitable collaborators and task types in the latent space, and the next stage of self-organization emerges in an orderly and efficient manner.

Our work makes a four-fold contribution. (i). For the problem, we identify a new problem for agents: in large-scale decentralized MAS, the information explosion of participation and collaboration records slows down self-organization, and the problem also provides a stage on which statistical methods can be built. (ii). For methodology, we propose AgentCue, which combines a new latent space model for continually updated networks with the MAS: the

MAS generates the networks, and the statistical summaries of the latent space model guide the MAS. (iii). For theory, we establish the identifiability of the latent space model and non-asymptotic error bounds for its estimates when the counts added at each checkpoint depend on the earlier history of the MAS, and we show that a fixed number of gradient updates per checkpoint attains the statistical error rate. (iv). For experiments, we evaluate AgentCue on the Multi-Doc QA subset of LongBench v2 (Bai et al. 2025), and the results show that AgentCue achieves competitive effectiveness, efficiency, and interpretability.

2 Background and Related Work

Large language models support language understanding, reasoning, and code generation. With tools and environmental feedback, agents can plan and act (Yao et al. 2023). MAS let agents divide work and exchange information (Wu et al. 2024). Recent decentralized MAS extend collaboration to many workers, who choose sub-tasks and coordinate through a shared board (Zhan et al. 2026).

Multi-Agent Collaboration. Several systems organize collaboration through role assignment and structured communication. *CAMEL* pairs a user agent with an assistant agent to solve tasks through role-playing (Li et al. 2023), while *AutoGen* supports conversations among agents with configurable roles and tools (Wu et al. 2024). *ChatDev* and *MetaGPT* organize software development around specialized roles and workflows (Qian et al. 2024, Hong et al. 2024). *AgentCoder* separates programming, test design, and test execution across three agents (Huang et al. 2023). For data analysis, *LAMBDA* pairs a programmer with an inspector for code correction (Sun et al. 2026). AgentCoder and LAMBDA illustrate collaboration in single-digit teams. Our concern is how large-scale MAS can use accumulated participation and collaboration records to guide subsequent local choices.

Decentralized Multi-Agent Systems. Decentralized MAS allow workers to choose

sub-tasks and collaborators through shared context and communication (Li et al. 2026). *DeLM* combines an asynchronous task queue with verified shared context, so that workers claim sub-tasks and build on earlier findings (Mao & Mirhoseini 2026). *AutoScientists* lets workers form teams around hypotheses and share successes and failures (Gao et al. 2026). *Agensh*, the closest setting to ours, scales to 1,024 workers in the reconstruction of the pandoc codebase (Zhan et al. 2026). The workers of Agensh claim, execute, verify, and merge work through a shared workspace; reported execution logs illustrate reviewer selection from prior experience. As the number of workers grows and participation and collaboration records accumulate, workers face more potential collaborators and more records to search. The statistical challenge is to jointly summarize participation across task types and collaboration through the use or review of published results, while accounting for overall activity.

Self-Organization Learning. Several approaches adapt collaboration through experience. *MorphAgent* updates textual profiles (Lu et al. 2024), while *SAT* learns strategies for roles, participation, and information flow (Pappu et al. 2026). Auctions and accumulated payments govern the allocation of the right to act, agent mutation, and replacement in *EoM* (Qi et al. 2026); *AgentLance* adapts bidding strategies while an allocator selects workers using bids and reputation (Liu et al. 2026). *TheBotCompany* uses managers to hire, assign, and retire workers (Lyu et al. 2026); *LATTE* lets workers propose task-graph changes coordinated by a lead agent (Mieczkowski et al. 2026). In robotics, *SwarmCF* learns robot and task vectors from outcomes for decentralized allocation (Apartsin et al. 2026). We study the cumulative participation and collaboration records that the local choices of workers produce.

Graph-Guided Multi-Agent Collaboration. Graph-based methods select participants or communication links. *DyLAN* uses contribution scores to select participants (Liu et al. 2024), *G-Designer* learns task-dependent communication graphs (Zhang et al. 2025), and *SelfOrg* rebuilds graphs from evolving answers (Tastan et al. 2026). *AgentNet* combines

local routing and retrieval-based memory with adaptive connections (Yang et al. 2025), and *DEALS* routes tasks between neighbors by weighing unfinished tasks against success rates (Huang, Wang, Yin, Velasquez & Liu 2026); *MeshHeal* uses peer verification and routing adjustments to detect degradation and reintegrate recovered agents (Chen et al. 2026). *DyTopo* and *Proxifield* construct communication links through semantic matching (Lu et al. 2026, Tambwekar et al. 2026). In our setting, the collaboration network records the use and review of published results.

Statistical Studies of Human Collaboration and Latent Space Models. Statistical models of human collaboration analyze collaboration records. *Relational event models* connect interaction rates to past activity, and *relational hyperevent models* describe group interactions such as coauthorship (Butts 2008, Perry & Wolfe 2013, Lerner & Hâncean 2023). Latent space models describe network relationships through low-dimensional latent positions, with methods for large networks (Ma et al. 2020), temporal variation, multiple layers, and hierarchical node categories (Durante & Dunson 2014, Loyal & Chen 2023, Pu et al. 2026). Fan et al. (2024) jointly model one-mode and two-mode networks through shared latent positions and network-specific node effects, that is, node-level parameters for overall activity. We apply the construction of Fan et al. (2024) to the collaboration network and the participation network and account for overall activity in each network. We return the statistical summaries to workers to inform their subsequent local choices.

3 Methodology

3.1 Network Data Collection

We collect network data from the execution history of an Agensh-based MAS (Zhan et al. 2026). Workers pursue a common goal by choosing their own sub-tasks and collaborators.

The shared board makes sub-tasks in progress, published results, and accumulated findings available to all workers. Workers use the shared board to identify useful work, publish their contributions, and continue from their own results or the results of other workers. Self-organization therefore leaves records of who performs each piece of work and how published results are used or reviewed.

A piece of published work, which we call a work, is the basic observation unit. The record of a work, which we call a work record, identifies the work, the worker who produced the work, the publication order, and the exact versions of the published results that the work uses or reviews. The work records thus contain the participation and collaboration records of Section 1. Preprocessing removes duplicate work records and assigns task types to each work according to the published results of the work. For network analysis, we pre-specify task types as categories of sub-tasks for the common goal. For example, in cross-document reasoning, we define four task types: *extract*, *compare*, *inference*, and *review*. A work can belong to multiple task types when the work delivers several independently reusable results.

We index the resulting work records by r , the n workers by $i, \ell \in [n]$, and the m task types by $k \in [m]$, where $[n] = \{1, \dots, n\}$. Each work record r produces network increments $\Delta \mathbf{A}_r \in \{0, 1\}^{n \times n}$ and $\Delta \mathbf{B}_r \in \{0, 1\}^{n \times m}$. The entry $(\Delta \mathbf{A}_r)_{i\ell}$ is one when the work of work record r is produced by worker i and uses or reviews a published result of worker ℓ , or vice versa. We aggregate collaboration over worker-worker pairs, so $\Delta \mathbf{A}_r$ is symmetric with zero diagonal. Each work contributes at most once to a worker-worker pair, even if the work uses several results of the same worker. The entry $(\Delta \mathbf{B}_r)_{ik}$ is one when worker i produced the work of work record r and the work belongs to task type k . Multiple collaborators or task types can therefore update several entries, while each work contributes at most once to each worker-task-type pair. The network increments record participation and collaboration in published work, whether or not the work is correct.

A worker acts in sessions, that is, separate activations of the worker. We place checkpoints by the number of completed sessions of the MAS instead of elapsed time, because the elapsed time of a session differs greatly across the host machines that run the MAS. At checkpoint $t \in [T]$, where T is the number of checkpoints, let S_t be the number of work records published up to checkpoint t . The cumulative network observations are:

$$\mathbf{A}^{(t)} = \sum_{r=1}^{S_t} \Delta \mathbf{A}_r, \quad \mathbf{B}^{(t)} = \sum_{r=1}^{S_t} \Delta \mathbf{B}_r.$$

Their entries $A_{i\ell}^{(t)}$ and $B_{ik}^{(t)}$ count collaboration and participation through checkpoint t . We retain all worker–worker and worker–task-type pairs, including those with zero counts. Successive checkpoints share the accumulated work records, and later local choices can depend on earlier published results. As illustrated in Figure 2, the two networks $\mathbf{A}^{(t)}$ and $\mathbf{B}^{(t)}$ provide complementary observations on the same workers: $\mathbf{A}^{(t)}$ describes whose results support the work of other workers, and $\mathbf{B}^{(t)}$ describes participation across task types. We jointly model these networks below.

3.2 Latent Space Model and Estimation

We connect the two networks through shared worker latent positions, as in the one-mode and two-mode construction of Fan et al. (2024). Let $\mathbf{z}_i^{(t)}, \mathbf{w}_k^{(t)} \in \mathbb{R}^d$ denote worker and task-type latent positions, where d is the latent dimension. The node effects $\alpha_i^{(t)}$ and $\beta_i^{(t)}$ describe the overall collaboration and participation activity of worker i , and the node effect $\gamma_k^{(t)}$ describes the overall participation that task type k attracts. The latent space model treats the network counts as Poisson variables with cumulative means $\theta_{i\ell}^{A,(t)} > 0$ and $\theta_{ik}^{B,(t)} > 0$:

$$\begin{aligned} A_{i\ell}^{(t)} &\sim \text{Poisson}(\theta_{i\ell}^{A,(t)}), \quad \log \theta_{i\ell}^{A,(t)} = \alpha_i^{(t)} + \alpha_\ell^{(t)} + (\mathbf{z}_i^{(t)})^\top \mathbf{z}_\ell^{(t)}, \\ B_{ik}^{(t)} &\sim \text{Poisson}(\theta_{ik}^{B,(t)}), \quad \log \theta_{ik}^{B,(t)} = \beta_i^{(t)} + \gamma_k^{(t)} + (\mathbf{z}_i^{(t)})^\top \mathbf{w}_k^{(t)}. \end{aligned} \tag{1}$$

Node effects $\alpha_i^{(t)}, \beta_i^{(t)}, \gamma_k^{(t)}$ capture overall activity, while the inner products $(\mathbf{z}_i^{(t)})^\top \mathbf{z}_\ell^{(t)}$ and $(\mathbf{z}_i^{(t)})^\top \mathbf{w}_k^{(t)}$ describe pair-specific variation beyond the node effects. Sharing $\mathbf{z}_i^{(t)}$ lets both

networks inform the latent position of each worker. All parameters vary with t and describe cumulative experience at checkpoint t .

Define the parameter tuple $\Theta^{(t)} = (\boldsymbol{\alpha}^{(t)}, \boldsymbol{\beta}^{(t)}, \boldsymbol{\gamma}^{(t)}, \mathbf{Z}^{(t)}, \mathbf{W}^{(t)})$, where $\boldsymbol{\alpha}^{(t)} = (\alpha_i^{(t)})_{i=1}^n$, $\boldsymbol{\beta}^{(t)} = (\beta_i^{(t)})_{i=1}^n$, and $\boldsymbol{\gamma}^{(t)} = (\gamma_k^{(t)})_{k=1}^m$ are column vectors, and $\mathbf{Z}^{(t)} = [\mathbf{z}_1^{(t)}, \dots, \mathbf{z}_n^{(t)}]^\top \in \mathbb{R}^{n \times d}$ and $\mathbf{W}^{(t)} = [\mathbf{w}_1^{(t)}, \dots, \mathbf{w}_m^{(t)}]^\top \in \mathbb{R}^{m \times d}$ contain the latent positions as rows. The two Poisson negative log-likelihoods $\mathcal{L}_A^{(t)}$ and $\mathcal{L}_B^{(t)}$ omit terms independent of the parameters:

$$\begin{aligned}\mathcal{L}_A^{(t)}(\Theta^{(t)}) &= \sum_{1 \leq i < \ell \leq n} \left\{ \theta_{i\ell}^{A,(t)} - A_{i\ell}^{(t)} \log \theta_{i\ell}^{A,(t)} \right\}, \\ \mathcal{L}_B^{(t)}(\Theta^{(t)}) &= \sum_{i=1}^n \sum_{k=1}^m \left\{ \theta_{ik}^{B,(t)} - B_{ik}^{(t)} \log \theta_{ik}^{B,(t)} \right\}.\end{aligned}\tag{2}$$

The joint objective $\mathcal{L}^{(t)}$ is their sum:

$$\mathcal{L}^{(t)}(\Theta^{(t)}) = \mathcal{L}_A^{(t)}(\Theta^{(t)}) + \mathcal{L}_B^{(t)}(\Theta^{(t)}).\tag{3}$$

Let $\mathbf{1}_n$ and $\mathbf{1}_m$ denote vectors of ones. We minimize (3) subject to $\mathbf{1}_n^\top \mathbf{Z}^{(t)} = \mathbf{0}^\top$, $\mathbf{1}_m^\top \mathbf{W}^{(t)} = \mathbf{0}^\top$, and $\mathbf{1}_n^\top \boldsymbol{\beta}^{(t)} = 0$. These centering constraints are imposed for parameter identifiability. When $\mathbf{Z}^{(t)}$ has full column rank, the node effects $\boldsymbol{\alpha}^{(t)}, \boldsymbol{\beta}^{(t)}, \boldsymbol{\gamma}^{(t)}$ are uniquely determined, and the latent positions $\mathbf{Z}^{(t)}, \mathbf{W}^{(t)}$ are identifiable up to a common orthogonal transformation. Proposition 1 in Supplementary A.1 gives the result and proof.

For numerical optimization, we follow the projected-gradient implementation of He et al. (2025) and add a small ridge penalty on $\mathbf{Z}^{(t)}$ and $\mathbf{W}^{(t)}$. Supplementary B.5 gives the penalized objective, the numerical scaling, and the stopping rule. Substituting the numerical estimate $\hat{\Theta}^{(t)}$, that is, the output of the optimization at checkpoint t , into (1) gives fitted means $\hat{\theta}_{i\ell}^{A,(t)}$ and $\hat{\theta}_{ik}^{B,(t)}$. At the first checkpoint, we initialize the latent positions $\mathbf{Z}^{(t)}$ and $\mathbf{W}^{(t)}$ with independent standard normal entries and the node effects $\boldsymbol{\alpha}^{(t)}, \boldsymbol{\beta}^{(t)}, \boldsymbol{\gamma}^{(t)}$ from the observed network counts, then apply the centering constraints. At each subsequent checkpoint t , we initialize the optimization with the preceding estimate $\hat{\Theta}^{(t-1)}$.

3.3 Summary Guiding in Self-Organization

In Agensh, workers search the work records on the shared board before they choose sub-tasks and collaborators (Zhan et al. 2026). Each session of a worker has a fixed budget of actions on the shared board, and every read of the work records of another worker costs one action. Many workers also retrieve the same unchanged work records to compare the same sub-tasks and collaborators, a pattern that Section 1 calls repeated searches. In AgentCue, the MAS instead estimates, once per checkpoint, the parameters of the latent space model (1) from the work records on behalf of all workers and publishes the estimates $\hat{\alpha}^{(t)}, \hat{\beta}^{(t)}, \hat{\gamma}^{(t)}, \hat{\mathbf{Z}}^{(t)}, \hat{\mathbf{W}}^{(t)}$, which we call the *statistical summary* at checkpoint t , together with a summary version that identifies the network observations behind the estimates. When a session first requests the statistical summary, the worker reads the latest statistical summary with one action and keeps the same summary version for the rest of the session. Every later *local comparison*, that is, a calculation of fitted means in (1) from the statistical summary that the worker holds, costs no action on the shared board. Figure 2 gives an overview of AgentCue.

Each fitted mean comes with an explanation, because the logarithm of each fitted mean in (1) splits into two kinds of terms:

$$\log \hat{\theta}_{ik}^{B,(t)} = \underbrace{\hat{\beta}_i^{(t)} + \hat{\gamma}_k^{(t)}}_{\text{activity}} + \underbrace{(\hat{\mathbf{z}}_i^{(t)})^\top \hat{\mathbf{w}}_k^{(t)}}_{\text{affinity}}, \quad \log \hat{\theta}_{i\ell}^{A,(t)} = \underbrace{\hat{\alpha}_i^{(t)} + \hat{\alpha}_\ell^{(t)}}_{\text{activity}} + \underbrace{(\hat{\mathbf{z}}_i^{(t)})^\top \hat{\mathbf{z}}_\ell^{(t)}}_{\text{affinity}}. \quad (4)$$

We call $\hat{\theta}_{ik}^{B,(t)}$ the *fitted participation mean* of worker i in task type k and $\hat{\theta}_{i\ell}^{A,(t)}$ the *fitted collaboration mean* between workers i and ℓ . We call the node effects *activity terms* and the inner products of latent positions *affinity terms*. The affinity terms measure whether worker i leans toward task type k and whether workers i and ℓ collaborate more often than their activity terms imply. A large network count alone cannot tell an active worker from a close collaborator, and the split in (4) separates the two. If all affinity terms $(\hat{\mathbf{z}}_i^{(t)})^\top \hat{\mathbf{w}}_k^{(t)}$ vanish, every worker spreads participation over the task types in the same proportions, so the affinity terms describe the division of labor.

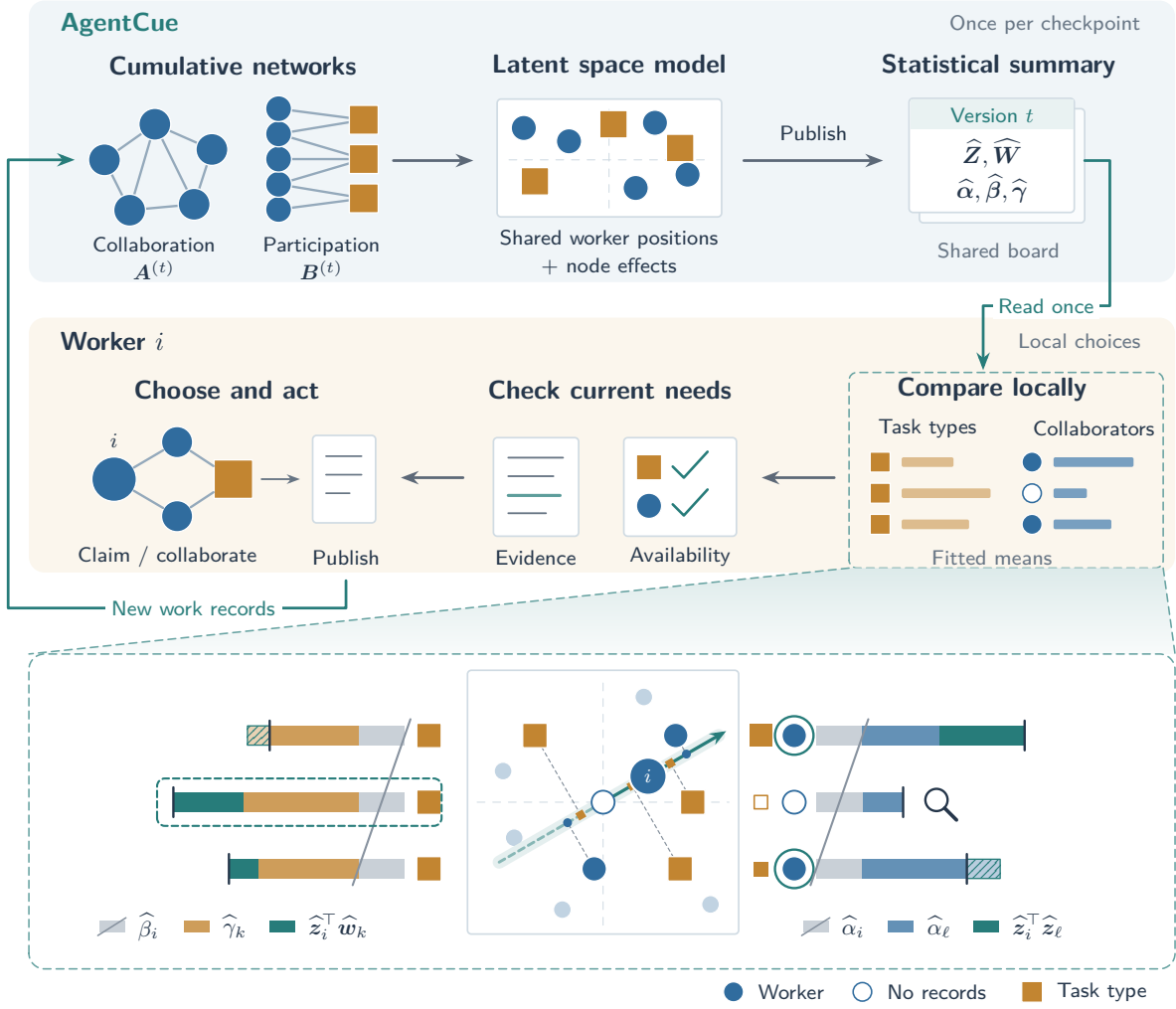


Figure 2: **Overview of AgentCue.** Upper and middle bands: once per checkpoint, the MAS publishes the statistical summary, and worker i reads the statistical summary once. Lower band: the local comparison, enlarged. Bars are log fitted means split as in (4); the struck-through term cancels, and hatched segments are negative affinity terms. The latent-space map shows the projections onto the line of sight of worker i . Small squares scale with the fitted participation mean of each candidate in the task type of the inputs that the sub-task needs; the dashed box and the rings mark the shortlist; the magnifier marks a work-record search. Superscripts (t) are omitted.

A local comparison of worker i involves two fitted means: two fitted participation means of worker i , or two fitted collaboration means between worker i and two candidates, that is,

two other workers whom worker i considers as collaborators. In both ratios, the activity term of worker i cancels:

$$\begin{aligned}\frac{\hat{\theta}_{ik}^{B,(t)}}{\hat{\theta}_{ik'}^{B,(t)}} &= \underbrace{\exp(\hat{\gamma}_k^{(t)} - \hat{\gamma}_{k'}^{(t)})}_{\text{task types}} \underbrace{\exp\{(\hat{\mathbf{z}}_i^{(t)})^\top (\hat{\mathbf{w}}_k^{(t)} - \hat{\mathbf{w}}_{k'}^{(t)})\}}_{\text{worker } i}, \\ \frac{\hat{\theta}_{i\ell}^{A,(t)}}{\hat{\theta}_{i\ell'}^{A,(t)}} &= \underbrace{\exp(\hat{\alpha}_\ell^{(t)} - \hat{\alpha}_{\ell'}^{(t)})}_{\text{candidates}} \underbrace{\exp\{(\hat{\mathbf{z}}_i^{(t)})^\top (\hat{\mathbf{z}}_\ell^{(t)} - \hat{\mathbf{z}}_{\ell'}^{(t)})\}}_{\text{pair}}.\end{aligned}\tag{5}$$

The first factors in the two rows of (5), which we call the task-type factor and the candidate factor, are common to all workers and compare how much participation task types k and k' attract and how active candidates ℓ and ℓ' are. The second factors, which we call the worker factor and the pair factor, belong to worker i alone. Because the centering constraint $\mathbf{1}_n^\top \widehat{\mathbf{Z}}^{(t)} = \mathbf{0}^\top$ makes the affinity terms of all workers with a task type sum to zero, a worker factor close to one means that worker i resembles the average worker on task types k and k' , and a pair factor far from one separates even two equally active candidates.

The affinity terms have a geometric reading. For any $\mathbf{x} \in \mathbb{R}^d$, the inner product $(\hat{\mathbf{z}}_i^{(t)})^\top \mathbf{x}$ equals the length of $\hat{\mathbf{z}}_i^{(t)}$ times the signed length of the projection of $\mathbf{x}$ onto the line through the origin along $\hat{\mathbf{z}}_i^{(t)}$, which we call the *line of sight* of worker i . The projections of the task-type latent positions and of the latent positions of the other workers onto the line of sight order the task types and the other workers by affinity with worker i , so one latent position serves both the sub-task choice and the collaborator choice of worker i . Under the ridge penalty of Section 3.2, a worker without work records stays near the origin, so the small fitted means of the worker come from the activity terms and reflect missing work records, not missing ability.

The lower band of Figure 2 enlarges the local comparisons of worker i . To choose a sub-task, worker i compares the fitted participation means of worker i across the task types of the open sub-tasks. To choose a collaborator for a sub-task, worker i compares candidates by two fitted means, namely the fitted participation mean of each candidate in the task

type of the inputs that the sub-task needs, for example *extract* for a sub-task of task type *inference*, and the fitted collaboration mean between worker i and each candidate. The local comparisons leave a shortlist of sub-tasks and candidates. Worker i then spends actions only to check the *current needs*, which the statistical summary cannot settle: whether a sub-task remains open, whether a candidate is occupied, and whether a published result supplies the required evidence. The statistical summary answers questions about the past, and the shared board answers questions about the present.

The statistical summary proposes, and the worker chooses. When the statistical summary gives weak support, for example for a candidate without work records, when candidates tie, when a sub-task is new, or when the current needs contradict the statistical summary, worker i searches only the work records that the choice requires and states a concrete reason for the choice. Each choice leaves a Choice Record, which stores the summary version, whether the statistical summary informed the choice, and the reason for the choice. The statistical summary guides the choices, the choices create new work records, and the new work records revise the statistical summary at the next checkpoint; along the loop, the Choice Records trace every choice to a summary version and a reason.

3.4 Feedback of the Statistical Summary

In the usual analysis with latent space models, the network does not respond to the estimates. In AgentCue, the statistical summary guides the choices that produce the next work records, so later estimates partly describe the effect of earlier estimates. This section analyzes the feedback and asks what the statistical summary describes in the long run. We follow one worker i through four steps of the loop in the participation network.

Step 1: the statistical summary reports past shares. The statistical summary gives

worker i a *fitted participation share* for each task type k :

$$\hat{\pi}_{ik}^{(t)} = \frac{\hat{\theta}_{ik}^{B,(t)}}{\sum_{k'=1}^m \hat{\theta}_{ik'}^{B,(t)}} = \frac{\exp\{\hat{\gamma}_k^{(t)} + (\hat{\mathbf{z}}_i^{(t)})^\top \hat{\mathbf{w}}_k^{(t)}\}}{\sum_{k'=1}^m \exp\{\hat{\gamma}_{k'}^{(t)} + (\hat{\mathbf{z}}_i^{(t)})^\top \hat{\mathbf{w}}_{k'}^{(t)}\}}. \quad (6)$$

The activity term $\hat{\beta}_i^{(t)}$ of worker i cancels in (6), as in (5). The *observed participation share* $p_{ik}^{(t)} = B_{ik}^{(t)} / \sum_{k'=1}^m B_{ik'}^{(t)}$ is the fraction of the participation counts of worker i that falls on task type k up to checkpoint t ; let $\hat{\boldsymbol{\pi}}_i^{(t)}$ and $\mathbf{p}_i^{(t)}$ denote the vectors of the two shares over task types. At a minimizer of the joint objective (3), the fitted participation means of each worker add up to the participation count of the worker, so $\hat{\boldsymbol{\pi}}_i^{(t)}$ differs from $\mathbf{p}_i^{(t)}$ only through the low-rank latent space and the latent positions shared with the collaboration network. Step 3 therefore takes $\hat{\boldsymbol{\pi}}_i^{(t)} = \mathbf{p}_i^{(t)}$.

Step 2: workers act on the shares. How a worker uses the statistical summary depends on the foundation model and the prompt, so we adopt a stylized choice rule: worker i chooses task type k with probability

$$\phi_{ik}(\hat{\boldsymbol{\pi}}_i^{(t)}) = \frac{(\hat{\pi}_{ik}^{(t)})^\omega \exp(u_{ik})}{\sum_{k'=1}^m (\hat{\pi}_{ik'}^{(t)})^\omega \exp(u_{ik'})}. \quad (7)$$

The *baseline preference* u_{ik} is the log-propensity of worker i for task type k without the statistical summary and combines the suitability of worker i for task type k with the current needs. The *reliance* $\omega \geq 0$, common to all workers, measures how strongly workers follow the statistical summary: in the odds of two task types, the reliance raises the task-type and worker factors of (5) to the power ω . With $\omega = 0$, a worker ignores the statistical summary; with $\omega = 1$, a worker treats the fitted participation share as a prior, as in Bayes' rule.

Step 3: cumulative counts feed the choices back. We hold u_{ik} fixed, take $\hat{\boldsymbol{\pi}}_i^{(t)} = \mathbf{p}_i^{(t)}$, and let worker i add the same number of participation counts before the first checkpoint and between consecutive checkpoints. Each checkpoint then moves the observed participation share a fraction $1/(t+1)$ of the way toward the choice probabilities $\phi_i(\mathbf{p}_i^{(t)})$:

$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \frac{1}{t+1} \left\{ \phi_i(\mathbf{p}_i^{(t)}) - \mathbf{p}_i^{(t)} + \boldsymbol{\xi}_i^{(t+1)} \right\}, \quad (8)$$

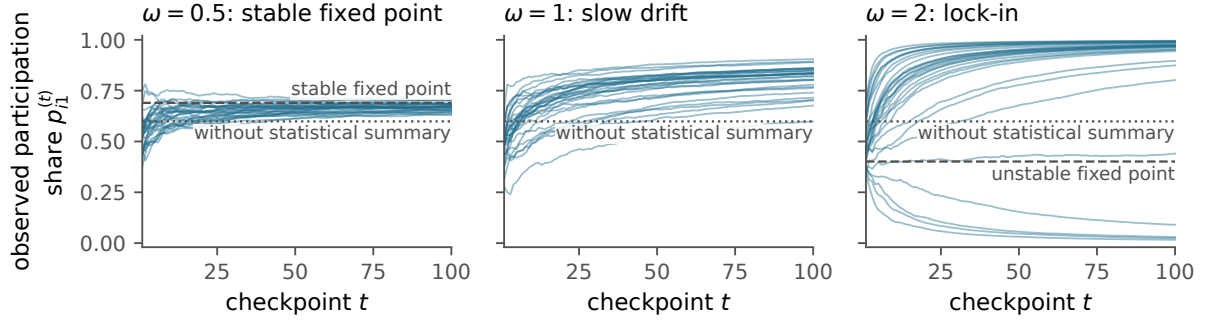


Figure 3: Simulation of (8) in the setting of Section 3.4; each line is one run. Dotted lines: fixed point without the statistical summary ($\omega = 0$). Dashed lines: interior fixed point, stable for $\omega = 0.5$ and unstable for $\omega = 2$; for $\omega = 1$, no interior fixed point exists.

where $\phi_i = (\phi_{i1}, \dots, \phi_{im})^\top$, and the noise $\xi_i^{(t+1)}$, the share of the added participation counts minus $\phi_i(\mathbf{p}_i^{(t)})$, has mean zero given the history. The recursion (8) is a stochastic approximation, so the observed participation share can settle only at a stable fixed point, that is, a share $\mathbf{p}_i^\circ = \phi_i(\mathbf{p}_i^\circ)$ that attracts nearby shares (Borkar 2023). At any fixed point, stable or not, the statistical summary describes the very choices that the statistical summary induces, a property known as performative stability (Perdomo et al. 2020, Mendler-Dünner et al. 2020). Figure 3 simulates (8) in an illustrative setting: one worker, two task types with $u_{i1} - u_{i2} = 0.4$, and $\omega \in \{0.5, 1, 2\}$. Each of 30 independent runs of (8) starts from 32 participation counts split with probability 1/2 and adds 32 participation counts per checkpoint for 100 checkpoints.

Step 4: the fixed point in the latent space. A fixed point is interior when all shares are positive; for $\omega > 0$, each vertex, where worker i performs one task type only, is also a fixed point, and a run locks in when the run converges to a vertex. For $\omega \neq 1$, the interior fixed point satisfies $\log p_{ik}^\circ = u_{ik}/(1 - \omega) + c_i$, where c_i does not depend on k . Let $\bar{u}_k$ denote the average of u_{ik} over workers, and let $\tilde{u}_{ik}$ denote u_{ik} centered over workers and over task types. If the observed participation shares of all workers sit at the interior fixed points and

the matrix $(\tilde{u}_{ik})$ has rank at most d , the centering constraints give

$$\hat{\gamma}_k^{(t)} = \frac{\bar{u}_k}{1 - \omega} + c^{(t)}, \quad (\hat{\mathbf{z}}_i^{(t)})^\top \hat{\mathbf{w}}_k^{(t)} = \frac{\tilde{u}_{ik}}{1 - \omega}, \quad (9)$$

where $c^{(t)}$ absorbs the growth of the counts and does not depend on k . For $0 \leq \omega < 1$, the interior fixed point is stable and the vertices are unstable, and (9) magnifies the baseline preferences by $1/(1 - \omega) \geq 1$ and keeps the order and the signs of $\tilde{u}_{ik}$: the statistical summary sharpens the division of labor. The sharper division of labor forms more slowly, since near the fixed point each checkpoint closes a fraction $(1 - \omega)/(t + 1)$ of the gap instead of $1/(t + 1)$; in the left panel of Figure 3, most runs remain below the interior fixed point at $t = 100$. For $\omega = 1$, a run drifts slowly toward the vertex of the task type with the largest baseline preference. For $\omega > 1$, the factor $1/(1 - \omega)$ is negative, so the interior fixed point reverses the order of the baseline preferences and is unstable: the early observed participation share decides the vertex at which a run locks in. In the right panel of Figure 3, 4 of 30 runs head to the task type with the smaller baseline preference.

For $\omega = 0$, (9) recovers the baseline preferences, as in the usual analysis with latent space models. For $0 \leq \omega < 1$, the feedback only rescales $\hat{\gamma}_k^{(t)}$ and the affinity terms by $1/(1 - \omega)$, so the order and the signs of the affinity terms keep the usual interpretation. In the MAS, the current needs counteract lock-in, because the open sub-tasks of a crowded task type run out and the baseline preferences for the crowded task type fall; Section 5 examines lock-in.

4 Theory

We study estimation of cumulative means and the accuracy of warm-started gradient updates, which start each checkpoint from the preceding estimate. The statistical results control error relative to the current estimation target, defined below. The optimization results control error relative to the current exact minimizer, that is, a minimizer of the joint

objective (3) over the feasible parameter set in Assumption 2.

Throughout, $n \geq 3$, $m \geq 1$, and $d \geq 1$ is fixed. For the work records indexed by r in Section 3.1, let $\mathcal{F}_r$ denote the information available up to and including work record r . With increments $\Delta \mathbf{A}_r, \Delta \mathbf{B}_r$ and the cumulative number S_t of work records, we have $\mathbf{A}^{(t)} = \sum_{r=1}^{S_t} \Delta \mathbf{A}_r$ and $\mathbf{B}^{(t)} = \sum_{r=1}^{S_t} \Delta \mathbf{B}_r$. The increments are $\mathcal{F}_r$ -measurable binary matrices. The finite random counts S_t are nondecreasing in t . We define the estimation targets by summing the expected network increments conditional on the preceding history:

$$\theta_{\star, i\ell}^{A,(t)} = \sum_{r=1}^{S_t} \mathbb{E}\{(\Delta \mathbf{A}_r)_{i\ell} \mid \mathcal{F}_{r-1}\}, \quad \theta_{\star, ik}^{B,(t)} = \sum_{r=1}^{S_t} \mathbb{E}\{(\Delta \mathbf{B}_r)_{ik} \mid \mathcal{F}_{r-1}\}.$$

The estimation targets can depend on the realized history and generally differ from the unconditional means $\mathbb{E}\mathbf{A}^{(t)}, \mathbb{E}\mathbf{B}^{(t)}$. Write the true parameter tuple as $\Theta_\star^{(t)} = (\boldsymbol{\alpha}_\star^{(t)}, \boldsymbol{\beta}_\star^{(t)}, \boldsymbol{\gamma}_\star^{(t)}, \mathbf{Z}_\star^{(t)}, \mathbf{W}_\star^{(t)})$, where $\log \theta_{\star, i\ell}^{A,(t)} = \alpha_{\star, i}^{(t)} + \alpha_{\star, \ell}^{(t)} + (\mathbf{z}_{\star, i}^{(t)})^\top \mathbf{z}_{\star, \ell}^{(t)}$ and $\log \theta_{\star, ik}^{B,(t)} = \beta_{\star, i}^{(t)} + \gamma_{\star, k}^{(t)} + (\mathbf{z}_{\star, i}^{(t)})^\top \mathbf{w}_{\star, k}^{(t)}$. The parameters satisfy the centering constraints $\mathbf{1}_n^\top \mathbf{Z}_\star^{(t)} = \mathbf{0}^\top$, $\mathbf{1}_m^\top \mathbf{W}_\star^{(t)} = \mathbf{0}^\top$, and $\mathbf{1}_n^\top \boldsymbol{\beta}_\star^{(t)} = \mathbf{0}$. Supplementary A.2 shows that the parameters can be centered without changing the estimation targets $\theta_{\star, i\ell}^{A,(t)}$ and $\theta_{\star, ik}^{B,(t)}$.

Assumption 1. There is a constant $K > 0$, independent of n, m, T , such that, for every work record r , $\sum_{i < \ell} (\Delta \mathbf{A}_r)_{i\ell} + \sum_{i, k} (\Delta \mathbf{B}_r)_{ik} \leq K$.

Assumption 1 bounds the contribution of each work record to the network increments $\Delta \mathbf{A}_r$ and $\Delta \mathbf{B}_r$. A work record may involve multiple collaborators and task types, but each work record updates at most K worker–worker and worker–task-type pairs in total.

Assumption 2. There are constants $R_z, R_w, \kappa > 0$, independent of n, m, T , such that, for every $t \in [T]$, the centered true parameters satisfy $\max_i \|\mathbf{z}_{\star, i}^{(t)}\|_2 \leq R_z$, $\max_k \|\mathbf{w}_{\star, k}^{(t)}\|_2 \leq R_w$, and $\lambda_{\min}\{(\mathbf{Z}_\star^{(t)})^\top \mathbf{Z}_\star^{(t)} / n\} \geq \kappa$. The feasible parameter set consists of tuples $\tilde{\Theta}$ with finite node effects that satisfy the same centering constraints and $\max_i \|\tilde{\mathbf{z}}_i\|_2 \leq 2R_z$, $\max_k \|\tilde{\mathbf{w}}_k\|_2 \leq 2R_w$.

Assumption 2 bounds the norms of the worker and task-type latent positions $\mathbf{z}_{\star, i}^{(t)}$ and

$\mathbf{w}_{\star,k}^{(t)}$. The lower bound on $\lambda_{\min}\{(\mathbf{Z}_{\star}^{(t)})^{\top} \mathbf{Z}_{\star}^{(t)}/n\}$ ensures that worker latent positions vary sufficiently in every latent dimension. The larger radii $2R_z$ and $2R_w$ define the radius constraints, that is, the norm bounds of the feasible parameter set, and keep the true latent positions strictly inside the radius constraints.

For a feasible parameter tuple $\tilde{\Theta}$, let $\theta_{i\ell}^A(\tilde{\Theta})$ and $\theta_{ik}^B(\tilde{\Theta})$ denote the cumulative means in (1) evaluated at $\tilde{\Theta}$. We consider feasible estimates $\hat{\Theta}^{(t)}$ satisfying:

$$\mathcal{L}^{(t)}(\hat{\Theta}^{(t)}) \leq \inf_{\tilde{\Theta} \text{ feasible}} \mathcal{L}^{(t)}(\tilde{\Theta}) + \varepsilon_t, \quad 0 \leq \varepsilon_t \leq \varepsilon_0, \quad (10)$$

where ε_t is the optimization error allowed at checkpoint t and ε_0 is a fixed constant.

For $\Theta = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{Z}, \mathbf{W})$ and $\Theta' = (\boldsymbol{\alpha}', \dots, \mathbf{W}')$, define the normalized distance:

$$\begin{aligned} \text{dist}_{n,m}^2(\Theta, \Theta') &= \frac{\|\boldsymbol{\alpha} - \boldsymbol{\alpha}'\|_2^2 + \|\boldsymbol{\beta} - \boldsymbol{\beta}'\|_2^2}{n} + \frac{\|\boldsymbol{\gamma} - \boldsymbol{\gamma}'\|_2^2}{m} \\ &\quad + \min_{\mathbf{Q}^{\top} \mathbf{Q} = \mathbf{I}_d} \left\{ \frac{\|\mathbf{Z} - \mathbf{Z}'\mathbf{Q}\|_{\text{F}}^2}{n} + \frac{\|\mathbf{W} - \mathbf{W}'\mathbf{Q}\|_{\text{F}}^2}{m} \right\}, \end{aligned} \quad (11)$$

where $\|\cdot\|_{\text{F}}$ is the Frobenius norm and $\mathbf{I}_d$ is the $d \times d$ identity matrix. The square root $\text{dist}_{n,m}$ is a metric on equivalence classes under a common orthogonal transformation. The same orthogonal $d \times d$ matrix $\mathbf{Q}$ acts on $\mathbf{Z}, \mathbf{W}$, and may differ across checkpoints. We write $\Theta\mathbf{Q} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{Z}\mathbf{Q}, \mathbf{W}\mathbf{Q})$.

Assumption 3. There are constants $0 < c_0 \leq C_0 < \infty$, independent of n, m, T , such that, for every $t \in [T]$ and all workers i and task types k :

$$c_0 t \leq \sum_{\ell \neq i} \theta_{\star, i\ell}^{A,(t)} \leq C_0 t, \quad c_0 t \leq \sum_k \theta_{\star, ik}^{B,(t)} \leq C_0 t, \quad c_0 nt/m \leq \sum_i \theta_{\star, ik}^{B,(t)} \leq C_0 nt/m. \quad (12)$$

Assumption 3 specifies the scale of cumulative network activity. For each worker, the sums $\sum_{\ell \neq i} \theta_{\star, i\ell}^{A,(t)}$ and $\sum_k \theta_{\star, ik}^{B,(t)}$ of estimation targets for collaboration and participation are of order t . The sum $\sum_{i,k} \theta_{\star, ik}^{B,(t)}$ is therefore of order nt , and each task type is required to have a sum of estimation targets $\sum_i \theta_{\star, ik}^{B,(t)}$ of order nt/m .

In the results below and Supplementary A, C, c, n_0 denote positive constants that may depend only on $d, K, c_0, C_0, R_z, R_w, \kappa, \varepsilon_0$. Under Assumption 2, Lemma S1 shows that Assumption 3 implies the following bounds for $n \geq n_0$:

$$c \frac{t}{n} \leq \theta_{\star, i\ell}^{A,(t)} \leq C \frac{t}{n} \quad (i < \ell), \quad c \frac{t}{m} \leq \theta_{\star, ik}^{B,(t)} \leq C \frac{t}{m} \quad (i, k). \quad (13)$$

Assumptions 1 and 3 imply that the cumulative number of work records satisfies $S_t \asymp nt$, that is, $cnt \leq S_t \leq Cnt$. The proof of $S_t \asymp nt$ is given at the beginning of Supplementary A.3.

Theorem 1. *Suppose Assumptions 1–3 hold and $n \geq n_0$. For every $\delta \in (0, 1)$, there is an event of probability at least $1 - \delta$ on which all checkpoints with $t \geq C(1 + m/n)\{\log((2n + m)/\delta) + \log \log(e^e + t)\}$ admit exact minimizers, and every feasible estimate with (10) obeys:*

$$\text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_\star^{(t)}) \leq C \frac{n+m}{nt} \left\{ \log \frac{2n+m}{\delta} + \log \log(e^e + t) \right\}. \quad (14)$$

$$\frac{1}{n^2} \sum_{i < \ell} \left\{ \log \frac{\theta_{i\ell}^A(\hat{\Theta}^{(t)})}{\theta_{\star, i\ell}^{A,(t)}} \right\}^2 + \frac{1}{nm} \sum_{i,k} \left\{ \log \frac{\theta_{ik}^B(\hat{\Theta}^{(t)})}{\theta_{\star, ik}^{B,(t)}} \right\}^2 \leq C \frac{n+m}{nt} \left\{ \log \frac{2n+m}{\delta} + \log \log(e^e + t) \right\}. \quad (15)$$

For a fixed $a \in (0, 1)$, let C_a denote a positive constant that may additionally depend on a . For any fixed T satisfying $T \geq C_a(1 + m/n) \log\{(2n + m)/\delta\}$, there is an event of probability at least $1 - \delta$ on which the maxima of $\text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_\star^{(t)})$ and the left-hand side of (15) over $\lceil aT \rceil \leq t \leq T$ are both at most $C_a(n + m) \log\{(2n + m)/\delta\}/(nT)$.

Theorem 1 quantifies the accuracy of the statistical summaries estimated from the cumulative networks. Bound (14) controls the average squared errors of the estimated node effects $\hat{\alpha}^{(t)}, \hat{\beta}^{(t)}, \hat{\gamma}^{(t)}$ and latent positions $\widehat{\mathbf{Z}}^{(t)}, \widehat{\mathbf{W}}^{(t)}$, with the latent positions compared up to a common orthogonal transformation. Bound (15) controls the average squared errors of the corresponding log-means, that is, the logarithms of the cumulative means, in both networks. Together, Bounds (14) and (15) provide statistical guarantees for the estimated parameters and for the participation and collaboration patterns that the estimated parameters describe. Apart from logarithmic factors, both error bounds scale as $(n + m)/(nt) = 1/t + m/(nt)$.

Under Assumption 3, each worker accumulates activity of order t , while each task type accumulates participation of order nt/m . The two terms in the rate are the reciprocals of the two activity scales. When $m = O(n)$, the rate is of order $1/t$. When $m \gg n$, the term $m/(nt)$ dominates, because less participation data is available for each task type. Thus, a larger number m of task types requires more work records to obtain the same error guarantee. For fixed n and m , both bounds tend to zero as t increases.

Remark 1. *The first conclusion of Theorem 1 holds simultaneously at all checkpoints satisfying the stated lower bound on t , which we call the information threshold, and the simultaneous control adds the $\log \log(e^e + t)$ term. For a fixed T , the second conclusion bounds the largest error over $\lceil aT \rceil \leq t \leq T$ without the $\log \log(e^e + t)$ term and therefore gives a bound of order $(n+m)/(nT)$, up to logarithmic factors, uniformly over $\lceil aT \rceil \leq t \leq T$.*

Let $\hat{\Theta}_\star^{(t)}$ denote an exact minimizer of $\mathcal{L}^{(t)}$ over the feasible parameter set, whose existence at the checkpoints covered by Theorem 1 is established by Lemma S3. We analyze the optimization updates from a checkpoint t_0 at which $\hat{\Theta}^{(t_0)}$ satisfies (10). The required lower bound on t_0 is specified in Theorem 2. For $t \geq 2$, define the checkpoint increments $\Delta \mathbf{A}^{(t)} = \mathbf{A}^{(t)} - \mathbf{A}^{(t-1)}$ and $\Delta \mathbf{B}^{(t)} = \mathbf{B}^{(t)} - \mathbf{B}^{(t-1)}$. The checkpoint increments aggregate the contributions of the work records added between checkpoints $t-1$ and t .

Assumption 4. There is a constant $K_1 > 0$, independent of n, m, T , such that, for every checkpoint $t = t_0 + 1, \dots, T$:

$$\frac{1}{n} \sum_i \left\{ \sum_{\ell \neq i} \Delta A_{i\ell}^{(t)} + \sum_k \Delta B_{ik}^{(t)} \right\}^2 + \frac{m}{n^2} \sum_k \left\{ \sum_i \Delta B_{ik}^{(t)} \right\}^2 \leq K_1^2 \left(1 + \frac{m}{n} \right). \quad (16)$$

Assumption 1 bounds each work record's contribution, whereas Assumption 4 controls the checkpoint increments $\Delta \mathbf{A}^{(t)}$ and $\Delta \mathbf{B}^{(t)}$. The two terms in (16) average squared increments over workers and task types, with the task-type term scaled by m/n to reflect the count scale nt/m of each task type under Assumption 3. The factor $1 + m/n$ accommodates

sparse integer counts: when $m \geq n$, one new participation per worker, each assigned to a distinct task type, with no new collaborations gives the two terms 1 and m/n .

Following [He et al. \(2025\)](#), we prescribe parameter-specific step sizes for the theoretical analysis, scaled according to the cumulative count scales in Assumption 3. Let $\mathbf{P}$ denote the orthogonal projection enforcing the centering constraints: it subtracts the column means from $\mathbf{Z}$ and $\mathbf{W}$ and the mean from $\boldsymbol{\beta}$, while leaving $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ unchanged. Starting from a feasible estimate $\hat{\Theta}^{(t_0)}$ satisfying (10), initialize each subsequent checkpoint at $\hat{\Theta}^{(t,0)} = \hat{\Theta}^{(t-1)}$. For a step-size constant $\eta > 0$ and an iteration count J , both independent of n, m, T , perform the following updates for $j = 0, \dots, J - 1$, with all gradients evaluated at $\hat{\Theta}^{(t,j)}$:

$$\hat{\Theta}^{(t,j+1)} = \mathbf{P} \left[\hat{\Theta}^{(t,j)} - \frac{\eta}{t} \left(\nabla_{\boldsymbol{\alpha}} \mathcal{L}^{(t)}, \nabla_{\boldsymbol{\beta}} \mathcal{L}^{(t)}, \frac{m}{n} \nabla_{\boldsymbol{\gamma}} \mathcal{L}^{(t)}, \nabla_{\mathbf{Z}} \mathcal{L}^{(t)}, \frac{m}{n} \nabla_{\mathbf{W}} \mathcal{L}^{(t)} \right)^\top \right]. \quad (17)$$

Remark 2. *Following the numerical implementation of [He et al. \(2025\)](#), we use adaptive Barzilai–Borwein step sizes in numerical optimization. Our implementation in Supplementary B.5 uses an adaptive scalar step size, obtained from Barzilai–Borwein step sizes and Armijo backtracking, to minimize the penalized objective (B.1).*

Set $\hat{\Theta}^{(t)} = \hat{\Theta}^{(t,J)}$, which we call the computed estimate at checkpoint t . Theorem 2 analyzes the step sizes and fixed iteration count in (17).

Theorem 2. *Suppose Assumptions 1–4 hold and $n \geq n_0$. For fixed $\delta \in (0, 1)$, choose a deterministic integer $t_0 < T$ satisfying $t_0 \geq C(n + m)^2 \log\{(2n + m)/\delta\}/n$, and let $\hat{\Theta}^{(t_0)}$ be a feasible estimate satisfying (10). There exist $\eta > 0$, an integer $J \geq 1$, and $q \in (0, 1)$, independent of n, m, T , such that, with probability at least $1 - \delta$, all iterates of (17) remain feasible and the following bounds hold simultaneously for $t_0 \leq t \leq T$:*

$$\text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \hat{\Theta}_\star^{(t)}) \leq C \frac{\varepsilon_0}{nt_0} q^{2(t-t_0)} + C \frac{n+m}{nt^2}, \quad (18)$$

$$0 \leq \mathcal{L}^{(t)}(\hat{\Theta}^{(t)}) - \mathcal{L}^{(t)}(\hat{\Theta}_\star^{(t)}) \leq C(1 + \varepsilon_0), \quad (19)$$

where the constants may additionally depend on K_1 .

Theorem 2 shows that a fixed number J of updates per checkpoint suffices to maintain a bounded optimization error. In (18), the contribution of the initial error decays geometrically, while the term $(n+m)/(nt^2)$ accounts for changes in the exact minimizer as new work records are added. Bound (19) ensures that the computed estimates satisfy a fixed optimization-error bound throughout $t_0 \leq t \leq T$. Combining Bound (19) with Theorem 1 gives the following corollary, which bounds the statistical error of the computed estimates relative to the true parameters $\Theta_\star^{(t)}$.

Corollary 1. *Fix $\delta \in (0, 1)$ and $a \in (0, 1)$. Suppose the conditions of Theorem 2 hold and $\lceil aT \rceil \geq t_0$. Then the estimates $\hat{\Theta}^{(t)}$ produced by (17), with η and J chosen as in Theorem 2, satisfy, with probability at least $1 - \delta$:*

$$\max_{\lceil aT \rceil \leq t \leq T} \text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_\star^{(t)}) \leq C_a \frac{n+m}{nT} \log \frac{2n+m}{\delta}. \quad (20)$$

The same bound holds for the maximum of the normalized squared log-mean error on the left-hand side of (15) over these checkpoints.

Corollary 1 shows that the estimates obtained with a fixed number of updates per checkpoint attain the statistical error rate in Theorem 1. Under Assumptions 1–4 and condition (10) at t_0 , with constants uniform along a sequence of (n, m, T, t_0) , a sufficient condition for uniform consistency in probability over $\lceil aT \rceil \leq t \leq T$ is $\{(n+m)^2 \log(2n+m)\}/(nt_0) \rightarrow 0$ with $t_0 \leq \lceil aT \rceil$. Supplementary A.5 gives the derivation.

Each gradient evaluation in (17) costs $O\{n(n+m)d\}$ operations when all cumulative means $\theta_{i\ell}^A$ and θ_{ik}^B are computed directly. The J updates at each of the $T - t_0$ subsequent checkpoints therefore cost $O\{J(T - t_0)n(n+m)d\}$ operations in total, excluding initialization.

5 Experiments

5.1 LongBench v2 Multi-Doc QA

LongBench v2 evaluates understanding and reasoning over realistic long contexts (Bai et al. 2025). We use all 125 questions in its Multi-Doc QA subset, covering academic, financial, governmental, legal, and multi-news documents. DeLM also evaluates decentralized evidence aggregation on this subset (Mao & Mirhoseini 2026). Each question requires the combination of evidence from multiple documents, so both the division of labor and the reuse of published results become observable. We use the four task types in Section 3.1: *extract* for recording source-grounded evidence, *compare* for relating objects along specified dimensions, *inference* for calculations, deductions, and answer synthesis, and *review* for checking the exact version of another worker’s published result. A work may deliver several independently reusable results and contribute to several task types. All workers may perform any task type and choose their own sub-tasks and collaborators.

We compare two methods: *AgentCue*, in which workers receive the statistical summary, and *History only*, in which workers receive no statistical summary. A replicate is one realization of the following paired design. The replicate begins with 32 warm-up sessions, which follow the History-only rules, and then copies the resulting shared board and the private memories, that is, the notes that each worker keeps across sessions, into two branches, the AgentCue branch and the History-only branch. Each branch adds 224 sessions. A trajectory consists of the shared warm-up and one branch, and we call the 256 sessions of a trajectory the effective sessions of the trajectory; a replicate runs 480 actual sessions because the two trajectories share the warm-up. The MAS uses 32 persistent workers with GPT-5.6 Sol at high reasoning effort and at most 32 concurrent sessions. Every completed session counts, including sessions without published work or a correct answer. Both methods expose the

same sources, published results, work records, review outcomes, and current workload. A worker designates a published answer as the final answer to a question. The first designated answer to a question must use a relevant published result of another worker, and designation does not require a PASS verdict, that is, an approval in a review.

Within these common rules, workers first prioritize uncovered questions, answer-critical gaps, and the continuation of useful existing works. Workers in the AgentCue branch additionally use the statistical summary to narrow local choices, as in Section 3.3, and search the work records when the statistical summary gives weak support or the current needs contradict the statistical summary. Each new session reads the latest statistical summary once and keeps the same summary version throughout the session. In the AgentCue branch, the MAS estimates the parameters of the latent space model with $d = 2$ at eight checkpoints, at 32, 64, $\dots$, 256 completed effective sessions; estimation at intermediate checkpoints runs in the background while sessions continue. The three replicates, R1–R3, each evaluate the AgentCue branch followed by the History-only branch from their shared warm-up. Supplementary B.1–B.5 give the design, prompts, and estimation procedure.

Table 1: Final results over three paired replicates (mean $\pm$ sample standard deviation), with 256 effective sessions per trajectory and 125 questions.

Measure	AgentCue	History only
Correct answers	76.0 $\pm$ 1.0	72.0 $\pm$ 1.0
Accuracy (%)	60.8 $\pm$ 0.8	57.6 $\pm$ 0.8

Accuracy compares each question’s final designated answer with the withheld benchmark key, using all 125 questions as the denominator. AgentCue achieves mean accuracy of 60.8%, compared with 57.6% for History only (Table 1). The paired differences are 4.0, 4.0, and 1.6 percentage points, with mean 3.2 and sample standard deviation 1.39. Both methods

designate answers for all 125 questions in every replicate. Supplementary B.6 reports token use and elapsed time.

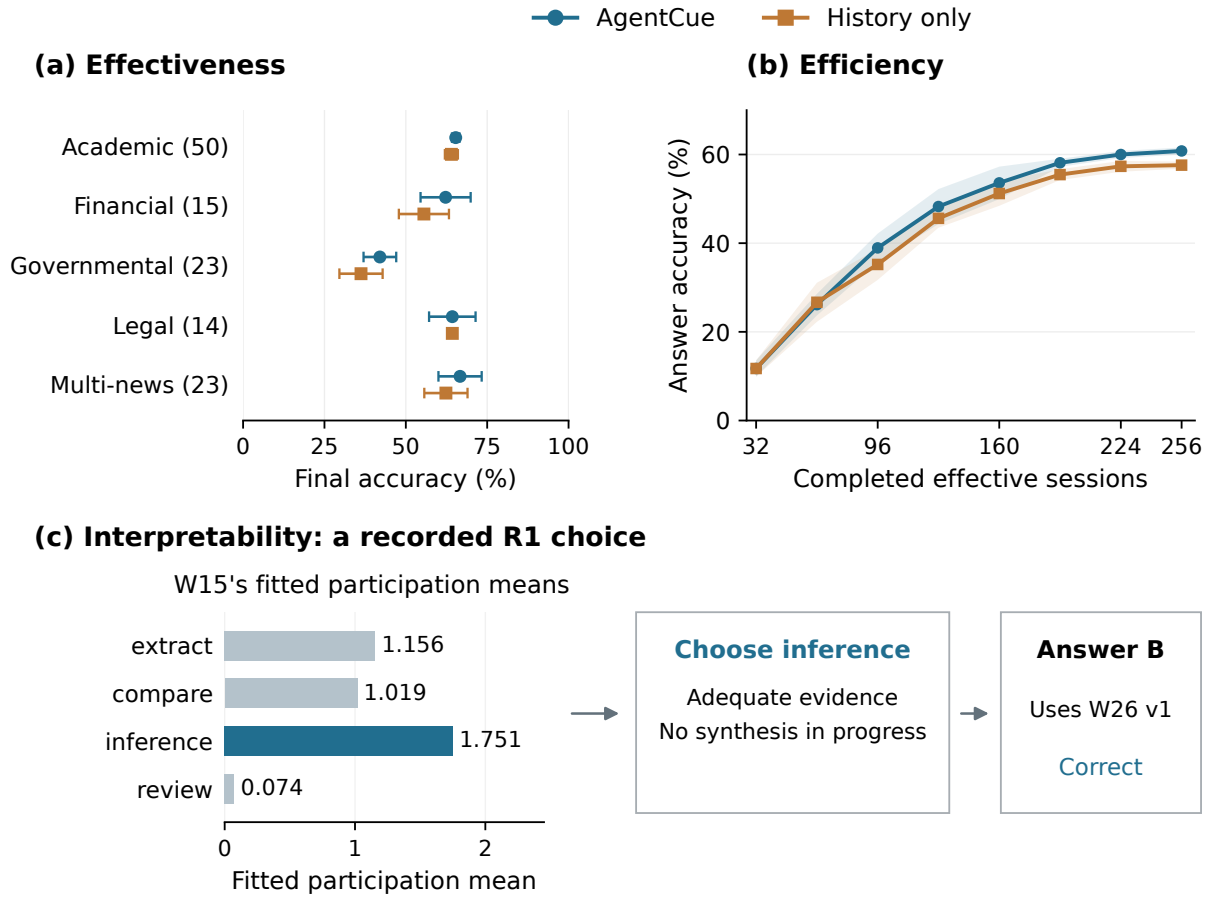


Figure 4: Effectiveness, efficiency, and interpretability of AgentCue. (a) Final accuracy by domain; parentheses give question counts. (b) Accuracy over the common effective-session budget, using all 125 questions. Points and lines show means across three paired replicates; error bars and shading show one sample standard deviation. (c) One recorded choice in R1 links the fitted participation means of W15, the choice of *inference*, the use of version 1 of the published result of W26, and the correct designated answer, where W_i denotes worker i .

Figure 4 connects the three evaluation dimensions. Final mean accuracy is higher for AgentCue in four domains and equal in legal documents, with the largest differences in financial and governmental documents. The accuracy curves in panel (b) coincide at 32

effective sessions, the end of the shared warm-up. At 64 effective sessions, mean accuracy is 26.1% for AgentCue and 26.7% for History only; AgentCue has higher mean accuracy at the remaining six checkpoints. Accuracies at intermediate checkpoints include published results of sessions in progress. Panel (c) traces one recorded choice, detailed below.

Figure 5 links all 256 sessions with eight statistical summaries for the AgentCue trajectory in R1. Workers overlap in their task types and move among extract, compare, inference, and review, so the participation of no worker locks in to a single task type, the risk that Section 3.4 analyzes. The upper panels show jointly aligned estimates; the orthogonal alignment leaves every fitted mean unchanged. Supplementary B.7 explains the construction.

The example in Figure 4(c) concerns changing security threats to courts and public officials. W26 publishes source-grounded evidence during warm-up. W15 reads version 1 of the published result of W26 and the statistical summary at the first checkpoint, then identifies inference as the task type with the largest fitted participation mean of W15, 1.751. The recorded choice of W15 also cites adequate evidence published by other workers and the absence of an answer synthesis in progress. W15 uses W26’s result to designate the correct answer B while the first sessions of both branches still run, at 32 completed effective sessions. The Choice Record of W15 and the work records link the read of the statistical summary, the local comparison, the recorded choice, the result version, and the answer, and show how a worker combines fitted participation means with current needs and available evidence.

6 Discussion

As an application of statistics to agent systems, our method replaces repeated searches of the shared board in large-scale decentralized MAS with a statistical summary of participation and collaboration records. In the experiment on LongBench v2 Multi-Doc QA, the statistical summary raised mean accuracy under a common effective-session budget and made the

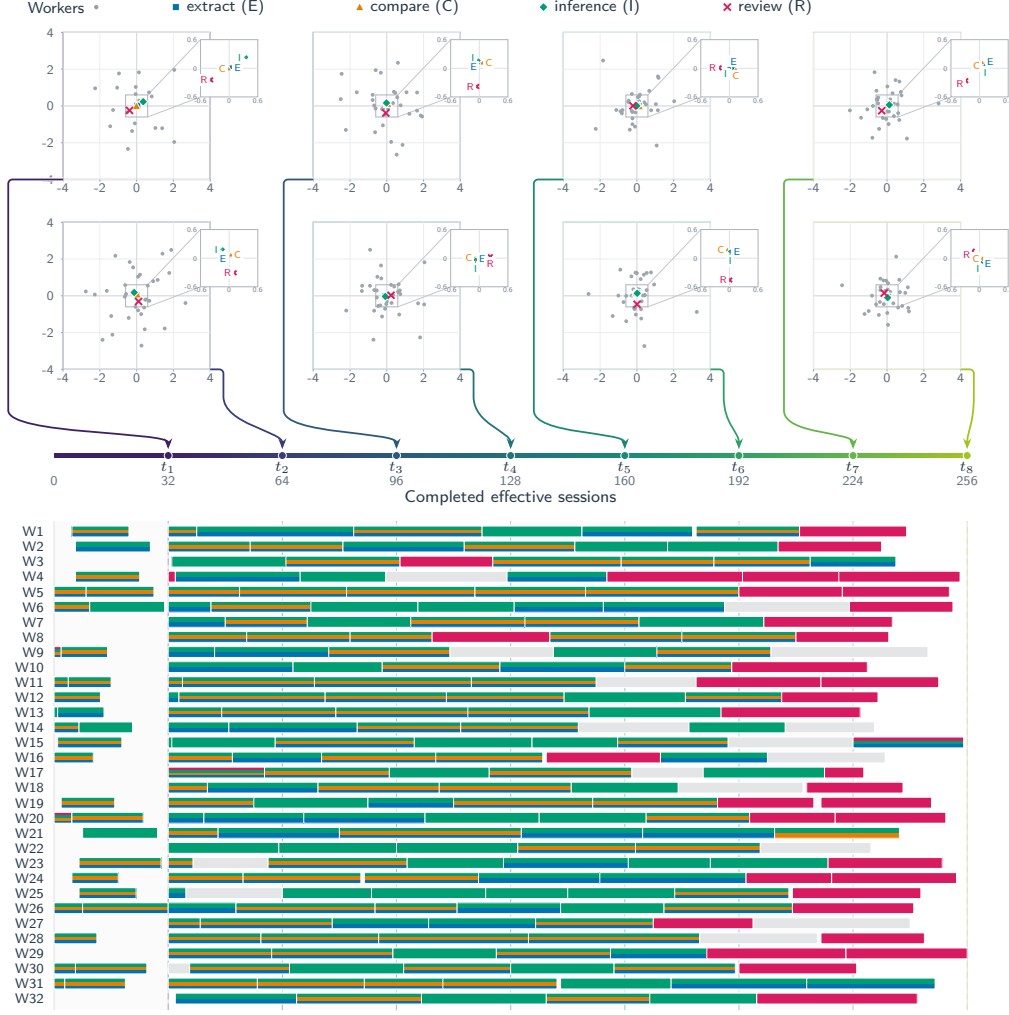


Figure 5: Worker sessions and statistical summaries for the AgentCue trajectory in R1. Upper panels show eight statistical summaries jointly aligned to the statistical summary at the final checkpoint by orthogonal Procrustes, with consecutive checkpoints paired by column. Linear axes show worker latent positions; insets enlarge the four task-type latent positions. The color scale links panels and arrows to the checkpoints $t_1, \dots, t_8$. Lower bars show all 256 sessions of 32 workers, mapped from timestamps to the number of completed effective sessions. Equal-height stripes indicate task types of work published in a session; gray denotes no publication.

recorded choices of workers interpretable. More broadly, our method builds a stage on which statistics acts inside agent systems: the MAS generates network data, the statistical

summary guides the MAS, and other statistical methods can join the loop.

We plan two extensions of our method. First, owing to the cost of foundation-model calls, our experiments use moderate-scale MAS, and a direct next step is to scale the experiments up to 1,024 workers, as in Agensh (Zhan et al. 2026). Second, as workers accumulate private memories, early collaboration records may no longer reflect current collaboration. Drawing on continual learning (Wang et al. 2024) and agent memory (Park et al. 2023), the latent space model can down-weight the early collaboration records through discounted edge counts or time-varying latent positions (Durante & Dunson 2014).

Disclosure Statement. The authors report there are no competing interests to declare.

Declaration of AI Use. OpenAI’s GPT-6 Astra was used to assist with literature searches and language editing, and GPT-5.6 Sol was used as the foundation model in the experiments.

Data Availability Statement. LongBench v2: <https://github.com/THUDM/LongBench>.

Supplementary Material. The supplementary file contains the theory and details of the LongBench v2 Multi-Doc QA experiment, including worker prompts, supplementary results, numerical estimation, computational resources, and figure construction. (PDF)

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Supplementary File of “AgentCue: Statistical Summaries for Guiding Decentralized Multi-Agent Self-Organization”

This supplementary file is organized as follows. Section A presents the theoretical development, including model identifiability, supporting lemmas, and proofs of the main results. Section B provides additional details of the LongBench v2 Multi-Doc QA experiment, covering the experimental design, worker prompts, network construction, supplementary results, numerical estimation, computational resources, and figure construction.

A Theoretical Development

This section provides the theoretical results for the latent space model and its estimation. Section A.1 establishes model identifiability up to a common orthogonal transformation of the latent positions. Section A.2 develops supporting results on centering, concentration, parameter recovery, and local curvature. Section A.3 proves the statistical error bounds in Theorem 1 and discusses their extension to larger cumulative activity. Section A.4 proves the optimization bounds in Theorem 2 for a fixed number of updates at each checkpoint. Section A.5 proves Corollary 1, which gives statistical error bounds and a sufficient condition for uniform consistency of computed estimates. Constants follow the convention in Section 4.

A.1 Identifiability

As in Fan et al. (2024), identifiability follows from equal complete linear predictor matrices.

Proposition 1 (Identifiability). *Fix a checkpoint t . Consider two parameter tuples $\Theta^{(t)} = (\alpha^{(t)}, \beta^{(t)}, \gamma^{(t)}, \mathbf{Z}^{(t)}, \mathbf{W}^{(t)})$ and $\Theta'^{(t)} = (\alpha'^{(t)}, \beta'^{(t)}, \gamma'^{(t)}, \mathbf{Z}'^{(t)}, \mathbf{W}'^{(t)})$, where $\alpha^{(t)}, \beta^{(t)} \in \mathbb{R}^n$, $\gamma^{(t)} \in \mathbb{R}^m$, $\mathbf{Z}^{(t)} \in \mathbb{R}^{n \times d}$, $\mathbf{W}^{(t)} \in \mathbb{R}^{m \times d}$, and the primed parameters have the same dimensions.*

Suppose $\text{rank}(\mathbf{Z}^{(t)}) = d$ and both tuples satisfy:

$$\mathbf{1}_n^\top \mathbf{Z}^{(t)} = \mathbf{1}_n^\top \mathbf{Z}'^{(t)} = \mathbf{0}^\top, \quad \mathbf{1}_m^\top \mathbf{W}^{(t)} = \mathbf{1}_m^\top \mathbf{W}'^{(t)} = \mathbf{0}^\top, \quad \mathbf{1}_n^\top \boldsymbol{\beta}^{(t)} = \mathbf{1}_n^\top \boldsymbol{\beta}'^{(t)} = 0.$$

If their complete linear predictor matrices are equal, that is:

$$\boldsymbol{\alpha}^{(t)} \mathbf{1}_n^\top + \mathbf{1}_n (\boldsymbol{\alpha}^{(t)})^\top + \mathbf{Z}^{(t)} (\mathbf{Z}^{(t)})^\top = \boldsymbol{\alpha}'^{(t)} \mathbf{1}_n^\top + \mathbf{1}_n (\boldsymbol{\alpha}'^{(t)})^\top + \mathbf{Z}'^{(t)} (\mathbf{Z}'^{(t)})^\top, \quad (\text{A.1})$$

$$\boldsymbol{\beta}^{(t)} \mathbf{1}_m^\top + \mathbf{1}_m (\boldsymbol{\beta}^{(t)})^\top + \mathbf{Z}^{(t)} (\mathbf{W}^{(t)})^\top = \boldsymbol{\beta}'^{(t)} \mathbf{1}_m^\top + \mathbf{1}_m (\boldsymbol{\beta}'^{(t)})^\top + \mathbf{Z}'^{(t)} (\mathbf{W}'^{(t)})^\top, \quad (\text{A.2})$$

then there is an orthogonal matrix $\mathbf{Q}^{(t)} \in \mathbb{R}^{d \times d}$ such that:

$$(\boldsymbol{\alpha}'^{(t)}, \boldsymbol{\beta}'^{(t)}, \boldsymbol{\gamma}'^{(t)}) = (\boldsymbol{\alpha}^{(t)}, \boldsymbol{\beta}^{(t)}, \boldsymbol{\gamma}^{(t)}), \quad \mathbf{Z}'^{(t)} = \mathbf{Z}^{(t)} \mathbf{Q}^{(t)}, \quad \mathbf{W}'^{(t)} = \mathbf{W}^{(t)} \mathbf{Q}^{(t)}.$$

Proof. Right-multiplying (A.1) by $\mathbf{1}_n$ and using centered worker latent positions gives:

$$n(\boldsymbol{\alpha}^{(t)} - \boldsymbol{\alpha}'^{(t)}) + \{\mathbf{1}_n^\top (\boldsymbol{\alpha}^{(t)} - \boldsymbol{\alpha}'^{(t)})\} \mathbf{1}_n = \mathbf{0}.$$

Summing its entries yields $\mathbf{1}_n^\top (\boldsymbol{\alpha}^{(t)} - \boldsymbol{\alpha}'^{(t)}) = 0$. Back-substitution gives $\boldsymbol{\alpha}^{(t)} = \boldsymbol{\alpha}'^{(t)}$.

Left-multiplying (A.2) by $\mathbf{1}_n^\top$ and using the centering of $\boldsymbol{\beta}^{(t)}, \boldsymbol{\beta}'^{(t)}, \mathbf{Z}^{(t)}, \mathbf{Z}'^{(t)}$ gives $n(\boldsymbol{\gamma}^{(t)} - \boldsymbol{\gamma}'^{(t)})^\top = \mathbf{0}^\top$, so $\boldsymbol{\gamma}^{(t)} = \boldsymbol{\gamma}'^{(t)}$. Right-multiplying the same equation by $\mathbf{1}_m$, then using this equality and the centering of the task-type latent positions, gives $m(\boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}'^{(t)}) = \mathbf{0}$. Hence $\boldsymbol{\beta}^{(t)} = \boldsymbol{\beta}'^{(t)}$. Removing the equal node effects from (A.1) and (A.2) leaves:

$$\begin{aligned} \mathbf{Z}^{(t)} (\mathbf{Z}^{(t)})^\top &= \mathbf{Z}'^{(t)} (\mathbf{Z}'^{(t)})^\top, \\ \mathbf{Z}^{(t)} (\mathbf{W}^{(t)})^\top &= \mathbf{Z}'^{(t)} (\mathbf{W}'^{(t)})^\top. \end{aligned}$$

The Gram-matrix equality implies that $\mathbf{Z}^{(t)}$ and $\mathbf{Z}'^{(t)}$ have the same column space. Since $\mathbf{Z}^{(t)}$ has full column rank, there is a matrix $\mathbf{Q}^{(t)} \in \mathbb{R}^{d \times d}$ such that $\mathbf{Z}'^{(t)} = \mathbf{Z}^{(t)} \mathbf{Q}^{(t)}$. Substituting this relation into the Gram-matrix equality and multiplying by a left inverse of $\mathbf{Z}^{(t)}$ and its transpose gives $\mathbf{Q}^{(t)} (\mathbf{Q}^{(t)})^\top = \mathbf{I}_d$, where $\mathbf{I}_d$ is the identity matrix. Thus $\mathbf{Q}^{(t)}$ is orthogonal. Substituting the same relation into the cross-product equality and multiplying by a left inverse of $\mathbf{Z}^{(t)}$ gives $(\mathbf{W}^{(t)})^\top = \mathbf{Q}^{(t)} (\mathbf{W}'^{(t)})^\top$. Therefore $\mathbf{W}'^{(t)} = \mathbf{W}^{(t)} \mathbf{Q}^{(t)}$. $\square$

A.2 Supporting Lemmas and Proofs

We first verify invariance under centering. Let $\bar{\mathbf{z}} = n^{-1} \sum_i \mathbf{z}_i$, $\bar{\mathbf{w}} = m^{-1} \sum_k \mathbf{w}_k$, and $\bar{b} = n^{-1} \sum_i (\beta_i + \mathbf{z}_i^\top \bar{\mathbf{w}})$. Set:

$$\begin{aligned} \mathbf{z}_i^c &= \mathbf{z}_i - \bar{\mathbf{z}}, & \mathbf{w}_k^c &= \mathbf{w}_k - \bar{\mathbf{w}}, & \alpha_i^c &= \alpha_i + \mathbf{z}_i^\top \bar{\mathbf{z}} - \frac{1}{2} \|\bar{\mathbf{z}}\|_2^2, \\ \beta_i^c &= \beta_i + \mathbf{z}_i^\top \bar{\mathbf{w}} - \bar{b}, & \gamma_k^c &= \gamma_k + \bar{\mathbf{z}}^\top \mathbf{w}_k - \bar{\mathbf{z}}^\top \bar{\mathbf{w}} + \bar{b}. \end{aligned}$$

The positions and β^c have zero sums. Expanding $(\mathbf{z}_i^c)^\top \mathbf{z}_\ell^c$ and $(\mathbf{z}_i^c)^\top \mathbf{w}_k^c$ gives:

$$\alpha_i^c + \alpha_\ell^c + (\mathbf{z}_i^c)^\top \mathbf{z}_\ell^c = \alpha_i + \alpha_\ell + \mathbf{z}_i^\top \mathbf{z}_\ell, \quad \beta_i^c + \gamma_k^c + (\mathbf{z}_i^c)^\top \mathbf{w}_k^c = \beta_i + \gamma_k + \mathbf{z}_i^\top \mathbf{w}_k.$$

Thus all log-means and means are preserved. Subtracting the row average increases a uniform row-norm bound by at most a factor of two. Assumption 2 is imposed on the centered positions and their spectrum.

Lemma S1. *Fix $u > 0$. With latent row norms bounded by $2R_z, 2R_w$, if every row sum of the mean matrices $(\theta_{i\ell}^A)$ and (θ_{ik}^B) lies between $c_0 u$ and $C_0 u$, and every column sum of (θ_{ik}^B) lies between $c_0 n u / m$ and $C_0 n u / m$, then for all $n \geq n_0$, where n_0 is a sufficiently large fixed constant, $c u / n \leq \theta_{i\ell}^A \leq C u / n$ and $c u / m \leq \theta_{ik}^B \leq C u / m$. Constants depend only on the radii and marginal bounds. $\mathbf{Z}, \mathbf{W}$ may be rank deficient.*

Proof. In the **A** block, with $x_i = e^{\alpha_i}$, $x_+ = \sum_i x_i$, $s_i = \sum_{\ell \neq i} \theta_{i\ell}^A$, bounded positions give:

$$e^{-4R_z^2} x_i (x_+ - x_i) \leq s_i \leq e^{4R_z^2} x_i (x_+ - x_i), \quad c_0 u \leq s_i \leq C_0 u. \quad (\text{A.3})$$

If the largest x_i were at least $x_+/2$, then:

$$(n-1)c_0 u \leq \sum_{j \neq i} s_j \leq e^{4R_z^2} x_+ (x_+ - x_i) \leq 2e^{4R_z^2} x_i (x_+ - x_i) \leq 2e^{8R_z^2} C_0 u.$$

Taking $n_0 > 1 + 2e^{8R_z^2} C_0 / c_0$ rules out this possibility. Hence $x_+/2 < x_+ - x_i \leq x_+$ for every i . Summing (A.3) gives $x_+^2 \asymp n u$. The individual bounds then give $x_i x_+ \asymp u$, so $x_i \asymp \sqrt{u/n}$.

Multiplying by the bounded inner-product exponential proves the **A** claim.

For the $\mathbf{B}$ block, set $x_i = e^{\beta_i}$, $y_k = e^{\gamma_k}$, $x_+ = \sum_i x_i$, and $y_+ = \sum_k y_k$. Write s_i, c_k, s_+ for the row, column, and total sums, and $a = e^{-4R_z R_w}$, $b = e^{4R_z R_w}$. Bounded inner products give:

$$ax_i y_+ \leq s_i \leq bx_i y_+, \quad ay_k x_+ \leq c_k \leq by_k x_+, \quad ax_+ y_+ \leq s_+ \leq bx_+ y_+.$$

Using $x_i y_k = (x_i y_+)(y_k x_+)/ (x_+ y_+)$ and $s_i \asymp u$, $c_k \asymp nu/m$, $s_+ \asymp nu$ yields:

$$\frac{a^2}{b^2} \frac{s_i c_k}{s_+} \leq \theta_{ik}^B \leq \frac{b^2}{a^2} \frac{s_i c_k}{s_+}, \quad \theta_{ik}^B \asymp u/m.$$

□

The martingales below are indexed by work record r . Checkpoint t enters through the cumulative index S_t . Assumption 3 gives:

$$\sum_{\ell \neq i} \theta_{*,i\ell}^{A,(t)} \asymp t, \quad \sum_k \theta_{*,ik}^{B,(t)} \asymp t, \quad \sum_i \theta_{*,ik}^{B,(t)} \asymp nt/m. \quad (\text{A.4})$$

The entrywise bounds in (13) follow from Lemma S1 and are used to bound curvature.

Define the weighted matrix increment and its centered cumulative sum:

$$\mathbf{H}_r = \begin{bmatrix} \frac{\sqrt{n}}{2} \Delta \mathbf{A}_r, & \sqrt{m} \Delta \mathbf{B}_r \end{bmatrix}, \quad \mathbf{M}_q = \sum_{r=1}^q \{ \mathbf{H}_r - \mathbb{E}(\mathbf{H}_r \mid \mathcal{F}_{r-1}) \}. \quad (\text{A.5})$$

Then $\mathbf{M}_q$ is an $n \times (n+m)$ matrix martingale. Write $\|\cdot\|_{\text{op}}$ for the operator norm. The factor $1/2$ compensates for symmetric duplication in $\mathbf{A}$, while $\sqrt{n}, \sqrt{m}$ scale the two blocks. At checkpoint t , the first block of $\mathbf{M}_{S_t}$ is $\sqrt{n}/2$ times the observed $\mathbf{A}^{(t)}$ minus the matrix of estimation targets $(\theta_{*,i\ell}^{A,(t)})$. The second block is the corresponding $\mathbf{B}$ residual scaled by $\sqrt{m}$.

Lemma S2. *Under Assumptions 1 and 3, for every $\delta \in (0, 1)$, the following holds simultaneously at all checkpoints under study with probability at least $1 - \delta$:*

$$\|\mathbf{M}_{S_t}\|_{\text{op}}^2 \leq C(n+m) \left[t \left\{ \log \frac{2n+m}{\delta} + \log \log(e^e + t) \right\} + \left\{ \log \frac{2n+m}{\delta} + \log \log(e^e + t) \right\}^2 \right]. \quad (\text{A.6})$$

Whenever the information threshold in Theorem 1 holds, the same event also gives:

$$ct \leq \sum_{\ell \neq i} A_{i\ell}^{(t)} \leq Ct, \quad ct \leq \sum_k B_{ik}^{(t)} \leq Ct, \quad c \frac{nt}{m} \leq \sum_i B_{ik}^{(t)} \leq C \frac{nt}{m}. \quad (\text{A.7})$$

For fixed $T \geq 1$ and $a \in (0, 1)$, the following bound holds simultaneously for $\lceil aT \rceil \leq t \leq T$, with probability at least $1 - \delta$:

$$\|\mathbf{M}_{S_t}\|_{\text{op}}^2 \leq C(n+m) \left\{ T \log \frac{2n+m}{\delta} + \log^2 \frac{2n+m}{\delta} \right\}. \quad (\text{A.8})$$

If $T \geq C_a(1 + m/n) \log\{(2n+m)/\delta\}$, (A.7) also holds on this event.

Proof. We first bound individual increments and conditional variances. Assumption 1 gives at most $2K$ nonzero entries in $\mathbf{H}_r$, with:

$$\|\mathbf{H}_r\|_{\text{F}}^2 = \frac{n}{2} \sum_{i < \ell} (\Delta \mathbf{A}_r)_{i\ell} + m \sum_{i,k} (\Delta \mathbf{B}_r)_{ik} \leq K(n+m).$$

Conditional expectation preserves the corresponding operator-norm bound, so centered increments have norm at most $2\sqrt{K(n+m)}$. For any vector $\mathbf{x}$, applying Cauchy–Schwarz to the nonzero entries in each column gives:

$$\mathbf{x}^\top \mathbf{H}_r \mathbf{H}_r^\top \mathbf{x} \leq 2K \sum_i x_i^2 \left\{ \frac{n}{4} \sum_{\ell \neq i} (\Delta \mathbf{A}_r)_{i\ell} + m \sum_k (\Delta \mathbf{B}_r)_{ik} \right\}.$$

Set $\mathbf{X}_r = \mathbf{H}_r - \mathbb{E}(\mathbf{H}_r \mid \mathcal{F}_{r-1})$. Conditional centering gives:

$$\begin{aligned} \mathbb{E}(\mathbf{X}_r \mathbf{X}_r^\top \mid \mathcal{F}_{r-1}) &= \mathbb{E}(\mathbf{H}_r \mathbf{H}_r^\top \mid \mathcal{F}_{r-1}) - \mathbb{E}(\mathbf{H}_r \mid \mathcal{F}_{r-1}) \mathbb{E}(\mathbf{H}_r \mid \mathcal{F}_{r-1})^\top \\ &\preceq \mathbb{E}(\mathbf{H}_r \mathbf{H}_r^\top \mid \mathcal{F}_{r-1}). \end{aligned}$$

The same identity holds with the factors transposed. Applying the preceding column bound and its row analogue, then (A.4), yields:

$$\begin{aligned} \left\| \sum_{r \leq S_t} \mathbb{E}(\mathbf{X}_r \mathbf{X}_r^\top \mid \mathcal{F}_{r-1}) \right\|_{\text{op}} &\leq 2K \max_i \left\{ \frac{n}{4} \sum_{\ell \neq i} \theta_{*,i\ell}^{A,(t)} + m \sum_k \theta_{*,ik}^{B,(t)} \right\} \leq C(n+m)t, \\ \left\| \sum_{r \leq S_t} \mathbb{E}(\mathbf{X}_r^\top \mathbf{X}_r \mid \mathcal{F}_{r-1}) \right\|_{\text{op}} &\leq 2K \max \left\{ \frac{n}{4} \max_\ell \sum_{i \neq \ell} \theta_{*,i\ell}^{A,(t)}, m \max_k \sum_i \theta_{*,ik}^{B,(t)} \right\} \leq Cnt. \end{aligned}$$

For a scalar row or column contribution $0 \leq Y_r \leq K$:

$$\begin{aligned} \sum_{r \leq S_t} \text{Var}(Y_r \mid \mathcal{F}_{r-1}) &\leq \sum_{r \leq S_t} \mathbb{E}(Y_r^2 \mid \mathcal{F}_{r-1}) \\ &\leq K \sum_{r \leq S_t} \mathbb{E}(Y_r \mid \mathcal{F}_{r-1}) \leq \begin{cases} Ct, & \text{worker row,} \\ Cnt/m, & \text{task-type column.} \end{cases} \end{aligned}$$

Define the matrix variance process:

$$V_q = \max \left\{ \left\| \sum_{r \leq q} \mathbb{E}(\mathbf{X}_r \mathbf{X}_r^\top \mid \mathcal{F}_{r-1}) \right\|_{\text{op}}, \left\| \sum_{r \leq q} \mathbb{E}(\mathbf{X}_r^\top \mathbf{X}_r \mid \mathcal{F}_{r-1}) \right\|_{\text{op}} \right\}.$$

The rectangular matrix Freedman inequality of [Tropp \(2011, Corollary 1.3\)](#) states that, if $\|\mathbf{X}_r\|_{\text{op}} \leq R$ almost surely, then for any $u, v > 0$:

$$\mathbb{P} \{ \exists q \geq 1 : \|\mathbf{M}_q\|_{\text{op}} \geq u, V_q \leq v \} \leq (2n + m) \exp \left\{ -\frac{u^2}{2(v + Ru/3)} \right\}.$$

For $x > 0$ and variance budget v , the threshold $\sqrt{2vx} + (2R/3)x$ therefore has failure probability at most $(2n + m)e^{-x}$. This maximal event covers all work-record indices within the budget and hence applies to random S_t . The scalar Freedman inequality for both signs controls the $2n + m$ marginal processes of the $\mathbf{A}, \mathbf{B}$ row sums and $\mathbf{B}$ column sums.

For the time-uniform result, partition checkpoints into $2^j \leq t < 2^{j+1}$, $j \geq 0$. On interval j , the variance budgets are $C(n + m)2^{j+1}$, $C2^{j+1}$, and $C(n/m)2^{j+1}$. Allocate total failure probability at most $C\delta/(j + 1)^2$ to its matrix and marginal events, with the allocation constant chosen so that the sum is at most δ . Within this proof, take:

$$x_j = \log \frac{C(2n + m)(j + 1)^2}{\delta} \leq C \left\{ \log \frac{2n + m}{\delta} + \log \log(e^e + t) \right\}.$$

The matrix deviation is bounded by $C\sqrt{n + m}\{\sqrt{tx_j} + x_j\}$. Squaring proves [\(A.6\)](#). Worker row deviations are bounded by $C\{\sqrt{tx_j} + x_j\}$ and task-type column deviations by $C\{\sqrt{(nt/m)x_j} + x_j\}$. When $\min\{t, nt/m\} \geq Cx_j$, each deviation is at most a fixed fraction of the corresponding sum of estimation targets, which gives [\(A.7\)](#). The information threshold in [Theorem 1](#) suffices.

For the bound over $\lceil aT \rceil \leq t \leq T$, apply the same maximal inequality directly over all work-record indices with variance budgets $C(n + m)T$, CT , and CnT/m . Taking $x = \log\{C(2n + m)/\delta\}$ gives [\(A.8\)](#) and the marginal deviations. On $t \geq aT$, the corresponding

sums of estimation targets are at least caT for worker rows and $canT/m$ for task-type columns, so the stated lower bound on T suffices for (A.7). $\square$

Lemma S3. *Suppose (A.7) holds, $\min\{t, nt/m\}$ exceeds a fixed constant, and $n \geq n_0$. Every estimate satisfying (10) then obeys:*

$$c \frac{t}{n} \leq \hat{\theta}_{il}^{A,(t)} \leq C \frac{t}{n}, \quad c \frac{t}{m} \leq \hat{\theta}_{ik}^{B,(t)} \leq C \frac{t}{m}. \quad (\text{A.9})$$

Under these conditions, the infimum in (10) is attained at a finite feasible parameter.

Proof. Let $D^{(t)}$ be one observed row sum of $\mathbf{A}^{(t)}$ or $\mathbf{B}^{(t)}$, or one column sum of $\mathbf{B}^{(t)}$, and let $\widehat{D}^{(t)}$ be the corresponding fitted sum. Whenever $D^{(t)} > 0$, we claim:

$$D^{(t)} \left\{ \frac{\widehat{D}^{(t)}}{D^{(t)}} - 1 - \log \frac{\widehat{D}^{(t)}}{D^{(t)}} \right\} \leq \varepsilon_t. \quad (\text{A.10})$$

For row i of $\mathbf{A}^{(t)}$, increasing only $\hat{\alpha}_i^{(t)}$ by h multiplies its fitted means by e^h and changes the joint objective by $\widehat{D}^{(t)}(e^h - 1) - D^{(t)}h$. Taking $h = \log(D^{(t)}/\widehat{D}^{(t)})$ decreases the joint objective by $\widehat{D}^{(t)} - D^{(t)} + D^{(t)} \log(D^{(t)}/\widehat{D}^{(t)})$. The perturbation is feasible, so the decrease is at most ε_t .

For row i of $\mathbf{B}^{(t)}$, increase $\hat{\beta}_i^{(t)}$ by $h(1 - 1/n)$, decrease every other $\hat{\beta}_j^{(t)}$ by h/n , and increase every $\hat{\gamma}_k^{(t)}$ by h/n . This preserves $\mathbf{1}_n^\top \widehat{\boldsymbol{\beta}}^{(t)} = 0$ and increases only that row's log-means by h . For column k , change only $\hat{\gamma}_k^{(t)}$. These three perturbations prove (A.10).

Since $\inf_{u>0, u \notin [1/2, 2]} (u - 1 - \log u) > 1/10$, (A.10) and $D^{(t)} \geq 10\varepsilon_0$ give $D^{(t)}/2 \leq \widehat{D}^{(t)} \leq 2D^{(t)}$. Lemma S1 then yields (A.9) and $e^{\hat{\alpha}_i^{(t)}} \asymp \sqrt{t/n}$, without requiring full-rank fitted positions or positive individual counts.

For existence, the joint objective is bounded below and finite at the true parameter, so choose a feasible minimizing sequence. Its tail has optimization error at most $\max\{1, \varepsilon_0\}$

and satisfies the preceding bounds. For each tail element $\tilde{\Theta}^{(t)}$, centering gives:

$$\begin{aligned}\tilde{\gamma}_k^{(t)} &= \frac{1}{n} \sum_i \log \theta_{ik}^B(\tilde{\Theta}^{(t)}), \quad \tilde{\beta}_i^{(t)} = \frac{1}{m} \sum_k \log \theta_{ik}^B(\tilde{\Theta}^{(t)}) - \frac{1}{nm} \sum_{j,k} \log \theta_{jk}^B(\tilde{\Theta}^{(t)}), \\ \max_i \left| \tilde{\alpha}_i^{(t)} - \frac{1}{2} \log(t/n) \right| + \max_i |\tilde{\beta}_i^{(t)}| &\leq C, \quad \max_k \left| \tilde{\gamma}_k^{(t)} - \log(t/m) \right| \leq C.\end{aligned}$$

At fixed t , these bounds place the tail in a compact feasible set. Continuity gives a subsequence attaining the infimum. $\square$

For parameter recovery, compare $\Theta = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{Z}, \mathbf{W})$ and $\Theta' = (\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}', \mathbf{Z}', \mathbf{W}')$. Both are centered and satisfy the radius constraints, and $\lambda_{\min}\{(\mathbf{Z}')^\top \mathbf{Z}'/n\} \geq \kappa$. Define the differences of the complete linear predictor matrices:

$$\begin{aligned}\mathbf{D}_A &= (\boldsymbol{\alpha} - \boldsymbol{\alpha}') \mathbf{1}_n^\top + \mathbf{1}_n (\boldsymbol{\alpha} - \boldsymbol{\alpha}')^\top + \mathbf{Z} \mathbf{Z}^\top - \mathbf{Z}' (\mathbf{Z}')^\top, \\ \mathbf{D}_B &= (\boldsymbol{\beta} - \boldsymbol{\beta}') \mathbf{1}_m^\top + \mathbf{1}_m (\boldsymbol{\gamma} - \boldsymbol{\gamma}')^\top + \mathbf{Z} \mathbf{W}^\top - \mathbf{Z}' (\mathbf{W}')^\top.\end{aligned}\tag{A.11}$$

The diagonal of $\mathbf{D}_A$ is used for algebraic completion. The joint objective includes only the entries of $\mathbf{A}$ with $i < \ell$ and all entries of $\mathbf{B}$, which we call observed entries. For any $\mathbf{U}, \mathbf{X} \in \mathbb{R}^{n \times d}$ with $\mathbf{X}$ of full column rank and smallest singular value $\sigma_d(\mathbf{X})$, [Tu et al. \(2016, Lemma 5.4\)](#) give:

$$\min_{\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}_d} \|\mathbf{U} - \mathbf{X} \mathbf{Q}\|_F^2 \leq \frac{\|\mathbf{U} \mathbf{U}^\top - \mathbf{X} \mathbf{X}^\top\|_F^2}{2(\sqrt{2} - 1) \sigma_d^2(\mathbf{X})}.\tag{A.12}$$

Lemma S4. *For $\mathbf{D}_A$ in (A.11), we have:*

$$\sum_i (\mathbf{D}_A)_{ii}^2 \leq \left\{ \frac{2}{n} + \frac{8R_z^2}{(\sqrt{2} - 1)n\kappa} \right\} \|\mathbf{D}_A\|_F^2.\tag{A.13}$$

Consequently, for all $n \geq n_0$ with a sufficiently large fixed n_0 :

$$\|\mathbf{D}_A\|_F^2 \leq 4 \sum_{i < \ell} (\mathbf{D}_A)_{ii}^2.\tag{A.14}$$

The two parameter tuples also satisfy $\text{dist}_{n,m}^2(\Theta, \Theta') \leq C\{\|\mathbf{D}_A\|_F^2/n^2 + \|\mathbf{D}_B\|_F^2/(nm)\}$, where C depends only on R_w, κ . Together with (A.14), this implies that, for $n \geq n_0$, equal means at observed entries identify the parameters up to a common orthogonal transformation.

Proof. Set $\mathbf{u} = \boldsymbol{\alpha} - \boldsymbol{\alpha}'$ and $\mathbf{G} = \mathbf{Z}\mathbf{Z}^\top - \mathbf{Z}'(\mathbf{Z}')^\top$. Centering gives $\mathbf{G}\mathbf{1}_n = \mathbf{0}$, so:

$$\|\mathbf{D}_A\|_F^2 = 2n\|\mathbf{u}\|_2^2 + 2(\mathbf{1}_n^\top \mathbf{u})^2 + \|\mathbf{G}\|_F^2. \quad (\text{A.15})$$

By (A.12), choose $\mathbf{Q}$ such that $\|\mathbf{Z} - \mathbf{Z}'\mathbf{Q}\|_F \leq \|\mathbf{G}\|_F / \sqrt{2(\sqrt{2} - 1)n\kappa}$. Applying $\|\mathbf{z}_i\|_2^2 - \|\mathbf{z}'_i\|_2^2 \leq 4R_z\|\mathbf{z}_i - \mathbf{Q}^\top \mathbf{z}'_i\|_2$ rowwise gives:

$$\left\{ \sum_i (\mathbf{D}_A)_{ii}^2 \right\}^{1/2} \leq 2\|\mathbf{u}\|_2 + \frac{4R_z\|\mathbf{G}\|_F}{\sqrt{2(\sqrt{2} - 1)n\kappa}} \leq \left\{ \frac{2}{n} + \frac{8R_z^2}{(\sqrt{2} - 1)n\kappa} \right\}^{1/2} (2n\|\mathbf{u}\|_2^2 + \|\mathbf{G}\|_F^2)^{1/2}.$$

Equation (A.15) proves (A.13). Choose a fixed n_0 such that the braced coefficient is at most 1/2. Subtracting the diagonal contribution from the complete norm proves (A.14).

Equations (A.15) and (A.12) allow the same $\mathbf{Q}$ to satisfy:

$$\frac{\|\boldsymbol{\alpha} - \boldsymbol{\alpha}'\|_2^2}{n} \leq \frac{\|\mathbf{D}_A\|_F^2}{2n^2}, \quad \frac{\|\mathbf{Z} - \mathbf{Z}'\mathbf{Q}\|_F^2}{n} \leq \frac{\|\mathbf{D}_A\|_F^2}{2(\sqrt{2} - 1)n^2\kappa}.$$

The three $\mathbf{B}$ -block components are mutually orthogonal, yielding:

$$\|\mathbf{D}_B\|_F^2 = m\|\boldsymbol{\beta} - \boldsymbol{\beta}'\|_2^2 + n\|\boldsymbol{\gamma} - \boldsymbol{\gamma}'\|_2^2 + \|\mathbf{Z}\mathbf{W}^\top - \mathbf{Z}'(\mathbf{W}')^\top\|_F^2.$$

The identity $\mathbf{Z}'\mathbf{Q}(\mathbf{W} - \mathbf{W}'\mathbf{Q})^\top = \mathbf{Z}\mathbf{W}^\top - \mathbf{Z}'(\mathbf{W}')^\top - (\mathbf{Z} - \mathbf{Z}'\mathbf{Q})\mathbf{W}^\top$, together with $\sigma_d^2(\mathbf{Z}'\mathbf{Q}) \geq n\kappa$ and $\|\mathbf{W}\|_{\text{op}}^2 \leq 4mR_w^2$, gives:

$$\frac{\|\mathbf{W} - \mathbf{W}'\mathbf{Q}\|_F^2}{m} \leq \frac{2\|\mathbf{Z}\mathbf{W}^\top - \mathbf{Z}'(\mathbf{W}')^\top\|_F^2}{nm\kappa} + \frac{8R_w^2}{n\kappa}\|\mathbf{Z} - \mathbf{Z}'\mathbf{Q}\|_F^2.$$

Summing gives dimension-independent coefficients $1/2 + 1/\{2(\sqrt{2} - 1)\kappa\} + 4R_w^2/\{(\sqrt{2} - 1)\kappa^2\}$ for the normalized $\mathbf{D}_A$ term and $1 + 2/\kappa$ for the normalized $\mathbf{D}_B$ term. $\square$

For small n , identifiability can fail. For example, take $n = 3, d = 1, \mathbf{W} = \mathbf{0}, \mathbf{Z} = (-1, 0, 1)^\top$, $\boldsymbol{\alpha} = \mathbf{0}$, and $\mathbf{Z}' = \mathbf{Z}/2, \boldsymbol{\alpha}' = (-3/8, 3/8, -3/8)^\top$. The two representations give identical off-diagonal $\mathbf{A}$ means. With the same $\boldsymbol{\beta}, \boldsymbol{\gamma}$, they also give identical $\mathbf{B}$ means, but $\mathbf{Z}$ and $\mathbf{Z}'$ are not related by an orthogonal transformation.

The next deterministic bound serves estimation and comparison of successive objectives.

Lemma S5. Fix nonnegative observations $\mathbf{A}, \mathbf{B}$, with $\mathbf{A}$ symmetric and zero-diagonal. For $u > 0$, suppose their row and column sums satisfy (A.7) with t replaced by u , and $\min\{u, nu/m\}$ exceeds a fixed constant. Let the feasible reference Θ' be centered, satisfy the radius constraints and $\lambda_{\min}\{(\mathbf{Z}')^\top \mathbf{Z}'/n\} \geq \kappa$, and have means between $cu/n, Cu/n$ for $\mathbf{A}$ and $cu/m, Cu/m$ for $\mathbf{B}$. Let $\hat{\Theta}$ be feasible with global optimization error at most $\varepsilon \leq \varepsilon_0$, where ε_0 is fixed. Define the mean matrices by $(\Theta_A(\Theta'))_{i\ell} = \theta_{i\ell}^A(\Theta')$ for $i \neq \ell$, $(\Theta_A(\Theta'))_{ii} = 0$, and $(\Theta_B(\Theta'))_{ik} = \theta_{ik}^B(\Theta')$, and set:

$$\mathbf{M} = \left[\frac{\sqrt{n}}{2} \{\mathbf{A} - \Theta_A(\Theta')\}, \sqrt{m} \{\mathbf{B} - \Theta_B(\Theta')\} \right]. \quad (\text{A.16})$$

For $n \geq n_0$:

$$\text{dist}_{n,m}^2(\hat{\Theta}, \Theta') \leq C \frac{\|\mathbf{M}\|_{\text{op}}^2}{nu^2} + C \frac{\varepsilon}{nu}. \quad (\text{A.17})$$

The normalized squared log-mean error obeys the same bound.

Proof. The feasible marginal perturbations in Lemma S3 apply to these observations. Together with Lemma S1, they give $\hat{\theta}_{i\ell}^A \asymp u/n$ and $\hat{\theta}_{ik}^B \asymp u/m$.

Objective difference. In (A.11), take $\Theta = \hat{\Theta}$ and use the current reference Θ' . At observed entries, $(\mathbf{D}_A)_{i\ell} = \log\{\hat{\theta}_{i\ell}^A/\theta_{i\ell}^A(\Theta')\}$, with the analogous definition for $\mathbf{D}_B$. At a given entry, let Y be the observed count, $\hat{\theta}, \theta'$ the fitted and reference means, and $D = \log(\hat{\theta}/\theta')$. Then:

$$\hat{\theta} - Y \log \hat{\theta} - (\theta' - Y \log \theta') = \theta'(e^D - 1 - D) - (Y - \theta')D.$$

Summing, using feasibility of the reference and the optimization error bound, yields:

$$\begin{aligned} & \sum_{i < \ell} \theta_{i\ell}^A(\Theta') \{e^{(\mathbf{D}_A)_{i\ell}} - 1 - (\mathbf{D}_A)_{i\ell}\} + \sum_{i,k} \theta_{ik}^B(\Theta') \{e^{(\mathbf{D}_B)_{ik}} - 1 - (\mathbf{D}_B)_{ik}\} \\ & \leq \sum_{i < \ell} \{A_{i\ell} - \theta_{i\ell}^A(\Theta')\} (\mathbf{D}_A)_{i\ell} + \sum_{i,k} \{B_{ik} - \theta_{ik}^B(\Theta')\} (\mathbf{D}_B)_{ik} + \varepsilon. \end{aligned} \quad (\text{A.18})$$

Set $\mathbf{F} = [\mathbf{D}_A/\sqrt{n}, \mathbf{D}_B/\sqrt{m}]$. The noise sums in (A.18) equal $\langle \mathbf{M}, \mathbf{F} \rangle = \text{tr}(\mathbf{M}^\top \mathbf{F})$: the zero-diagonal $\mathbf{A}$ block of $\mathbf{M}$ has factor $1/2$ against double counting. Taylor's formula gives:

$$\theta'(e^D - 1 - D) = D^2 \int_0^1 (1-v)(\theta')^{1-v} \hat{\theta}^v dv \geq \frac{1}{2} \min\{\theta', \hat{\theta}\} D^2.$$

The reference and fitted mean bounds, followed by (A.14), therefore yield:

$$cu\|\mathbf{F}\|_{\text{F}}^2 \leq cu \left\{ \frac{1}{n} \sum_{i < \ell} (\mathbf{D}_A)_{i\ell}^2 + \frac{1}{m} \|\mathbf{D}_B\|_{\text{F}}^2 \right\} \leq \langle \mathbf{M}, \mathbf{F} \rangle + \varepsilon,$$

where the first constant is decreased if needed.

Low-rank noise bound. Equation (A.11) shows that the column space of $\mathbf{F}$ is contained in the span of $\widehat{\mathbf{Z}}, \mathbf{Z}', \widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}', \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}', \mathbf{1}_n$. Thus $\text{rank}(\mathbf{F}) \leq 2d + 3$. For a singular value decomposition $\mathbf{F} = \sum_j \sigma_j \mathbf{a}_j \mathbf{b}_j^\top$:

$$|\langle \mathbf{M}, \mathbf{F} \rangle| \leq \sum_j \sigma_j |\mathbf{a}_j^\top \mathbf{M} \mathbf{b}_j| \leq \|\mathbf{M}\|_{\text{op}} \sum_j \sigma_j \leq \sqrt{2d+3} \|\mathbf{M}\|_{\text{op}} \|\mathbf{F}\|_{\text{F}}.$$

The last step uses Cauchy–Schwarz and fixed d .

Young’s inequality now gives:

$$cu\|\mathbf{F}\|_{\text{F}}^2 \leq C\|\mathbf{M}\|_{\text{op}}\|\mathbf{F}\|_{\text{F}} + \varepsilon \leq \frac{cu}{2}\|\mathbf{F}\|_{\text{F}}^2 + \frac{C}{u}\|\mathbf{M}\|_{\text{op}}^2 + \varepsilon.$$

Thus $\|\mathbf{F}\|_{\text{F}}^2 \leq C\|\mathbf{M}\|_{\text{op}}^2/u^2 + C\varepsilon/u$. Lemma S4 gives (A.17). The normalized log-mean error is at most $\|\mathbf{F}\|_{\text{F}}^2/n$. $\square$

For local optimization, write a centered perturbation as $\mathbf{h} = (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{U}, \mathbf{V})$, with rows $\mathbf{u}_i^\top, \mathbf{v}_k^\top$ in $\mathbf{U}, \mathbf{V}$, and $\mathbf{1}_n^\top \mathbf{b} = 0, \mathbf{1}_n^\top \mathbf{U} = \mathbf{0}^\top, \mathbf{1}_m^\top \mathbf{V} = \mathbf{0}^\top$. Use the norm:

$$\|\mathbf{h}\|_{n,m}^2 = \frac{\|\mathbf{a}\|_2^2 + \|\mathbf{b}\|_2^2 + \|\mathbf{U}\|_{\text{F}}^2}{n} + \frac{\|\mathbf{c}\|_2^2 + \|\mathbf{V}\|_{\text{F}}^2}{m}. \quad (\text{A.19})$$

The inner product $\langle \cdot, \cdot \rangle_{n,m}$ is the polarization of the norm $\|\cdot\|_{n,m}$ on perturbations, and Equation (11) minimizes the discrepancy over common orthogonal transformations.

Lemma S6. *The minimum in (11) is attained, and $\text{dist}_{n,m}$ satisfies the triangle inequality. For two tuples Θ, Θ' , let $\mathbf{Q}$ optimally align Θ' to Θ and set $\Theta_0 = \Theta' \mathbf{Q}$. For $\mathbf{h} = \Theta - \Theta_0$, we have $\|\mathbf{h}\|_{n,m} = \text{dist}_{n,m}(\Theta, \Theta_0)$ and:*

$$(\mathbf{Z}_0)^\top \mathbf{U}/n + (\mathbf{W}_0)^\top \mathbf{V}/m \quad \text{is symmetric.} \quad (\text{A.20})$$

Equivalently, $\mathbf{h}$ is orthogonal to every tangent direction $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{Z}_0\boldsymbol{\Omega}, \mathbf{W}_0\boldsymbol{\Omega})$ of common orthogonal transformations, where $\boldsymbol{\Omega}^\top = -\boldsymbol{\Omega}$.

Proof. Compactness of the orthogonal group and continuity give an optimal orthogonal matrix. For three tuples $\Theta_1, \Theta_2, \Theta_3$, let $\mathbf{Q}_{12}$ align Θ_2 to Θ_1 and $\mathbf{Q}_{23}$ align Θ_3 to Θ_2 . Invariance of (A.19) gives:

$$\begin{aligned} \text{dist}_{n,m}(\Theta_1, \Theta_3) &\leq \|\Theta_1 - \Theta_3 \mathbf{Q}_{23} \mathbf{Q}_{12}\|_{n,m} \\ &\leq \|\Theta_1 - \Theta_2 \mathbf{Q}_{12}\|_{n,m} + \|(\Theta_2 - \Theta_3 \mathbf{Q}_{23}) \mathbf{Q}_{12}\|_{n,m} \\ &= \text{dist}_{n,m}(\Theta_1, \Theta_2) + \text{dist}_{n,m}(\Theta_2, \Theta_3). \end{aligned}$$

For $\boldsymbol{\Omega}^\top = -\boldsymbol{\Omega}$, optimal alignment implies:

$$0 = \frac{d}{ds} \|\Theta - \Theta_0 \exp(s\boldsymbol{\Omega})\|_{n,m}^2 \Big|_{s=0} = -2 \text{tr} \left\{ \left(\frac{\mathbf{Z}_0^\top \mathbf{U}}{n} + \frac{\mathbf{W}_0^\top \mathbf{V}}{m} \right)^\top \boldsymbol{\Omega} \right\}.$$

Orthogonality to every skew-symmetric matrix is equivalent to symmetry, proving (A.20). $\square$

Lemma S7. Fix a centered reference Θ_0 with row norms bounded by $2R_z, 2R_w$ and $\lambda_{\min}(\mathbf{Z}_0^\top \mathbf{Z}_0/n) \geq \kappa_1 > 0$, where κ_1 is fixed. For a centered perturbation $\mathbf{h} = (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{U}, \mathbf{V})$, define the first-order log-mean changes:

$$j_{A,i\ell}(\mathbf{h}) = a_i + a_\ell + \mathbf{u}_i^\top \mathbf{z}_{0,\ell} + \mathbf{z}_{0,i}^\top \mathbf{u}_\ell, \quad j_{B,ik}(\mathbf{h}) = b_i + c_k + \mathbf{u}_i^\top \mathbf{w}_{0,k} + \mathbf{z}_{0,i}^\top \mathbf{v}_k. \quad (\text{A.21})$$

Every such $\mathbf{h}$ satisfies:

$$\frac{1}{n^2} \sum_{i < \ell} j_{A,i\ell}(\mathbf{h})^2 + \frac{1}{nm} \sum_{i,k} j_{B,ik}(\mathbf{h})^2 \leq C \|\mathbf{h}\|_{n,m}^2. \quad (\text{A.22})$$

If $\mathbf{h}$ also satisfies (A.20), then for $n \geq n_0$ with a sufficiently large fixed n_0 :

$$\frac{1}{n^2} \sum_{i < \ell} j_{A,i\ell}(\mathbf{h})^2 + \frac{1}{nm} \sum_{i,k} j_{B,ik}(\mathbf{h})^2 \geq c \|\mathbf{h}\|_{n,m}^2. \quad (\text{A.23})$$

Proof. Upper bound. Applying $(x_1 + \dots + x_4)^2 \leq 4 \sum_j x_j^2$ and counting each worker at most $n - 1$ times gives:

$$\begin{aligned} \sum_{i < \ell} j_{A,il}(\mathbf{h})^2 &\leq 4(n-1) \{ \|\mathbf{a}\|_2^2 + 4R_z^2 \|\mathbf{U}\|_F^2 \}, \\ \sum_{i,k} j_{B,ik}(\mathbf{h})^2 &\leq 4 \{ m \|\mathbf{b}\|_2^2 + n \|\mathbf{c}\|_2^2 + 4mR_w^2 \|\mathbf{U}\|_F^2 + 4nR_z^2 \|\mathbf{V}\|_F^2 \}. \end{aligned}$$

Dividing by n^2, nm and adding proves (A.22).

First-order limit of the distance. Consider $\Theta_0 + \varepsilon \mathbf{h}$ as $\varepsilon \rightarrow 0$. We first establish:

$$\lim_{\varepsilon \rightarrow 0} \frac{\text{dist}_{n,m}^2(\Theta_0 + \varepsilon \mathbf{h}, \Theta_0)}{\varepsilon^2} = \min_{\mathbf{\Omega}^\top = -\mathbf{\Omega}} \|\mathbf{h} - (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{Z}_0 \mathbf{\Omega}, \mathbf{W}_0 \mathbf{\Omega})\|_{n,m}^2. \quad (\text{A.24})$$

The choice $\mathbf{Q} = \mathbf{I}_d$ bounds the minimum distance by $|\varepsilon| \|\mathbf{h}\|_{n,m}$. For optimal $\mathbf{Q}_\varepsilon$, the triangle inequality gives:

$$\frac{\|\mathbf{Z}_0(\mathbf{I}_d - \mathbf{Q}_\varepsilon)\|_F}{\sqrt{n}} \leq \frac{|\varepsilon| \|\mathbf{U}\|_F}{\sqrt{n}} + \text{dist}_{n,m}(\Theta_0 + \varepsilon \mathbf{h}, \Theta_0) \leq 2|\varepsilon| \|\mathbf{h}\|_{n,m}.$$

The spectral bound gives $\|\mathbf{Q}_\varepsilon - \mathbf{I}_d\|_F \leq 2|\varepsilon| \|\mathbf{h}\|_{n,m} / \sqrt{\kappa_1}$. Hence $\mathbf{\Omega}_\varepsilon = (\mathbf{Q}_\varepsilon - \mathbf{I}_d)/\varepsilon$ is bounded, and $\mathbf{\Omega}_\varepsilon + \mathbf{\Omega}_\varepsilon^\top = -\varepsilon \mathbf{\Omega}_\varepsilon^\top \mathbf{\Omega}_\varepsilon$. Every subsequential limit is skew-symmetric, so substitution into the distance gives:

$$\liminf_{\varepsilon \rightarrow 0} \frac{\text{dist}_{n,m}^2(\Theta_0 + \varepsilon \mathbf{h}, \Theta_0)}{\varepsilon^2} \geq \min_{\mathbf{\Omega}^\top = -\mathbf{\Omega}} \|\mathbf{h} - (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{Z}_0 \mathbf{\Omega}, \mathbf{W}_0 \mathbf{\Omega})\|_{n,m}^2.$$

Conversely, each skew-symmetric $\mathbf{\Omega}$ gives the rotation $\exp(\varepsilon \mathbf{\Omega}) = \mathbf{I}_d + \varepsilon \mathbf{\Omega} + O(\varepsilon^2)$ and the matching upper bound. Full column rank of $\mathbf{Z}_0$ makes the quadratic coercive in $\mathbf{\Omega}$, so the minimum exists. For $\mathbf{h}$ satisfying (A.20), $\mathbf{\Omega} = \mathbf{0}$ attains the minimum $\|\mathbf{h}\|_{n,m}^2$.

Lower bound. Apply Lemma S4 and (A.14) to $\Theta_0 + \varepsilon \mathbf{h}$ and Θ_0 . For any fixed $\mathbf{h}$ and sufficiently small $|\varepsilon|$, the row bounds can be enlarged to $3R_z, 3R_w$, with constants independent of $\mathbf{h}, n, m$.

The observed log-mean differences are:

$$(\mathbf{D}_A)_{il} = \varepsilon j_{A,il}(\mathbf{h}) + \varepsilon^2 \mathbf{u}_i^\top \mathbf{u}_\ell, \quad (\mathbf{D}_B)_{ik} = \varepsilon j_{B,ik}(\mathbf{h}) + \varepsilon^2 \mathbf{u}_i^\top \mathbf{v}_k.$$

For $\mathbf{h}$ satisfying (A.20), division by ε^2 and (A.24) give:

$$\begin{aligned}\|\mathbf{h}\|_{n,m}^2 &= \lim_{\varepsilon \rightarrow 0} \frac{\text{dist}_{n,m}^2(\Theta_0 + \varepsilon \mathbf{h}, \Theta_0)}{\varepsilon^2} \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left\{ \frac{1}{n^2} \sum_{i < \ell} \left(\frac{(\mathbf{D}_A)_{i\ell}}{\varepsilon} \right)^2 + \frac{1}{nm} \sum_{i,k} \left(\frac{(\mathbf{D}_B)_{ik}}{\varepsilon} \right)^2 \right\} \\ &= C \left\{ \frac{1}{n^2} \sum_{i < \ell} j_{A,i\ell}(\mathbf{h})^2 + \frac{1}{nm} \sum_{i,k} j_{B,ik}(\mathbf{h})^2 \right\}.\end{aligned}$$

This proves (A.23). $\square$

As in (A.16), the diagonal of $\Theta_A(\Theta)$ is set to zero. At checkpoint t , define residuals $\mathbf{E}_A^{(t)}(\Theta) = \Theta_A(\Theta) - \mathbf{A}^{(t)}$ and $\mathbf{E}_B^{(t)}(\Theta) = \Theta_B(\Theta) - \mathbf{B}^{(t)}$. Differentiating the joint objective block by block gives:

$$\begin{aligned}(\nabla_{\alpha} \mathcal{L}^{(t)})_i &= \sum_{\ell \neq i} (\mathbf{E}_A^{(t)}(\Theta))_{i\ell}, & \nabla_{\beta} \mathcal{L}^{(t)} &= \mathbf{E}_B^{(t)}(\Theta) \mathbf{1}_m, & \nabla_{\gamma} \mathcal{L}^{(t)} &= \{\mathbf{E}_B^{(t)}(\Theta)\}^{\top} \mathbf{1}_n, \\ \nabla_{\mathbf{Z}} \mathcal{L}^{(t)} &= \mathbf{E}_A^{(t)}(\Theta) \mathbf{Z} + \mathbf{E}_B^{(t)}(\Theta) \mathbf{W}, & \nabla_{\mathbf{W}} \mathcal{L}^{(t)} &= \{\mathbf{E}_B^{(t)}(\Theta)\}^{\top} \mathbf{Z}.\end{aligned}$$

Each $\mathbf{A}$ edge appears once in the joint objective, and both endpoint derivatives enter the corresponding rows. On the centered parameter space with (A.19), $\mathcal{L}^{(t)}/(nt)$ has gradient:

$$\nabla_{n,m} \{\mathcal{L}^{(t)}/(nt)\} = \mathbf{P} \left(\frac{\nabla_{\alpha} \mathcal{L}^{(t)}}{t}, \frac{\nabla_{\beta} \mathcal{L}^{(t)}}{t}, \frac{m \nabla_{\gamma} \mathcal{L}^{(t)}}{nt}, \frac{\nabla_{\mathbf{Z}} \mathcal{L}^{(t)}}{t}, \frac{m \nabla_{\mathbf{W}} \mathcal{L}^{(t)}}{nt} \right). \quad (\text{A.25})$$

The block weights make $\mathbf{P}$ orthogonal and give, for centered $\mathbf{h}$:

$$\left\langle \nabla_{n,m} \{\mathcal{L}^{(t)}/(nt)\}, \mathbf{h} \right\rangle_{n,m} = D\mathcal{L}^{(t)}(\Theta)[\mathbf{h}]/(nt).$$

Thus (17) is the gradient update in this weighted norm.

Let $\dot{\eta}_{A,i\ell}, \dot{\eta}_{B,ik}$ be the first-order changes in (A.21) at the current $\mathbf{Z}, \mathbf{W}$. Along $\Theta + s\mathbf{h}$, the log-means have second derivatives $2\mathbf{u}_i^{\top} \mathbf{u}_{\ell}$ and $2\mathbf{u}_i^{\top} \mathbf{v}_k$. Differentiating once more yields:

$$\begin{aligned}D^2 \{\mathcal{L}^{(t)}/(nt)\}(\Theta)[\mathbf{h}, \mathbf{h}] &= \frac{1}{nt} \left\{ \sum_{i < \ell} \theta_{i\ell}^A(\Theta) \dot{\eta}_{A,i\ell}^2 + \sum_{i,k} \theta_{ik}^B(\Theta) \dot{\eta}_{B,ik}^2 \right\} \\ &\quad + \frac{1}{nt} \text{tr}(\mathbf{U}^{\top} \mathbf{E}_A^{(t)}(\Theta) \mathbf{U}) + \frac{2}{nt} \text{tr}(\mathbf{U}^{\top} \mathbf{E}_B^{(t)}(\Theta) \mathbf{V}).\end{aligned} \quad (\text{A.26})$$

The mean-weighted terms and residual terms jointly determine local curvature.

Lemma S8. Suppose $n \geq n_0$. Let Θ_0 be centered with fixed latent row bounds and $\lambda_{\min}(\mathbf{Z}_0^\top \mathbf{Z}_0/n) \geq \kappa_1 > 0$. Suppose its means satisfy $ct/n \leq \theta_{i\ell}^A(\Theta_0) \leq Ct/n$ and $ct/m \leq \theta_{ik}^B(\Theta_0) \leq Ct/m$. There are fixed $b_0, r_0, \mu, L > 0$ such that the following residual condition suffices for the local bounds below:

$$\frac{\|\Theta_A(\Theta_0) - \mathbf{A}^{(t)}\|_{\text{op}}}{t} + \sqrt{\frac{m}{n}} \frac{\|\Theta_B(\Theta_0) - \mathbf{B}^{(t)}\|_{\text{op}}}{t} \leq b_0. \quad (\text{A.27})$$

For every centered Θ with $\|\Theta - \Theta_0\|_{n,m} \leq r_0/\sqrt{n+m}$ and every centered perturbation $\mathbf{h}$:

$$|D^2\{\mathcal{L}^{(t)}/(nt)\}(\Theta)[\mathbf{h}, \mathbf{h}]| \leq L\|\mathbf{h}\|_{n,m}^2. \quad (\text{A.28})$$

If $\mathbf{h}$ also satisfies (A.20) at Θ_0 , then:

$$D^2\{\mathcal{L}^{(t)}/(nt)\}(\Theta)[\mathbf{h}, \mathbf{h}] \geq \mu\|\mathbf{h}\|_{n,m}^2. \quad (\text{A.29})$$

All constants are independent of n, m, t, T .

Proof. Write $\mathbf{h}_0 = \Theta - \Theta_0$ within this proof. The norm definition bounds each worker scalar or latent row change by $\sqrt{n}\|\mathbf{h}_0\|_{n,m}$, and each task-type scalar or row change by $\sqrt{m}\|\mathbf{h}_0\|_{n,m}$. All these changes are therefore at most r_0 . Expanding the inner products around the bounded reference positions gives:

$$\max_{i < \ell} |\log\{\theta_{i\ell}^A(\Theta)/\theta_{i\ell}^A(\Theta_0)\}| \vee \max_{i,k} |\log\{\theta_{ik}^B(\Theta)/\theta_{ik}^B(\Theta_0)\}| \leq C\sqrt{n+m}\|\mathbf{h}_0\|_{n,m} \leq Cr_0. \quad (\text{A.30})$$

For example, the $\mathbf{A}$ inner-product change is $(\mathbf{z}_i - \mathbf{z}_{0,i})^\top \mathbf{z}_\ell + \mathbf{z}_{0,i}^\top (\mathbf{z}_\ell - \mathbf{z}_{0,\ell})$, where the current row $\mathbf{z}_\ell$ is bounded by the reference radius plus r_0 . Thus the local means remain within fixed multiples of $t/n, t/m$.

Mean-weighted terms. The first line of (A.26) is comparable to $n^{-2} \sum_{i < \ell} \dot{\eta}_{A,i\ell}^2 + (nm)^{-1} \sum_{i,k} \dot{\eta}_{B,ik}^2$. The derivative changes from the reference obey:

$$\begin{aligned} \frac{\|\mathbf{U}(\mathbf{Z} - \mathbf{Z}_0)^\top + (\mathbf{Z} - \mathbf{Z}_0)\mathbf{U}^\top\|_{\text{F}}}{n} &\leq 2 \frac{\|\mathbf{U}\|_{\text{F}}}{\sqrt{n}} \frac{\|\mathbf{Z} - \mathbf{Z}_0\|_{\text{F}}}{\sqrt{n}}, \\ \frac{\|\mathbf{U}(\mathbf{W} - \mathbf{W}_0)^\top + (\mathbf{Z} - \mathbf{Z}_0)\mathbf{V}^\top\|_{\text{F}}}{\sqrt{nm}} &\leq \frac{\|\mathbf{U}\|_{\text{F}}}{\sqrt{n}} \frac{\|\mathbf{W} - \mathbf{W}_0\|_{\text{F}}}{\sqrt{m}} + \frac{\|\mathbf{Z} - \mathbf{Z}_0\|_{\text{F}}}{\sqrt{n}} \frac{\|\mathbf{V}\|_{\text{F}}}{\sqrt{m}}. \end{aligned}$$

Their sum is at most $C\|\mathbf{h}_0\|_{n,m}\|\mathbf{h}\|_{n,m}$. Lemma S7 and the reverse triangle inequality therefore bound the mean-weighted terms below by $c_F\|\mathbf{h}\|_{n,m}^2$ for $\mathbf{h}$ satisfying (A.20) when r_0 is small. The upper bound (A.22), with enlarged fixed radii, gives $C_F\|\mathbf{h}\|_{n,m}^2$ for every $\mathbf{h}$.

Residual terms. By $|e^x - 1| \leq e^{|x|}|x|$ and (A.30), entrywise mean changes are at most $C(t/n)\sqrt{n+m}\|\mathbf{h}_0\|_{n,m}$ in $\mathbf{A}$ and $C(t/m)\sqrt{n+m}\|\mathbf{h}_0\|_{n,m}$ in $\mathbf{B}$. The operator norm of a symmetric matrix is at most its maximum absolute row sum, and the operator norm is always at most the Frobenius norm. Consequently:

$$\frac{\|\Theta_A(\Theta) - \Theta_A(\Theta_0)\|_{\text{op}}}{t} + \sqrt{\frac{m}{n}} \frac{\|\Theta_B(\Theta) - \Theta_B(\Theta_0)\|_{\text{op}}}{t} \leq C\sqrt{n+m}\|\mathbf{h}_0\|_{n,m} \leq Cr_0. \quad (\text{A.31})$$

Adding (A.27) bounds the same standardized residual sum at the current parameter by $b_0 + Cr_0$. The operator-norm definition gives:

$$\begin{aligned} \frac{|\text{tr}(\mathbf{U}^\top \mathbf{E}_A^{(t)}(\Theta) \mathbf{U})|}{nt} &\leq \frac{\|\mathbf{E}_A^{(t)}(\Theta)\|_{\text{op}}}{t} \frac{\|\mathbf{U}\|_{\text{F}}^2}{n}, \\ \frac{2|\text{tr}(\mathbf{U}^\top \mathbf{E}_B^{(t)}(\Theta) \mathbf{V})|}{nt} &\leq 2\sqrt{\frac{m}{n}} \frac{\|\mathbf{E}_B^{(t)}(\Theta)\|_{\text{op}}}{t} \frac{\|\mathbf{U}\|_{\text{F}}}{\sqrt{n}} \frac{\|\mathbf{V}\|_{\text{F}}}{\sqrt{m}}. \end{aligned}$$

Using $2ab \leq a^2 + b^2$, the absolute residual contribution is at most $C(b_0 + Cr_0)\|\mathbf{h}\|_{n,m}^2$. Choose b_0 , then r_0 , sufficiently small that the residual contribution is at most $c_F\|\mathbf{h}\|_{n,m}^2/2$. The lower bound under (A.20) holds with $\mu = c_F/2$, and the full-space absolute upper bound holds with a fixed $L \geq C_F + c_F/2$. The constants depend only on the fixed mean bounds, radii, and κ_1 . $\square$

Self-adjointness and (A.28) bound the Hessian operator norm by L . Integration along segments gives an L -Lipschitz gradient on the local ball. Equation (A.29) applies to directions satisfying (A.20) at the reference.

A.3 Proof of Theorem 1

Proof. We first derive the order of the cumulative number S_t of work records. Each work record has total contribution at least one and, by Assumption 1, at most K . Taking

conditional expectations given $\mathcal{F}_{r-1}$ and summing along each path through $r \leq S_t$ gives:

$$S_t \leq \sum_{i < \ell} \theta_{\star, i\ell}^{A,(t)} + \sum_{i,k} \theta_{\star, ik}^{B,(t)} \leq K S_t.$$

The total $\mathbf{A}$ mean is half the sum of its row means, while the total $\mathbf{B}$ mean is the sum of its row means. By (12), the middle quantity lies between $(3c_0/2)nt$ and $(3C_0/2)nt$. Hence $S_t \asymp nt$, pathwise, including random S_t .

Work on the time-uniform event of Lemma S2. At any checkpoint satisfying the information threshold in Theorem 1, (A.7) holds and the smallest marginal sum is sufficiently large. Lemma S3 gives a finite exact minimizer. Apply Lemma S5 with $u = t$ and $\Theta' = \Theta_{\star}^{(t)}$. The matrix in (A.16) is then $\mathbf{M}_{S_t}$ from (A.5). Write $x_t = \log\{(2n+m)/\delta\} + \log\log(e^e + t)$ within this proof. Since $t \geq C(1+m/n)x_t$, $x_t \geq 1$, and ε_0 is fixed, (A.6) and (A.17) give:

$$\begin{aligned} \text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_{\star}^{(t)}) &\leq C \frac{\|\mathbf{M}_{S_t}\|_{\text{op}}^2}{nt^2} + C \frac{\varepsilon_0}{nt} \\ &\leq C \frac{(n+m)x_t}{nt} \left\{ 1 + \frac{x_t}{t} + \frac{\varepsilon_0}{(n+m)x_t} \right\} \leq C \frac{(n+m)x_t}{nt}. \end{aligned}$$

The same deterministic bound controls (15). This proves (14) on the common concentration event, without further failure probability. The matrix bound is uniform over low-rank directions and therefore allows history-dependent true parameters.

For the result over $\lceil aT \rceil \leq t \leq T$, use (A.8) and the corresponding marginal event. Write $x = \log\{(2n+m)/\delta\}$. For $t \geq \lceil aT \rceil$, Lemma S5 gives:

$$\text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_{\star}^{(t)}) \leq C_a \frac{n+m}{nT} x + C_a \frac{n+m}{nT^2} x^2 + C_a \frac{\varepsilon_0}{nT}.$$

The condition $T \geq C_a(1+m/n)x$ ensures the marginal bounds and lets the first term absorb the other two. Taking maxima on the same event proves the claimed bounds over these checkpoints. The log-mean bound follows from the same deterministic inequality. $\square$

Larger cumulative activity. Replace Assumption 3 by worker row sums of order $(n+m)t$

and task-type column sums of order $n(n+m)t/m$:

$$\begin{aligned} c_0(n+m)t &\leq \sum_{\ell \neq i} \theta_{\star, i\ell}^{A,(t)} \leq C_0(n+m)t, & c_0(n+m)t &\leq \sum_k \theta_{\star, ik}^{B,(t)} \leq C_0(n+m)t, \\ c_0 n(n+m)t/m &\leq \sum_i \theta_{\star, ik}^{B,(t)} \leq C_0 n(n+m)t/m. \end{aligned} \tag{A.32}$$

For $n \geq n_0$, Lemma S1 gives:

$$c \frac{(n+m)t}{n} \leq \theta_{\star, i\ell}^{A,(t)} \leq C \frac{(n+m)t}{n}, \quad c \frac{(n+m)t}{m} \leq \theta_{\star, ik}^{B,(t)} \leq C \frac{(n+m)t}{m}.$$

Keep the feasible parameter set, model (1), Assumption 1, and the optimization-error bound ε_0 unchanged. Under Assumptions 1, 2, and these larger activity scales, $S_t \asymp n(n+m)t$. With probability at least $1 - \delta$, every estimate satisfying (10), at every checkpoint with $nt \geq C\{\log((2n+m)/\delta) + \log \log(e^e + t)\}$, obeys:

$$\text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_{\star}^{(t)}) \leq \frac{C}{nt} \left\{ \log \frac{2n+m}{\delta} + \log \log(e^e + t) \right\}. \tag{A.33}$$

For fixed $a \in (0, 1)$ and $nT \geq C_a \log\{(2n+m)/\delta\}$, the maximum normalized squared parameter error over $\lceil aT \rceil \leq t \leq T$ is at most $C_a \log\{(2n+m)/\delta\}/(nT)$ on one event of probability at least $1 - \delta$.

To verify these claims, summing (A.32) gives $S_t \asymp n(n+m)t$. Write $\nu = n+m$ and $\ell_t = \log\{(2n+m)/\delta\} + \log \log(e^e + t)$ within this argument. The single-increment norm in (A.5) remains at most $C\sqrt{\nu}$, while the variance calculation in Lemma S2 replaces t by νt . The left and right variance bounds become $C\nu^2 t$ and $C\nu t$, hence the total bound is $C\nu^2 t$. Partitioning checkpoints into $2^j \leq t < 2^{j+1}$ and applying Freedman gives $\|\mathbf{M}_{S_t}\|_{\text{op}}^2 \leq C\{\nu^2 t \ell_t + \nu \ell_t^2\}$, with failure probabilities allocated by checkpoint index.

Worker margins now have scale νt , and task-type margins have scale $\nu t/m$. Their minimum scale $\nu t \min\{1, n/m\}$ lies between nt and $2nt$: use $n \leq \nu \leq 2n$ when $m \leq n$, and $m \leq \nu \leq 2m$ when $m \geq n$. Thus $nt \geq C\ell_t$ ensures marginal concentration and finite minimizers.

Applying Lemma S5 with $u = \nu t$ and $\Theta' = \Theta_{\star}^{(t)}$ yields:

$$\text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_{\star}^{(t)}) \leq C \frac{\|\mathbf{M}_{S_t}\|_{\text{op}}^2}{\nu \nu^2 t^2} + C \frac{\varepsilon_0}{\nu t} \leq C \frac{\ell_t}{nt} + C \frac{\ell_t^2}{\nu t^2} + C \frac{\varepsilon_0}{\nu t}.$$

The ratio of the second term in the last bound to the first is $\ell_t/(\nu t) \leq n/(C\nu)$. The fixed optimization error also lets the first term absorb the third. This proves (A.33). For fixed T , variance budget $C\nu^2 T$ gives the matrix bound $C\{\nu^2 T \log((2n+m)/\delta) + \nu \log^2((2n+m)/\delta)\}$. The marginal variance budgets are $C\nu T, Cn\nu T/m$. Apply the same deterministic calculation over $t \geq aT$. Finally, with $x = \log\{(2n+m)/\delta\} \geq 1$, the function $\log \log(e^e + t)/t$ is bounded for $t \geq 1$, so $\sup_{t \geq 1} \ell_t/t \leq Cx$. Thus $n \geq Cx$ ensures the corresponding information threshold for every $t \geq 1$, and the supremum of the bound in (A.33) is at most Cx/n . The maximum error over $\lceil aT \rceil \leq t \leq T$ has scale $x/(nT)$, corresponding to cumulative activity of order $n(n+m)T$. An optimization analysis under these larger activity scales requires an analogue of Assumption 4 and parameter-block step sizes.

A.4 Proof of Theorem 2

Proof. Work on the time-uniform event of Lemma S2. We establish local geometry at exact minimizers, bound the displacement between successive minimizers, and then prove finite-step tracking. Write $x = \log\{(2n+m)/\delta\}$ and $\ell_t = x + \log \log(e^e + t)$.

Step 1: Initial information and empirical local geometry. For any sufficiently small fixed $\rho > 0$, increasing the dimension-independent constant in the initial threshold, that is, the lower bound on t_0 in Theorem 2, ensures that, for every $t_0 \leq t \leq T$, all feasible estimates with global optimization error at most ε_0 satisfy:

$$\text{dist}_{n,m}(\hat{\Theta}^{(t)}, \Theta_\star^{(t)}) \leq \frac{\rho}{\sqrt{n+m}}. \quad (\text{A.34})$$

If $\rho \leq \min\{R_z, R_w\}/4$, Assumption 2 then bounds their latent row norms by $5R_z/4, 5R_w/4$, leaving margins of at least $3R_z/4, 3R_w/4$ inside the radius constraints. Their normalized worker Gram matrices have minimum eigenvalue at least $\kappa/4$. For every exact minimizer, the left side of (A.27) can also be made at most $b_0/4$.

The function ℓ_t/t decreases for $t \geq 1$, since $t\ell'_t = t/\{(e^e + t) \log(e^e + t)\} < 1/e < \ell_t$. For

$b_* > 0$, set $\nu = n + m$, $\tilde{\nu} = \nu^2/n$, and $t_* = b_*\tilde{\nu}x$. Here $\nu \leq \tilde{\nu} \leq \nu^2$ and $x \geq \log \nu \geq 1$, so $e^e + t_* \leq (e^e + b_*)e^{3x}$. Consequently:

$$\sup_{t \geq t_*} \frac{p\ell_t}{t} = \frac{\ell_{t_*}}{b_*x} \leq \frac{2 + \log\{\log(e^e + b_*) + 3\}}{b_*} \rightarrow 0 \quad (b_* \rightarrow \infty).$$

Choosing a fixed large b_* controls $\tilde{\nu}\ell_t/t$ uniformly over $t \geq t_0 \geq t_*$.

This choice also ensures the information threshold of Theorem 1. Finite exact minimizers therefore exist, and all feasible estimates with the fixed optimization error bound satisfy $\text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_\star^{(t)}) \leq C\nu\ell_t/(nt) \leq \rho^2/\nu$, proving (A.34). After optimal common alignment, each individual worker or task-type parameter row has error at most $\sqrt{\nu} \text{dist}_{n,m}(\hat{\Theta}^{(t)}, \Theta_\star^{(t)}) \leq \rho$. The true row bounds then prove the interior margins. Taking the infimum over unit vectors in $\|\mathbf{X}\mathbf{v}\|_2 \geq \|\mathbf{Y}\mathbf{v}\|_2 - \|(\mathbf{X} - \mathbf{Y})\mathbf{v}\|_2$ gives the singular-value perturbation bound:

$$\frac{\sigma_d(\widehat{\mathbf{Z}}^{(t)})}{\sqrt{n}} \geq \sqrt{\kappa} - \frac{\|\widehat{\mathbf{Z}}^{(t)} - \mathbf{Z}_\star^{(t)}\mathbf{Q}\|_{\text{op}}}{\sqrt{n}} \geq \sqrt{\kappa} - \text{dist}_{n,m}(\hat{\Theta}^{(t)}, \Theta_\star^{(t)}).$$

If also $\rho \leq \sqrt{\kappa}/2$, the right side is at least $\sqrt{\kappa}/2$. Squaring proves the worker Gram bound.

To bound empirical residuals, the first and second blocks of (A.5) are the true cumulative noise scaled by $\sqrt{n}/2$ and $\sqrt{m}$. Taking a block does not increase the operator norm, so:

$$\frac{\|\mathbf{A}^{(t)} - \Theta_A(\Theta_\star^{(t)})\|_{\text{op}}}{t} + \sqrt{\frac{m}{n}} \frac{\|\mathbf{B}^{(t)} - \Theta_B(\Theta_\star^{(t)})\|_{\text{op}}}{t} \leq \frac{3\|\mathbf{M}_{S_t}\|_{\text{op}}}{\sqrt{n}t} \leq C\sqrt{\frac{\nu\ell_t}{nt}}. \quad (\text{A.35})$$

The last inequality uses (A.6) and $t \geq C\ell_t$ to absorb the linear tail term. Common alignment preserves means. Expanding at each entry and using the radius constraints gives:

$$\max_{i < \ell} \left| \log \frac{\theta_{i\ell}^A(\hat{\Theta}_\star^{(t)})}{\theta_{\star,i\ell}^{A,(t)}} \right| \vee \max_{i,k} \left| \log \frac{\theta_{ik}^B(\hat{\Theta}_\star^{(t)})}{\theta_{\star,ik}^{B,(t)}} \right| \leq C\sqrt{\nu} \text{dist}_{n,m}(\hat{\Theta}_\star^{(t)}, \Theta_\star^{(t)}).$$

The entrywise, row-sum, and Frobenius calculations used in (A.31) now give:

$$\frac{\|\Theta_A(\hat{\Theta}_\star^{(t)}) - \Theta_A(\Theta_\star^{(t)})\|_{\text{op}}}{t} + \sqrt{\frac{m}{n}} \frac{\|\Theta_B(\hat{\Theta}_\star^{(t)}) - \Theta_B(\Theta_\star^{(t)})\|_{\text{op}}}{t} \leq C\sqrt{\nu} \text{dist}_{n,m}(\hat{\Theta}_\star^{(t)}, \Theta_\star^{(t)}).$$

Adding (A.35) bounds the empirical residual by $C\nu\sqrt{\ell_t/(nt)}$. Its square is $Cp\ell_t/t$, which the preceding threshold calculation makes arbitrarily small. In particular, the residual bound can be made at most $b_0/4$.

Step 2: Stationarity, uniqueness, and initialization error. Increasing the fixed threshold constant if necessary, we show that every exact minimizer for $t \geq t_0$ is unique up to a common orthogonal transformation and satisfies:

$$\nabla_{n,m}\{\mathcal{L}^{(t)}/(nt)\}(\hat{\Theta}_\star^{(t)}) = \mathbf{0}. \quad (\text{A.36})$$

An initial estimate with optimization error at most ε_0 also satisfies:

$$\text{dist}_{n,m}^2(\hat{\Theta}^{(t_0)}, \hat{\Theta}_\star^{(t_0)}) \leq C\varepsilon_0/(nt_0). \quad (\text{A.37})$$

Step 1 gives positive margins inside the radius constraints at every exact minimizer. Thus, for any centered $\mathbf{h}$, the perturbation $\hat{\Theta}_\star^{(t)} + s\mathbf{h}$ is feasible for sufficiently small positive and negative s . The joint objective along this line is minimized at $s = 0$, so its derivative in s vanishes there. Since the derivative vanishes for every centered $\mathbf{h}$, (A.36) follows.

Two exact minimizers are within $2\rho/\sqrt{n+m}$ by (A.34). Choose $\rho < r_0/4$, optimally align them, and write $\mathbf{h} = \hat{\Theta}_{\star,2}^{(t)} - \hat{\Theta}_{\star,1}^{(t)}$. Lemma S6 gives (A.20) for $\mathbf{h}$. Step 1 verifies Lemma S8 along the connecting segment. Stationarity and equal minimum values give:

$$0 = \frac{\mathcal{L}^{(t)}(\hat{\Theta}_{\star,2}^{(t)}) - \mathcal{L}^{(t)}(\hat{\Theta}_{\star,1}^{(t)})}{nt} \geq \frac{\mu}{2}\|\mathbf{h}\|_{n,m}^2.$$

Thus $\mathbf{h} = \mathbf{0}$, proving uniqueness up to a common orthogonal transformation.

The initial estimate and its exact minimizer both satisfy (A.34). Their optimally aligned difference lies in the same local ball and satisfies (A.20) at the exact minimizer. Repeating the integral lower bound proves (A.37):

$$\frac{\mu}{2} \text{dist}_{n,m}^2(\hat{\Theta}^{(t_0)}, \hat{\Theta}_\star^{(t_0)}) \leq \frac{\mathcal{L}^{(t_0)}(\hat{\Theta}^{(t_0)}) - \mathcal{L}^{(t_0)}(\hat{\Theta}_\star^{(t_0)})}{nt_0} \leq \frac{\varepsilon_0}{nt_0}.$$

Step 3: Objective increments and a common local region. At the same parameter tuple Θ , subtracting successive objectives gives:

$$\mathcal{L}^{(t)}(\Theta) - \mathcal{L}^{(t-1)}(\Theta) = -\sum_{i < \ell} \Delta A_{i\ell}^{(t)} \log \theta_{i\ell}^A(\Theta) - \sum_{i,k} \Delta B_{ik}^{(t)} \log \theta_{ik}^B(\Theta). \quad (\text{A.38})$$

For a parameter tuple with fixed latent row bounds, the normalized gradient change satisfies:

$$\begin{aligned} & \left\| \nabla_{n,m} \{ \mathcal{L}^{(t)} / (nt) \} (\Theta) - \frac{t-1}{t} \nabla_{n,m} \{ \mathcal{L}^{(t-1)} / (n(t-1)) \} (\Theta) \right\|_{n,m}^2 \\ & \leq \frac{C}{t^2} \left[\frac{1}{n} \sum_i \left\{ \sum_{\ell \neq i} \Delta A_{i\ell}^{(t)} + \sum_k \Delta B_{ik}^{(t)} \right\}^2 + \frac{m}{n^2} \sum_k \left\{ \sum_i \Delta B_{ik}^{(t)} \right\}^2 \right]. \end{aligned} \quad (\text{A.39})$$

The exponential terms cancel in (A.38). Up to the minus sign, its gradient has blocks:

$$(\Delta \mathbf{A}^{(t)} \mathbf{1}_n, \Delta \mathbf{B}^{(t)} \mathbf{1}_m, (\Delta \mathbf{B}^{(t)})^\top \mathbf{1}_n, \Delta \mathbf{A}^{(t)} \mathbf{Z} + \Delta \mathbf{B}^{(t)} \mathbf{W}, (\Delta \mathbf{B}^{(t)})^\top \mathbf{Z}).$$

Write $r_i^{(t)} = \sum_{\ell \neq i} \Delta A_{i\ell}^{(t)} + \sum_k \Delta B_{ik}^{(t)}$ and $c_k^{(t)} = \sum_i \Delta B_{ik}^{(t)}$. The scalar blocks are bounded by $r_i^{(t)}$ and $c_k^{(t)}$, and nonnegativity gives:

$$\begin{aligned} \left\| (\Delta \mathbf{A}^{(t)} \mathbf{Z} + \Delta \mathbf{B}^{(t)} \mathbf{W})_i \right\|_2 & \leq \sum_{\ell \neq i} \Delta A_{i\ell}^{(t)} \|\mathbf{z}_\ell\|_2 + \sum_k \Delta B_{ik}^{(t)} \|\mathbf{w}_k\|_2 \leq C r_i^{(t)}, \\ \left\| ((\Delta \mathbf{B}^{(t)})^\top \mathbf{Z})_k \right\|_2 & \leq \sum_i \Delta B_{ik}^{(t)} \|\mathbf{z}_i\|_2 \leq C c_k^{(t)}. \end{aligned}$$

The weights in (A.25) give $(1/n)t^{-2} = 1/(nt^2)$ for worker blocks and $(1/m)\{m/(nt)\}^2 = m/(n^2t^2)$ for task-type blocks. Projection cannot increase the norm, proving (A.39). The factor $(t-1)/t$ puts both objectives over nt .

For $t > t_0$, we next use (16) and the initial threshold to show $\text{dist}_{n,m}(\hat{\Theta}_\star^{(t)}, \hat{\Theta}_\star^{(t-1)}) \leq r_0/(4\sqrt{n+m})$. We also verify Lemma S8 for the new objective $\mathcal{L}^{(t)}/(nt)$ with the old minimizer as reference. With $\nu = n+m$, (16) gives $\sum_i (r_i^{(t)})^2 \leq K_1^2 \nu$ and $\sum_k (c_k^{(t)})^2 \leq K_1^2 n \nu / m$. By nonnegativity:

$$\|\Delta \mathbf{A}^{(t)}\|_F^2 = 2 \sum_{i < \ell} (\Delta A_{i\ell}^{(t)})^2 \leq \sum_i (r_i^{(t)})^2, \quad \|\Delta \mathbf{B}^{(t)}\|_F^2 \leq \sum_k (c_k^{(t)})^2.$$

The operator norm is at most the Frobenius norm, so:

$$\left\| \left[\frac{\sqrt{n}}{2} \Delta \mathbf{A}^{(t)}, \sqrt{m} \Delta \mathbf{B}^{(t)} \right] \right\|_{\text{op}}^2 \leq \frac{n}{4} \sum_i (r_i^{(t)})^2 + m \sum_k (c_k^{(t)})^2 \leq C n(n+m). \quad (\text{A.40})$$

Apply Lemma S5 to the new observations with $u = t-1$ and reference $\Theta_\star^{(t-1)}$. The marginal scales are comparable because $1 \leq t/(t-1) \leq 2$, and the residual matrix is $\mathbf{M}_{S_{t-1}} +$

$[\frac{\sqrt{n}}{2}\Delta\mathbf{A}^{(t)}, \sqrt{m}\Delta\mathbf{B}^{(t)}]$. The exact minimizer has zero optimization error. Equations (A.6) and (A.40) give:

$$\begin{aligned} \text{dist}_{n,m}^2(\hat{\Theta}_\star^{(t)}, \Theta_\star^{(t-1)}) &\leq C \frac{\|\mathbf{M}_{S_{t-1}}\|_{\text{op}}^2 + n\nu}{n(t-1)^2} \\ &\leq C \left\{ \frac{\nu\ell_{t-1}}{n(t-1)} + \frac{\nu\ell_{t-1}^2}{n(t-1)^2} + \frac{\nu}{(t-1)^2} \right\} \\ &= C \frac{\nu\ell_{t-1}}{n(t-1)} \left\{ 1 + \frac{\ell_{t-1}}{t-1} + \frac{n}{(t-1)\ell_{t-1}} \right\} \leq C \frac{\nu\ell_{t-1}}{n(t-1)}. \end{aligned} \quad (\text{A.41})$$

The last step uses $t-1 \geq C \max\{n, \ell_{t-1}\}$ and $\ell_{t-1} \geq 1$. The old minimizer satisfies the same bound relative to $\Theta_\star^{(t-1)}$. The triangle inequality and Step 1 give:

$$\begin{aligned} \text{dist}_{n,m}(\hat{\Theta}_\star^{(t)}, \hat{\Theta}_\star^{(t-1)}) &\leq \text{dist}_{n,m}(\hat{\Theta}_\star^{(t)}, \Theta_\star^{(t-1)}) + \text{dist}_{n,m}(\hat{\Theta}_\star^{(t-1)}, \Theta_\star^{(t-1)}) \\ &\leq C \sqrt{\frac{\nu\ell_{t-1}}{n(t-1)}} \leq \frac{r_0}{4\sqrt{\nu}}. \end{aligned}$$

At the old minimizer, fitted means have scales $(t-1)/n, (t-1)/m$, comparable to the new scales. Step 1 gives the radii, interior margins, and spectral bound. The new residual is the old residual minus the checkpoint increment, so the left side of (A.27) is at most:

$$\frac{t-1}{t} \frac{b_0}{4} + \frac{\|\Delta\mathbf{A}^{(t)}\|_{\text{F}}}{t} + \sqrt{\frac{m}{n}} \frac{\|\Delta\mathbf{B}^{(t)}\|_{\text{F}}}{t} \leq \frac{b_0}{4} + C \frac{\sqrt{n+m}}{t}.$$

The initial threshold also implies $t \geq C(n+m) \geq C\sqrt{n+m}$, making this bound at most b_0 . Choose fixed b_0, r_0, μ, L covering both sets of mean comparison constants if needed. Lemma S8 now applies to the new objective at the old reference.

Step 4: Displacement of adjacent minimizers. Optimally align the old minimizer to the new one, retaining the notation, and set $\mathbf{h} = \hat{\Theta}_\star^{(t)} - \hat{\Theta}_\star^{(t-1)}$. By Lemma S6, $\mathbf{h}$ satisfies (A.20) at the old minimizer and its norm equals their distance. Step 3 puts the connecting segment in the new objective's local ball at the old reference. Integrating the gradient difference gives:

$$\begin{aligned} &\left\langle \nabla_{n,m}\{\mathcal{L}^{(t)}/(nt)\}(\hat{\Theta}_\star^{(t)}) - \nabla_{n,m}\{\mathcal{L}^{(t)}/(nt)\}(\hat{\Theta}_\star^{(t-1)}), \mathbf{h} \right\rangle_{n,m} \\ &= \int_0^1 D^2\{\mathcal{L}^{(t)}/(nt)\}(\hat{\Theta}_\star^{(t-1)} + s\mathbf{h})[\mathbf{h}, \mathbf{h}] ds \geq \mu \|\mathbf{h}\|_{n,m}^2. \end{aligned}$$

The new and old minimizers have zero gradient for the new and old objectives, respectively. Combining the zero gradient of the old minimizer with (A.39), Cauchy–Schwarz yields:

$$\mu \|\mathbf{h}\|_{n,m}^2 \leq \left\| \nabla_{n,m} \{\mathcal{L}^{(t)}/(nt)\}(\hat{\Theta}_\star^{(t-1)}) \right\|_{n,m} \|\mathbf{h}\|_{n,m}.$$

The bound is immediate if $\|\mathbf{h}\|_{n,m} = 0$. Otherwise, divide the preceding inequality by $\|\mathbf{h}\|_{n,m}$ and square. Applying (A.39) and then Assumption 4 gives, for every $t_0 < t \leq T$:

$$\text{dist}_{n,m}^2(\hat{\Theta}_\star^{(t)}, \hat{\Theta}_\star^{(t-1)}) \leq \frac{C}{t^2} \left[\frac{1}{n} \sum_i \left\{ \sum_{\ell \neq i} \Delta A_{i\ell}^{(t)} + \sum_k \Delta B_{ik}^{(t)} \right\}^2 + \frac{m}{n^2} \sum_k \left\{ \sum_i \Delta B_{ik}^{(t)} \right\}^2 \right] \leq \frac{C(n+m)}{nt^2}. \quad (\text{A.42})$$

Step 5: Gradient contraction and the inner iteration count. In the local region from Steps 1–2, use the current exact minimizer as reference. For a centered parameter Θ with $\text{dist}_{n,m}(\Theta, \hat{\Theta}_\star^{(t)}) \leq r_0/\sqrt{n+m}$, we show that one update in (17) contracts distance by a fixed $q_0 \in (0, 1)$. Choose r_0 below the fixed boundary margin as well.

Optimally align Θ to $\hat{\Theta}_\star^{(t)}$, and write $\mathbf{h}$ for the resulting difference and $\mathbf{g}^{(t)} = \nabla_{n,m} \{\mathcal{L}^{(t)}/(nt)\}(\Theta)$. The difference $\mathbf{h}$ satisfies (A.20) at $\hat{\Theta}_\star^{(t)}$. A common orthogonal transformation preserves the joint objective and the local conditions. By stationarity, integration of (A.29) and (A.28) gives $\langle \mathbf{g}^{(t)}, \mathbf{h} \rangle_{n,m} \geq \mu \|\mathbf{h}\|_{n,m}^2$ and $\|\mathbf{g}^{(t)}\|_{n,m} \leq L \|\mathbf{h}\|_{n,m}$. Since $\mathbf{P}$ is included in the gradient, the update is $\Theta - \eta \mathbf{g}^{(t)}$, and:

$$\begin{aligned} \text{dist}_{n,m}^2(\Theta - \eta \mathbf{g}^{(t)}, \hat{\Theta}_\star^{(t)}) &\leq \|\mathbf{h} - \eta \mathbf{g}^{(t)}\|_{n,m}^2 \\ &\leq (1 - 2\eta\mu + \eta^2 L^2) \|\mathbf{h}\|_{n,m}^2 = q_0^2 \|\mathbf{h}\|_{n,m}^2, \end{aligned}$$

where $\eta = \mu/(2L^2)$ and $q_0 = \{1 - 3\mu^2/(4L^2)\}^{1/2} \in (0, 1)$.

After contraction, distance remains below $r_0/\sqrt{n+m}$. Under some optimal orthogonal matrix, every latent row changes by at most r_0 . The reference lies at least $3R_z/4, 3R_w/4$ inside the two boundaries. Choosing $r_0 \leq \min\{R_z, R_w\}/4$ leaves margins of at least $R_z/2, R_w/2$. The update is therefore feasible. Choose a fixed integer J with $q = q_0^J \leq 1/2$, for example $J \geq \log(1/2)/\log q_0$. All of μ, L, q_0, J, q depend only on fixed model constants.

Step 6: Remaining in the local region. Write $e_t = \text{dist}_{n,m}(\hat{\Theta}^{(t)}, \hat{\Theta}_\star^{(t)})$ in this proof and subsequent references. Step 1 and a sufficiently large threshold constant ensure $e_{t_0} \leq r_0/(4\sqrt{n+m})$. Bound (A.42) gives adjacent-minimizer distance at most $C\sqrt{(n+m)/n}/t$. The threshold also implies $t_0 \geq C(n+m)/\sqrt{n}$, so this distance can be made at most $r_0/(4\sqrt{n+m})$ by increasing a dimension-independent constant.

Suppose $e_{t-1} \leq r_0/(4\sqrt{n+m})$. The previous output initializes the new checkpoint, and the triangle inequality places it within $r_0/(2\sqrt{n+m})$ of the new exact minimizer. Every inner step contracts by Step 5 and stays local. After J steps, the factor is at most $q \leq 1/2$, restoring $e_t \leq r_0/(4\sqrt{n+m})$. Induction proves that every checkpoint output and inner iterate stays in the local region and inside the radius constraints.

Step 7: Tracking recursion. Iterating $e_t \leq qe_{t-1} + Cq\sqrt{(n+m)/n}/t$ gives:

$$e_t \leq q^{t-t_0}e_{t_0} + C\sqrt{\frac{n+m}{n}} \sum_{s=t_0+1}^t \frac{q^{t-s}}{s} \leq q^{t-t_0}e_{t_0} + \frac{C\sqrt{(n+m)/n}}{t(1-q)^2}.$$

The second inequality uses $\tau = t - s$, $t/s \leq \tau + 1$, and $\sum_{\tau \geq 0} (\tau + 1)q^\tau = (1 - q)^{-2}$. Squaring with $(a + b)^2 \leq 2a^2 + 2b^2$ and applying (A.37) proves (18).

Step 8: Subsequent global optimization errors. Align the computed estimate to the current exact minimizer. Their segment lies in the local region, where the reference gradient is zero. Integrating the Hessian upper bound (A.28) gives:

$$0 \leq \mathcal{L}^{(t)}(\hat{\Theta}^{(t)}) - \mathcal{L}^{(t)}(\hat{\Theta}_\star^{(t)}) \leq \frac{Lnt}{2}e_t^2. \quad (\text{A.43})$$

Substitute Step 7 and $e_{t_0}^2 \leq C\varepsilon_0/(nt_0)$ to obtain the first upper bound in (19). For $\tau = t - t_0$:

$$\frac{t}{t_0}q^{2(t-t_0)} = (1 + \tau/t_0)q^{2\tau} \leq (1 + \tau)q^{2\tau} \leq \sum_{j=0}^{\infty} (1 + j)q^{2j} = \frac{1}{(1 - q^2)^2}.$$

The initial threshold also gives $(n+m)/t \leq (n+m)/t_0 \leq C$, with a dimension-independent constant. Thus the global optimization error is at most $C(1 + \varepsilon_0)$ uniformly. All steps are

deterministic on the same statistical event, and (16) holds almost surely by Assumption 4. No additional failure probability is incurred. $\square$

A sufficient condition for Assumption 4. Let a fixed K_* bound all checkpoint increments:

$$\max_i \left\{ \sum_{\ell \neq i} \Delta A_{i\ell}^{(t)} + \sum_k \Delta B_{ik}^{(t)} \right\} \leq K_*, \quad \max_k \sum_i \Delta B_{ik}^{(t)} \leq K_*(1 + n/m).$$

Then (16) holds with $K_1 = \sqrt{2}K_*$. Using $r_i^{(t)}, c_k^{(t)}$ from Step 3, nonnegativity gives $\sum_k c_k^{(t)} = \sum_{i,k} \Delta B_{ik}^{(t)} \leq \sum_i r_i^{(t)} \leq nK_*$. Consequently:

$$\frac{1}{n} \sum_i (r_i^{(t)})^2 \leq K_*^2, \quad \frac{m}{n^2} \sum_k (c_k^{(t)})^2 \leq \frac{m}{n^2} (\max_k c_k^{(t)}) \sum_k c_k^{(t)} \leq K_*^2(1 + m/n).$$

The sum is at most $2K_*^2(1 + m/n)$, proving (16). The converse fails: when $m = n$, a single $\mathbf{A}$ edge of weight $\lfloor \sqrt{n} \rfloor$ gives a weighted squared sum at most two, while its maximum row increment grows with dimension.

Local stopping error. In the local region of Step 5, let $\mathbf{g}^{(t)} = \nabla_{n,m} \{\mathcal{L}^{(t)}/(nt)\}(\hat{\Theta}^{(t)})$. Then:

$$\text{dist}_{n,m}(\hat{\Theta}^{(t)}, \hat{\Theta}_*^{(t)}) \leq \|\mathbf{g}^{(t)}\|_{n,m}/\mu, \quad 0 \leq \mathcal{L}^{(t)}(\hat{\Theta}^{(t)}) - \mathcal{L}^{(t)}(\hat{\Theta}_*^{(t)}) \leq \frac{Lnt}{2\mu^2} \|\mathbf{g}^{(t)}\|_{n,m}^2.$$

For the aligned difference $\mathbf{h}$, Step 5 and Cauchy–Schwarz give:

$$\mu \|\mathbf{h}\|_{n,m}^2 \leq \langle \mathbf{g}^{(t)}, \mathbf{h} \rangle_{n,m} \leq \|\mathbf{g}^{(t)}\|_{n,m} \|\mathbf{h}\|_{n,m}.$$

For $\mathbf{h} \neq \mathbf{0}$, divide by $\|\mathbf{h}\|_{n,m}$ and apply (A.43). For $\mathbf{h} = \mathbf{0}$, both bounds hold directly.

For the numerical normalization, let $n_{\text{pair}} = n(n-1)/2 + nm$. If $\|\mathbf{P}\nabla \{\mathcal{L}^{(t)}/n_{\text{pair}}\}(\hat{\Theta}^{(t)})\|_\infty \leq \varepsilon$, the $n(d+2)$ worker and $m(d+1)$ task-type coordinates give:

$$\|\mathbf{g}^{(t)}\|_{n,m}^2 \leq \frac{n_{\text{pair}}^2 \varepsilon^2}{t^2} \left\{ d + 2 + (d+1) \frac{m^2}{n^2} \right\}.$$

Together with the preceding bounds, this converts the stopping rule in Supplementary B.5 into local distance and optimization errors.

A.5 Proof of Corollary 1

Proof. Allocate failure probability $\delta/2$ to Theorem 2 and $\delta/2$ to the event of Lemma S2 for $\lceil aT \rceil \leq t \leq T$, including (A.8). Their intersection has probability at least $1 - \delta$, without an independence requirement. Replacing $\log\{(2n + m)/\delta\}$ by $\log\{2(2n + m)/\delta\}$ adds only $\log 2$, absorbed into dimension-independent constants. On this intersection, all computed estimates are feasible and have global optimization error at most $C(1 + \varepsilon_0)$. The optimization event also gives the marginal bounds (A.7) for $\lceil aT \rceil \leq t \leq T$. Write $x = \log\{(2n + m)/\delta\}$. Applying Lemma S5 with $u = t$, $\Theta' = \Theta_\star^{(t)}$, and the fixed error bound gives, by (A.8) and $t \geq aT$:

$$\begin{aligned} \max_{\lceil aT \rceil \leq t \leq T} \text{dist}_{n,m}^2(\hat{\Theta}^{(t)}, \Theta_\star^{(t)}) &\leq C_a \left\{ \frac{(n + m)(Tx + x^2)}{nT^2} + \frac{1 + \varepsilon_0}{nT} \right\} \\ &\leq C_a \frac{(n + m)x}{nT}. \end{aligned}$$

The initial threshold gives $T \geq Cx$, and $x \geq 1$ absorbs the fixed optimization error. The same argument controls (15), proving (20).

For consistency, write $\nu = n + m$, $s_* = 2n + m$, and let b_* be the fixed threshold constant.

For a fixed cutoff $m_* > 4$, each fixed $\delta \in (0, 1)$ eventually gives, along the stated sequence:

$$\frac{\nu \log s_*}{nT} \leq \frac{\log s_*}{b_* \nu \log(s_*/\delta)} \leq \begin{cases} (b_* m_*)^{-1}, & \nu \geq m_*, \\ \log(2m_*)/\{b_* \log(1/\delta)\}, & \nu < m_*. \end{cases}$$

Given $\varepsilon > 0$, choose fixed $m_* > 4$ with $(b_* m_*)^{-1} < \varepsilon$, then fixed δ making the second bound smaller than ε . Hence $\nu \log s_*/(nT) \rightarrow 0$ for every such sequence.

The explicit sufficient condition $\{(n + m)^2 \log(2n + m)\}/(nt_0) \rightarrow 0$ ensures the initial threshold for every fixed δ , since $\log(s_*/\delta) \leq C_\delta \log s_*$ for a fixed C_δ . It also gives directly:

$$\frac{\nu \log s_*}{nT} \leq \frac{1}{\nu} \frac{\nu^2 \log s_*}{nt_0} \rightarrow 0.$$

Together with $t_0 \leq \lceil aT \rceil$, the explicit condition is the sufficient condition stated after Corollary 1. For every fixed δ , $\log\{(2n + m)/\delta\}$ and $\log(2n + m)$ differ by at most a fixed

factor, so (20) gives uniform consistency in probability over $\lceil aT \rceil \leq t \leq T$. If initialization succeeds with probability only $1 - \delta_{\text{init}}$, add its failure probability. When $\delta_{\text{init}} \rightarrow 0$, allocate half of any target failure probability to concentration. The initialization failure is eventually below the other half, yielding unconditional consistency. $\square$

B Additional Experiment Details: LongBench v2

Multi-Doc QA

This section supplements the LongBench v2 Multi-Doc QA experiment. Section B.1 describes the paired design, shared warm-up, per-session limits, and checkpoint schedule. Section B.2 reproduces the worker prompts and shared-board interface. Section B.3 defines the task types assigned to published results and explains how these results contribute to the collaboration and participation networks. Section B.4 reports outcomes and participation across replicates, together with workers’ recorded use of the statistical summaries. Section B.5 specifies the penalized objective, initialization, optimization procedure, and convergence diagnostics. Section B.6 reports token use, elapsed time, and estimation time. Section B.7 explains the construction of the figures and documents the recorded choice illustrated in the main text.

B.1 Paired Design

Each replicate begins with 32 warm-up sessions, which follow the History-only rules. Copies of the resulting shared board, private memories, and activation state initialize the AgentCue and History-only branches, each with 224 further sessions. A trajectory therefore contains 256 effective sessions, and a replicate uses 480 actual sessions. The AgentCue branch runs first; branches and replicates evolve independently. All sessions complete normally. Means and sample standard deviations summarize three replicates on the same 125 questions.

The MAS has 32 persistent workers, with at most one concurrent session per worker and at most 32 concurrent sessions in total. Initial activations are spaced 20 seconds apart; later sessions replace completed ones asynchronously. Each session allows 32 actions on the shared board and 1,200 seconds, with private memory limited to 8,000 characters. The question-order seed is 20260930; activation seeds are 20261006–20261008 for R1–R3. Model sampling seeds are uncontrolled. All questions are accessible from the start through different initial question orders. Sources are divided into chunks of at most 1,400 words without removing content; answer keys remain withheld from workers.

Both methods prioritize uncovered questions, answer-critical gaps, and useful existing works. Workers may perform any task type, revise their focus, and contribute to several questions per session. A first designated answer must use and explain a relevant published result of another worker; publishing an answer and designating the answer are separate actions. After all questions have answers, workers may review and correct the answers. Workers in the AgentCue branch also use the statistical summaries to narrow local choices, followed by checks of the current needs. Workers can choose otherwise with a concrete reason; no fixed number of alternatives is required.

The first checkpoint at 32 completed effective sessions, which is common to both branches, and the final checkpoint at 256 follow completion of all active sessions. Intermediate checkpoints at 64–224 include published results of sessions in progress. Estimation at each intermediate checkpoint runs serially in the background while sessions continue. A session in the AgentCue branch reads the latest statistical summary with one action when the session first requests the statistical summary, keeps the same summary version, and computes subsequent local comparisons. The MAS estimates the parameters of the latent space model only in the AgentCue branch. The MAS copies the shared board into the two branches before the MAS publishes the statistical summary at the first checkpoint, so the shared

board of the History-only branch never holds a statistical summary.

B.2 Worker Prompts

The boxes reproduce the prompt components used in R1–R3. Each session prompt contains the identity and phase of the worker, the phase instruction, the common policy, the shared-board interface, the execution instruction, and the private memory of the worker. In these quotations, `history_summary` denotes `AgentCue`; `common_warmup` and `history_only` supply no statistical summary; “function” and “type” denote a task type, “artifact” denotes a published result, “submit” denotes designate, “complete estimate” and “packet” denote the statistical summary, and the statuses `used` and `considered_not_used` record whether the statistical summary informed a choice. Only line wrapping and pagination change.

AgentCue: phase instruction.

A latest complete fitted summary is published before this branch dispatches. Read the latest published complete packet once at your first statistical read, then reuse that immutable session version/token. New checkpoints fit and publish in the background without pausing collaboration; a later global version does not invalidate your already-read packet.

Warm-up and History only: phase instruction.

No statistical summary exists or is queried in this phase. Do not call summary or perform fitted-parameter calculations; use current state, experience and actual artifacts.

Common worker policy.

You and 31 peers jointly advance the fixed pool of 125 Multi-Doc QA questions. Every worker may answer any question and perform any of the four functions. Choose your own work, decomposition, collaborators and handoffs. Roles are not assigned, and questions do not require a fixed sequence of functions.

Apply these priorities in order when choosing and continuing work:

1. ****Complete the question pool within the fixed effective-session budget.**** Aim to bring all 125 questions to source-grounded submitted answers. Give priority to work that moves unsubmitted questions toward actual submission: open a useful scope on a question without published results, fill an answer-critical gap, or carry existing results forward into an answer. Choose among these needs using current inputs, active claims and your ability to contribute; there is no required order of functions. When inputs are adequate, synthesize and submit promptly; peer PASS is not a prerequisite. When they are inadequate, contribute the needed evidence, comparison, reasoning or targeted check and leave a usable handoff. You need not perform every function or finish a whole question yourself. While suitable unsubmitted needs remain, defer routine polishing and repeated checks of submitted answers. A concrete error, evidence conflict or substantive quality benefit may justify submitted-question work; explain the tradeoff in the existing choice reason. After all

questions are submitted, use remaining authorized sessions for useful verification and correction.

2. ****Use familiar work and a revisable functional focus to serve that goal.**** Among useful contributions to the first priority, prefer continuing work whose source familiarity, reusable results or current function reduces rereading, switching and repeated reasoning. Continue when relevant work is ready. Hand off or switch when an important uncovered need fits you better, the current input is unavailable, or a peer is better placed to continue. Familiarity and focus help choose how to contribute; they do not justify neglecting suitable unsubmitted needs. No worker has an exclusive or permanent function.

3. ****In history_summary, use the current statistical summary first to narrow choices.**** Within the first two priorities, use the fitted worker-function relations to identify suitable functions among actual pending tasks, and worker-worker relations to identify plausible collaborators. Once the summary and current need identify a suitable candidate, inspect that question/task and the relevant member's current workload, then proceed using the required artifacts and sources. Avoid browsing unrelated histories, questions or member profiles, or expanding comparisons merely to fill a quota. If support is weak, candidates are tied, the task is new, or relevant state/experience contradicts the suggestion, inspect only the additional history needed and choose with a concrete reason. The summary guides your choice; it does not assign work or rank question difficulty. This third priority applies only in history_summary.

Use the supplied non-login shell; omit login or set login=false in exec_command. Do not request login=true, start a login-shell wrapper, or change shell settings. Run only the two quoted-heredoc interfaces below.

Avoid preventable board refusals. Copy exact question, task, attempt and chunk IDs from actual responses; chunk numbering need not start at 0000. Before proposing first_answer, check current question_status and prefer an unsubmitted question without an active answer integration. Reuse an existing suitable open task. Refresh question_status immediately before CLAIM and pass that revision; concurrent changes may still require one targeted refresh. If another answer integration is active, choose useful complementary work or another question instead of repeatedly proposing/claiming competing syntheses. A concrete submitted-question review/correction remains allowed.

Before finish, read every listed input artifact in this call, including your own earlier-session work; an overview, title or memory does not count. Copy short contiguous source quotations from the exact returned chunk and preserve punctuation and PDF hyphens. Whitespace and the seven Latin PDF ligatures (such as fi/fi) are equivalent; wording, punctuation, case, hyphens and source identity remain exact. If a result is supported by an actually read peer artifact, cite that A-... ID in evidence_refs and explain its contribution; do not copy its source quotes as if you had read those sources yourself. source_quotes may be empty when the actual artifact inputs supply the evidence. If you directly quote a source, read/search that source in this call first.

In history_summary, prefer summary_use with just status, version and a concrete interpretation. Put genuine numerical comparisons and fallback reasoning in interpretation; optional alternatives are descriptive text or valid worker/function or worker/task objects, not a required comparison quota. After a documented helper rejects input or a local calculation fails, use its error/recovery to correct the specified issue in this same session and original budget. This includes malformed JSON, wrong fields/identities, self-comparison, missing cache tokens and rejected board actions. Do not repeat the same rejected payload unchanged or claim that rejected work was published/submitted. Empty, incomplete or missing-ok tool displays also permit continuing; inspect current state before retrying an uncertain write.

Types describe independently usable delivered results:

- ****extract:**** a source-located evidence package or useful search result, with objects, locations, dates, units and applicable conditions.
- ****compare:**** an explicit alignment of objects, sources or periods on shared dimensions,

with supported similarities, differences, conflicts, non-comparability or missing information.

- ****inference:**** a new calculated result, deduction, candidate conclusion or final answer. State inputs or premises, the calculation or reasoning, the conclusion and its conditions. For a final answer, explain how the inputs support the option criteria. This function includes substantive reasoning and answer synthesis.
- ****review:**** an independent check of another worker's identified artifact and version, with checks, findings, evidence and a PASS/FAIL verdict. Evidence, comparisons, calculations and candidate conclusions may be checked before a question has a submitted answer.

A work may deliver several independently usable results with distinct acceptance criteria and therefore have multiple types. Finding quotations for a comparison, citing facts for a deduction, recalculating during verification and routine self-checks are supporting steps unless they produce additional independent results. Multiple calculations or answer deductions within one work count once toward inference. Copying or renaming the same content does not create a new contribution.

Start with the compact overview of ready work, active claims and published results. Its coverage_opportunities shows separate factual examples of unsubmitted questions with and without published artifacts, including recent artifact IDs and active-work counts. Check both unmet-work and continuation needs; artifact presence alone does not establish sufficient evidence. In history_summary, read the current complete summary once next, before selecting work or collaborators. Then use targeted question/task/member reads for the candidate you choose; a broad snapshot is optional. Before you have relevant experience, explore a useful available scope and, where relevant, use the summary's activity support. You may choose randomly among similarly suitable alternatives. Register a specific scope and expected result before extensive source reading, and recheck the latest claims.

Retain reusable search strategies, comparison frameworks, calculation methods, option criteria and verification checklists in private memory. Update your working focus from actual experience under the priorities above. Consider peers' published results and workload when choosing who should continue; do not wait for unavailable inputs or duplicate an active claim. Multiple genuine functional results may occur in one work or session.

For an unsubmitted question with published artifacts, inspect the relevant exact versions and decide whether to fill a specific gap or synthesize the answer; do not repeat completed work. A targeted pre-submission check may resolve an answer-critical risk. Review the exact version and state which result it enables. Do not repeat checks just to obtain peer PASS or balance type frequencies.

Publishing a synthesis with finish does not submit the answer. After publishing an eligible answer for an unsubmitted question, complete designate_final before moving on and check its designated=true response. That successful response confirms submission; a further state read is needed only if the result is unclear or a conflict requires it. You may also designate an eligible peer synthesis after actually reading its exact version. Do not wait for a peer review when the submission requirements are already met. If validation or a concurrent change prevents submission, inspect the relevant response/state and fix the specific issue within budget, or leave the exact answer/artifact IDs, blocking issue and next required action for continuation; do not claim it was submitted.

Publish useful intermediate results with source locations, actual input artifact IDs and versions, boundaries and limitations. In the same result, identify the question, what the result enables, the specific remaining gap and next work with its required input IDs. This lets another worker continue without reconstructing your search. Keep a handoff concise; a separate invitation or duplicated task is not required for every result. One work may use results from several peers. An invitation, a read or a decorative citation is not completed cooperation.

During common_warmup and history_only, do not query statistical summaries or perform fitted calculations. During history_summary, a current complete estimate is always published before your session starts; read its complete representation once for this session/version, then

compare useful functions or possible collaborators locally. It contains all 32 worker positions Z, four function positions W, node effects alpha/beta/gamma, checkpoint metadata and activity support. Reuse it for changed candidates, new tasks and refreshed question state. After common c32, collaboration continues while statistical snapshots are fitted in the background. Your first full summary read receives the latest published complete version; cache that immutable version for this whole session. A newer global publication does not invalidate your cached version or its CLAIM/INVITE reasons. Refresh current task, question and artifact state as needed. Zero activity is missing record support, not lack of ability. The statistical service supplies numerical representation; you choose work and collaborators.

Use the cached parameters for internal comparison or pass the returned local_cache token to the pure local_compute.py interface, which costs zero board actions. Its only cache read is this call's already obtained immutable parameters; no need to copy 140 numbers or fetch live state. For worker/function i,k compute $z_i \cdot w_k$ and $\exp(\beta_i + \gamma_k + z_i \cdot w_k)$; for different workers i,j compute $z_i \cdot z_j$ and $\exp(\alpha_i + \alpha_j + z_i \cdot z_j)$. These are fitted cumulative counts, not success probabilities or question difficulty. You do not need a candidate_summary request or returned task/worker alternatives to claim or invite. After statistical comparison, inspect current question/task state and member workload as needed. Before CLAIM refresh question_status; current revision and conflicts still apply. Record used/considered_not_used/single_option with the complete version and concrete reasoning. Weak support, ties, source experience or a new task can justify considered_not_used; unavailable means no fitted summary exists in this condition. Alternatives and adopted values are optional, not a comparison quota.

Every response shows remaining session actions and the team session budget. Choose a scope you can publish within that budget and reserve actions for actual input reads, finish and, for an eligible answer, designate_final. As the team budget shrinks, favor feasible completion of pending answers and precise answer-critical gaps; avoid broad work whose useful result cannot be delivered in the remaining budget. If this session cannot finish an answer, publish actual reusable partial findings and an explicit handoff before ending. Do not spend the end of a session repeatedly comparing candidates. Submission priority does not permit fabricated evidence, unsupported guesses or relaxing publication validation.

In common_warmup and history_only, choose using current work state, accumulated experience and relevant history. In history_summary, missing or invalid current parameters are a fault to report, not permission to continue without them. Workers who have not activated or delivered results lack statistical support; zero activity does not imply lack of ability. Retain autonomous exploration and targeted history fallback for weakly supported candidates. Expected cumulative participation and cooperation counts describe work history and relationships, not success rates. Statistical selection never replaces reading the chosen question, needed sources and actual input artifacts.

A first final answer must use at least one relevant published result from another worker and explain its substantive contribution. Peer PASS is not required before submission. All 125 questions and source collections are accessible from the start. Ground answers in the provided collections. Do not use outside knowledge as a substitute for evidence. Only the frozen board interface and supplied-data pure calculation interface are permitted. Local calculation cannot fetch new state or replace source/artifact reads.

Construct each request as a JavaScript object in the code-mode host and serialize it with ``JSON.stringify(payload)`` when building the documented quoted heredoc command. Source quotes, newlines and quotation marks must be serialized rather than manually escaped. A malformed JSON/object/action request at the fixed board entry is an ordinary rejection and consumes one of your 32 board actions. Correct it in this same session using the returned reason; it does not publish an artifact. At the action limit, finish without further board calls. At local_compute.py, rejected inputs cost zero board actions and return an ordinary error plus recovery instructions. A cooperation comparison must have different worker and peer IDs: explicitly remove your own worker ID from candidate peers. Use B for your own worker/function comparisons. Correct a rejected comparison locally and continue useful work in the original session.

Shared-board interface.

```
## Board interface
```

Shared-board requests must have exactly this quoted-heredoc form:

```
```sh
python3 board.py <<'JSON'
{"action":"snapshot"}
JSON
```
```

The only other allowed command is the pure `local_compute.py` quoted-JSON interface described below. Do not inspect files, execute arbitrary code, use external sources, network tools or agents. Your session identity is bound by the controller. Each session permits 32 board actions including invalid actions, and 1200 seconds. The transport uses non-login shells. Omit `login` or set `login=false`; do not set `login=true` or wrap a command in a login shell. Failed requests still cost their actual board action. Ordinary rejections return `failure_class="ordinary"` and, where applicable, `error_code/recovery` with the exact input/chunk or next read to correct. They do not automatically execute that read, fix a payload or select work. Technical/protocol failures stop new dispatch.

Every response includes the remaining team session budget. No new slot available means existing sessions may finish; do not assume another session will follow. `shared_context.session_action_budget` also gives this session's maximum, used and remaining actions. Keep actions for actual input reads, finish and designate_final.

Start with overview; in `history_summary` read summary once next, then request the relevant `question_status/task_status/member_profile` and actual inputs for the selected work. A broad snapshot is optional. If the summary and current state identify a suitable candidate, proceed with targeted reads instead of browsing unrelated board history. In `common_warmup` and `history_only`, skip summary and fitted calculations. All 125 questions are open from the beginning. The initial

question page is rotated by worker ID, not assigned. Use questions with offset and `next_offset` to browse all items; optional `query/sub_domain/status` filters.

Choose any question. Each question has its own source collection.

The common prompt prioritizes advancing unsubmitted questions to submission. Default question pages and functional ready-work examples sort unsubmitted questions first, without hiding submitted work. The overview also exposes submitted ready/review examples. You may select concrete submitted-question review/correction with a reason explaining its value relative to uncovered needs. No worker, function or question is assigned; artifact presence does not certify sufficient evidence.

Read actions:

- overview: compact ready-work counts/examples, active claims, unreviewed peer results and revisable working focuses. `coverage_opportunities` separately lists unsubmitted questions with and without published artifacts (counts and at most six rotated examples each), with recent artifact IDs/versions/types, published-synthesis counts and active-work/integration counts. Published results remain visible even after their task closes or is reviewed. This is included in the same overview action. Examples are factual leads, not judgments that an answer is ready or work assignments. Read `question_status` for complete exact versions and current claims; use targeted questions when the question text is needed.
- snapshot: optional question/project/offset.
- summary: one action returns the entire available 32-worker/four-function, two-dimensional representation: parameters {version,dimension,worker_ids,type_keys,alpha,beta,gamma,Z,W}, 140 numbers and 36 latent vectors, plus checkpoint/event cutoff, cumulative record interval, worker support and an own-session immutable `local_cache` token. No derived full matrix is sent. During `common_warmup` and `history_only`, do not call `summary`: these phases expose no fitted parameters or token and `summary` queries are rejected. In `history_summary` joint-c32 is published before the branch starts; later complete estimates publish asynchronously while

collaboration continues. A missing, corrupt or inconsistent published packet is a stop condition, not unavailable fallback. A newer checkpoint still being fitted is normal; the latest published complete packet remains available. No automatic parameter delivery occurs in other responses.

Cache the complete packet in this session. Same-version repeat requests still cost one board action and return only a cache reminder; changing candidate, task or question revision never requires another statistical read. The first actual full read pins the latest published version for this whole session. Background publication can advance latest_summary_version; session_summary_version identifies your already read packet. CLAIM/INVITE accept that exact actually-read version, and local_compute reuses its immutable token. A new session must perform its own first full read even if private memory contains an old packet.

- question_status: question. Returns exact final, artifacts, reviews, active work, stage and revision; one state-read action, no statistical re-fetch.

active_answer_integrations and first_answer_claim_possible_now report the current atomic conflict conditions, not evidence sufficiency. Overview ready-work examples also include claimable_now and blocking_answer_integration_ids, with blocked work still visible. These facts may change concurrently. Before proposing first_answer, inspect this question status; before CLAIM refresh its revision. If integration is active, use complementary useful work or another current need instead of duplicate synthesis claims.

- task_status: task_id. Current task specification/state/owner; no statistics.
- member_profile: worker. Current member workload/history and focus; no fitted parameters.
- questions: optional offset/question/query/sub_domain/status; status is unsubmitted, awaiting_PASS or peer_PASS.
- sources: question, optional offset. Returns real chunk IDs.
- search: project="MultiDoc", question, pattern. Returns source excerpts.
- read_source: project="MultiDoc", question, chunks=[one to three real IDs].
- read_artifact: attempt_id. Read each actual input's exact version before use.

Pure local calculation (zero board actions):

Use the exact summary.local_cache object returned by your complete summary read:

```
{"cache":{"version":"the actual returned version","parameter_sha256":"the actual returned hash"},"comparisons":[{"kind":"B","worker":1,"function":"MultiDoc/extract"}, {"kind":"A","worker":1,"peer":2}]}
```

Send that actual JSON (replace the two token placeholders) with exactly:

```
```sh
```

```
python3 local_compute.py <<'JSON'
```

```
...the actual JSON request described above...
```

```
JSON
```

```
```
```

The tool reads only the immutable parameter snapshot already obtained by this same bound call, then computes over it. You do not need to copy 140 numbers into each command. No caller-supplied path, other session, live board database/state, answer file, network, arbitrary code or board write is permitted. Alternatively, explicit summary.parameters plus comparisons may be supplied as numerical data; the trace checks the served version and hash. If local display or audit records are missing, continue your work in this original session; do not echo all parameters or repeat actions to supply audit evidence. Batch genuine comparisons (up to 4096 per request) and reuse the same token. Results give inner products and $\exp(\beta_i + \gamma_k + z_i \cdot w_k)$ or $\exp(\alpha_i + \alpha_j + z_i \cdot z_j)$, excluding self-cooperation. Numeric comparison/choice remains yours; the tool makes no work or role assignment. Pure calculations are recorded from command trace, consume session time/model usage and create no participation/cooperation. History-only has the identical tool but receives no fitted packet or token; do not fabricate them.

Propose:

```
{"action":"propose","project":"MultiDoc","question":"exact ID",
"expected_types":["extract","compare"],"description":"specific gap",
"acceptance":"inspectable result","inputs":[],"parent":null,"tags":[],"purpose":"evidence",
"target_attempt":null,"follow_up_reason":null}
```

purpose is evidence, first_answer, review or correction. Final-answer synthesis must use first_answer or correction. review is a focused task with review in expected_types and target_attempt set to its exact input artifact. correction requires follow_up_reason identifying the concrete issue.

Expected types describe intentions only. Use a separate work/task per question when batching. Use existing work rather than reposting it as a new task.

Claim:

```
{"action":"claim","task_id":"T-...","reason":"why this work now",
"partner":null,"question_revision":"from question_status","follow_up_reason":null,"summary_
use":{"status":"unavailable","version":null,
"alternatives":[],"interpretation":"concrete history or availability reason"}}
```

Submitted-question work is permitted before full coverage; claiming it requires a concrete follow_up_reason for an error, conflict or quality benefit relative to uncovered work.

Already-reviewed exact targets also need a new substantive reason. First-answer, version and duplicate-claim checks remain.

summary_use for an available model is, for example:

```
{"status":"used","version":"joint-c32","interpretation":"actual numerical basis and current
work need"}
```

Allowed statuses are used, considered_not_used, single_option and unavailable.

Available-model choices require one complete read of this version in this session and legal selected task/worker identities. The actual CLAIM/INVITE records the selected object; no previously returned candidate pair or competitor list is required. Optional alternatives/adopted values are descriptive. unavailable requires no fitted model and null version.

Always read actual input artifacts/exact versions and needed source text before reuse; statistics do not replace evidence.

Prefer omitting alternatives and adopted values: the three-field status/version/interpretation object above is sufficient. If supplied, alternatives is a list of nonempty descriptive strings or objects {worker:1,function:"MultiDoc/extract"} / {worker:1,task_id:"actual T-..."}; structured identities are validated. Text is saved as an unverified declaration, not proof of a calculation or successful use. Put the actual comparison and reason in interpretation.

Finish:

```
{"action":"finish","attempt_id":"A-...","lease":"returned lease",
"artifact":{"text":"actual result summary","inputs":[],"input_uses":{}},
"source_quotes":[{"project":"MultiDoc","chunk":"real chunk ID","quote":"literal text"}],
"final_choice":null,"results":[{"id":"e1","kind":"evidence",
"scope":"standalone","evidence_refs":["Q0"],"data":{"facts":["source-located fact"]}}]}
```

There are four task types: extract, compare, inference, review.

results[].kind uses the six delivery structures below; scope is standalone or supporting. Only standalone results add task types. All delivery data fields are required and must be nonempty. evidence_refs may be empty only for review; artifact.inputs and input_uses are empty when no published input is used. Use explicit "not applicable" or uncertainty where appropriate;

do not fabricate findings to fill fields. Lists below contain strings unless otherwise stated. evidence_refs lists Q0/Q1/... for source_quotes or actual input A-... IDs. To ground an unsuccessful search, include artifact.search_records as [{"question":"exact ID","pattern":"actual searched pattern"}] and cite S0/S1/...; the board checks that you performed that search in this session. Describe the search scope and uncertainty; no hit from one query is not proof of absence. Every non-review result needs evidence_refs. Each artifact input must be used by a result or be its review target, and input_uses maps every input ID to an explanation of its actual contribution. Supporting results do not add types.

- evidence -> extract: data.facts (list).

source_quotes use short contiguous text from the exact cited chunk. Only whitespace and seven Latin PDF ligatures (ff/ff, fi/fi, fl/fl, ffi/ffi, ffl/ffl, st/st, st/st) are

equivalent. No punctuation, case, hyphen, wording, cross-chunk or fuzzy substitution is accepted. The rule is recorded in each published source_quote_checks. Quote failures identify quote_index/chunk; unread-source/input failures name the precise read needed. If a read peer artifact is the evidence, cite its A-... ID with input_uses rather than copying its original-source quotes without a source read in this call.

- comparison -> compare: data.objects (at least two distinct names), dimensions (list), findings (list).
- calculation -> inference: data.inputs (list), procedure (string), result (string), units (string), time_basis (string).
- inference -> inference: data.premises (list), reasoning (list), conclusion (string), conditions (string).
- synthesis -> inference: data.choice (A/B/C/D), input_uses (list of actual input IDs), rationale (string), remaining_uncertainty (string). Set artifact.final_choice to the same choice. Standalone only. Use at least one relevant other-worker artifact and explain its contribution; it cannot be a decorative citation.
- review -> review: data.target (input A-... ID), version (integer), checks (list), findings (string), verdict (PASS/FAIL). Standalone only; target another worker's exact version. This automatically publishes the peer review and records its participation. There is no separate review action. Do not create a duplicate review summary task. Supporting recalculations do not add inference.

calculation, inference and synthesis are three delivery formats of the single inference task type. A work containing both contributes only one inference participation. Use inference in expected_types for any of these formats.

After publication, results are immutable. An identical finish retry is idempotent; a changed retry is rejected. Reopen is for genuine new corrective work, not for reposting a result. Publish actual partial findings and state remaining gaps; completion or correctness is not needed for a useful participation record. In artifact.text, make unfinished work easy to continue: identify what this result enables, the remaining answer-critical gap and next work with actual required input IDs. Do not create a redundant task or invitation just to record a handoff.

Other actions:

- focus: function (extract/compare/inference/review or null), reason. Revisable nonexclusive working focus; it creates no participation count.
 - designate_final: question, attempt_id of a published synthesis; peer review is not required first. Only relevant other-worker inputs satisfy collaboration. finish alone does not submit an answer. After publishing an eligible answer for an unsubmitted question, reserve and perform this action before moving on; result.designated=true confirms submission without an additional polling action. An eligible peer synthesis may also be designated after actually reading its exact version. If a rejection or concurrent change blocks submission, address the specific issue within budget or leave the artifact ID, blocking reason and next required action in the handoff. Report submission only after confirmation.
 - reopen: task_id, reason (specific supported correction).
 - invite: worker, task_id, reason, summary_use in the same format as claim.
- Invitations are nonexclusive; other workers may accept the work.

Return final JSON with exactly summary (string), published_attempts (list of your own actual A-... IDs), memory (string), useful_work_remaining (boolean). Preserve useful context in memory. Invalid/no-progress completions still consume a session. Request construction: in the code-mode host, use `const payload = {...}; const cmd = "python3 board.py <<'JSON'\n" + JSON.stringify(payload) + "\nJSON"; text(await tools.exec\_command({cmd, shell:"/bin/zsh", login:false}))`; The command must still contain exactly one documented helper and one quoted heredoc. If a value contains a line exactly `JSON`, use its JSON-escaped string representation; never place a literal delimiter line in the body. A malformed board request returns `failure\_class:ordinary`, `error\_code:INVALID\_BOARD\_REQUEST`, a raw-input SHA-256, recovery instruction and action-budget header; it consumes one action

without publishing or executing the requested action. Correct the payload in the same session. After 32 actions, `BOARD\_ACTION\_LIMIT` rejects without further execution. A malformed local calculation returns `failure\_class:ordinary`, `error\_code:INVALID\_LOCAL\_REQUEST`, the raw-input SHA-256, zero board actions and a correction instruction before any cache read or arithmetic. Local cache/calculation errors and uncertain displays are warnings; correct them within this same session. Exclude the acting worker from A comparison peers; use B for worker/function comparisons. A rejected query in a condition without a summary exposes no parameters and permits continuing with state/history. Missing delivery records are warnings and do not require replay or extra evidence from the worker. Actual forbidden commands, confirmed conflicting parameter versions/hashes, damaged execution provenance and a missing published summary remain integrity failures.

Execution instruction.

EXECUTION PARAMETERS: Use tools.exec_command with shell="/bin/zsh" and login=false. Omit workdir so it inherits this session workspace; do not invent a directory or shell path. Keep cmd exactly one documented quoted-JSON board.py or local_compute.py invocation. Use the returned recovery instruction to correct ordinary board rejections; do not repeat an unchanged rejected request. If tool output is empty, incomplete, or lacks ok, continue in this same session. For an uncertain write, read current state to check whether it already succeeded before any correction; do not blindly repeat writes. Correct local input errors within the existing time/action budget.

B.3 Published Results and Count Construction

Each work record stores the worker who produced the work, the source references, and the exact versions of the published results that the work uses or reviews. A fixed program checks required fields, quotations, and input references, and classifies independently reusable results as in Table B.1. The classification ignores benchmark correctness.

Each work contributes at most once to each task type in $\mathbf{B}^{(t)}$. Supporting quotations, review calculations, and self-checks add no separate task types. Several deductions or answer syntheses in one work count once as inference. Identical publication retries add no counts. Using or reviewing published results of other workers contributes once per work and worker–worker pair to $\mathbf{A}^{(t)}$; invitations and reads alone add nothing. A work may contribute to multiple task types and pairs.

Table B.1: Task types defined by independently reusable results.

| Task type | Required independently reusable result |
|-----------|---|
| Extract | Source-located facts, values, conditions, or a documented search result. |
| Compare | Named objects, shared dimensions, and evidence-supported similarities, differences, or conflicts. |
| Inference | Inputs or premises, reasoning, conclusion, and applicable conditions. Answer synthesis also gives an option and explains its use of published results of other workers. |
| Review | The exact version of another worker’s published result, checks, findings, evidence, and a PASS or FAIL verdict. |

B.4 Outcomes and Summary Use

Table B.2 reports replicate outcomes, and Tables B.3–B.4 summarize participation. All six trajectories designate answers for all 125 questions. All 32 workers perform extract and inference; compare involves 31–32 workers and review 25–31. PASS concerns a review of the designated answer version and differs from benchmark correctness.

Table B.2: Final results by replicate. Difference is in percentage points; PASS columns count designated answers with a PASS verdict (AgentCue/History only).

| Replicate | AgentCue | | History only | | Difference | PASS | Correct + PASS |
|-----------|----------|--------------|--------------|--------------|------------|-------|----------------|
| | Correct | Accuracy (%) | Correct | Accuracy (%) | | | |
| R1 | 76 | 60.8 | 71 | 56.8 | 4.0 | 36/36 | 22/20 |
| R2 | 77 | 61.6 | 72 | 57.6 | 4.0 | 33/41 | 18/23 |
| R3 | 75 | 60.0 | 73 | 58.4 | 1.6 | 92/47 | 55/24 |

Each AgentCue branch provides the statistical summary to all 224 new sessions. R1–R3

Table B.3: Participation and collaboration (mean $\pm$ sample standard deviation). Task-type counts are column sums of $\mathbf{B}^{(T)}$; collaboration is $\sum_{i<\ell} A_{i\ell}^{(T)}$.

| Measure | AgentCue | History only |
|------------------------------|------------------|------------------|
| Published works | 359.7 ± 26.8 | 348.0 ± 2.6 |
| Extract | 160.7 ± 3.2 | 162.3 ± 11.2 |
| Compare | 140.0 ± 18.4 | 153.3 ± 6.4 |
| Inference | 246.0 ± 56.0 | 250.7 ± 51.4 |
| Review | 71.3 ± 31.0 | 57.0 ± 5.3 |
| Collaboration count | 284.7 ± 46.9 | 248.7 ± 3.2 |
| Distinct collaborating pairs | 209.3 ± 26.0 | 190.3 ± 1.5 |

Table B.4: Participation by replicate. E/C/I/R denote extract/compare/inference/review; the worker column counts contributors to each task type.

| Replicate | Method | Works | E | C | I | R | Workers | E/C/I/R | Collaboration |
|-----------|--------------|-------|-----|-----|-----|-----|---------|-------------|---------------|
| R1 | AgentCue | 350 | 163 | 119 | 290 | 56 | | 32/32/32/28 | 274 |
| R1 | History only | 349 | 175 | 156 | 288 | 51 | | 32/32/32/25 | 250 |
| R2 | AgentCue | 339 | 162 | 153 | 183 | 51 | | 32/32/32/27 | 244 |
| R2 | History only | 345 | 154 | 146 | 192 | 59 | | 32/32/32/29 | 251 |
| R3 | AgentCue | 390 | 157 | 148 | 265 | 107 | | 32/31/32/31 | 336 |
| R3 | History only | 350 | 158 | 158 | 272 | 61 | | 32/32/32/30 | 245 |

record 285, 288, and 299 requests for local comparisons, which contain 1,687, 1,559, and 1,784 local comparisons. Recorded choices use the statistical summary 237, 197, and 214 times and consider but decline the statistical summary 58, 84, and 114 times; a session can make several choices. Workers read the first seven summary versions. The final statistical summary follows all worker activity.

B.5 Numerical Estimation

Each estimation uses the latent space model (1) with $d = 2$ on all 32 workers and four task types, including zero-count pairs. Let $\mathbf{x}$ stack the entries of $\Theta^{(t)}$. The penalized objective is

$$f^{(t)}(\mathbf{x}) = \frac{\mathcal{L}^{(t)}(\mathbf{x})}{624} + \frac{0.01}{32} \|\mathbf{Z}\|_F^2 + \frac{0.01}{4} \|\mathbf{W}\|_F^2, \quad 624 = \binom{32}{2} + 32 \times 4. \quad (\text{B.1})$$

Penalty coefficients are fixed throughout. Let $\mathbf{P}$ subtract column means from $\mathbf{Z}$ and $\mathbf{W}$ and the mean from β , leaving α and γ unchanged. Projected updates are

$$\mathbf{x}_{s+1} = \mathbf{P} \left\{ \mathbf{x}_s - \eta_s \mathbf{P} \nabla f^{(t)}(\mathbf{x}_s) \right\}. \quad (\text{B.2})$$

The step size η_s in (B.2) results from Armijo backtracking, which halves a trial step size until the sufficient-decrease condition with coefficient 10^{-4} holds. The first trial step size is 1. Write $\Delta \mathbf{x}_s = \mathbf{x}_{s+1} - \mathbf{x}_s$ and $\Delta \mathbf{g}_s = \mathbf{P} \nabla f^{(t)}(\mathbf{x}_{s+1}) - \mathbf{P} \nabla f^{(t)}(\mathbf{x}_s)$. Subsequent trial step sizes alternate across iterations between the two Barzilai–Borwein step sizes $\Delta \mathbf{x}_s^\top \Delta \mathbf{x}_s / (\Delta \mathbf{x}_s^\top \Delta \mathbf{g}_s)$ and $\Delta \mathbf{x}_s^\top \Delta \mathbf{g}_s / (\Delta \mathbf{g}_s^\top \Delta \mathbf{g}_s)$, restricted to $[10^{-6}, 10^4]$; when $\Delta \mathbf{x}_s^\top \Delta \mathbf{g}_s \leq 10^{-20}$, the trial step size is $1.5\eta_s$, capped at 10^4 .

The first checkpoint uses five random starts, with seeds 42–46, centered $N(\mathbf{0}, \mathbf{I}_2)$ latent positions, and node effects initialized from the observed network counts. The smallest finite penalized objective selects the estimate. Later checkpoints start from all parameter blocks of the previous estimate. Each start allows 20,000 iterations, stopping at $\|\mathbf{P} \nabla f^{(t)}(\mathbf{x}_s)\|_\infty \leq 10^{-5}$. All 24 estimates, eight checkpoints in each of the three AgentCue trajectories, satisfy

this stopping rule (Table B.5). The local stopping error in Supplementary A.4 converts this stopping rule into bounds on the distance to the exact minimizer and on optimization error.

Table B.5: Estimation diagnostics over eight checkpoints. Centering columns give maximum absolute sums of latent-position columns or node effects.

| Method | Estimates | Iterations | Gradient norm | $\mathbf{Z}$ | $\mathbf{W}$ | β |
|----------|-----------|------------|---|------------------------|------------------------|------------------------|
| AgentCue | 24 | 111–469 | 2.33×10^{-6} – 9.95×10^{-6} | 1.33×10^{-15} | 1.11×10^{-16} | 6.22×10^{-15} |

B.6 Computational Resources

Table B.6 reports the final token use and elapsed time of each 256-session trajectory. Cached input and reasoning output are subsets of input and output; cache-write input is zero. Worker spans, that is, the elapsed times of worker activity, cover the 224 new branch sessions and exclude the warm-up, the intervals between branches, and the estimation after completion. Warm-up spans are 11.57, 11.56, and 9.60 minutes. The seven additional estimations in the AgentCue branch take 0.307, 0.297, and 0.337 seconds per replicate, and estimation at intermediate checkpoints overlaps worker activity. AgentCue uses more tokens and elapsed time and improves accuracy under the common effective-session budget.

B.7 Figure Construction and Choice Record

Figure 4(a,b) reports means and sample standard deviations across R1–R3. Domain accuracies use their question counts; checkpoint accuracies use all 125 questions. Figure 5 jointly aligns R1’s worker and task-type latent positions to the statistical summary at the final checkpoint by orthogonal Procrustes, preserving inner products. Linear axes and insets retain the original scale. Session opening and closing times map to the cumulative number of completed effective sessions. Equal-height stripes mark task types of published work; gray

Table B.6: Computational resources (mean $\pm$ sample standard deviation). Tokens include warm-up; worker spans exclude it. Additional estimation time covers checkpoints 64–256; History only involves no estimation.

| Measure | AgentCue | History only |
|--------------------------------------|-------------------|-------------------|
| Input tokens (millions) | 415.97 ± 2.04 | 390.06 ± 2.93 |
| Cached input (millions) | 392.85 ± 1.91 | 367.76 ± 2.92 |
| Output tokens (millions) | 2.05 ± 0.04 | 1.94 ± 0.04 |
| Reasoning output (millions) | 0.60 ± 0.01 | 0.56 ± 0.02 |
| Post-warm-up worker span (minutes) | 29.81 ± 0.70 | 24.33 ± 3.35 |
| Additional estimation time (seconds) | 0.314 ± 0.021 | – |

marks no publication. The eight statistical summaries are published at their checkpoints.

Figure 4(c) follows one recorded choice in R1. W26 publishes evidence version 1 at 22 completed effective sessions. W15 reads the evidence and the statistical summary at the first checkpoint, computes the four fitted participation means shown, and identifies inference as the task type with the largest fitted participation mean of W15. The recorded reason of W15 also cites adequate evidence published by other workers and the absence of an answer synthesis in progress. W15 uses version 1 of the result of W26 to designate the correct answer B while new sessions remain active, at 32 completed effective sessions. The linked read of the statistical summary, local comparison, recorded choice, and answer illustrate how the statistical summary enters a choice.