

Costly Voting in the Hotelling–Downs Model

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Abstract. We study a partial-participation variation of the Hotelling–Downs model. Voters each have a cost to vote, and only vote when the comparative gain from their preferred candidate exceeds the cost. Under this model the median voter theorem breaks, and we study the extent of polarization under equilibria in different voters and cost distributions. We find that the main predictor of polarization is the reverse-hazard-rate of the cost distribution, indicating that the driver of polarization under our model is the willingness of voters to respond to changes in positions of candidates. We then extend the model by adding parameters governing alienation and candidate competitiveness, showing that our results are robust even when taking into account other realistic factors.

1 Introduction

Two developments mark democratic politics in recent decades: in the United States the two parties’ elected representatives have moved apart [63] and partisans with them [74], while across much of Europe partisans are sharply hostile to one another [76]; and in many democracies fewer citizens turn out to vote [49,37]. The canonical account of two-candidate competition predicts neither.

That account is the *Hotelling–Downs model* [50,28], in which two office-seeking candidates choose policy platforms to maximize vote share. Under full voter participation and single-peaked preferences it yields convergence to the median voter, whose ideal policy defeats any alternative in pairwise majority voting—the *median voter theorem*. Yet divergence, not convergence, is the recurrent empirical finding. In the United States, Ansolabehere et al. [7] and Lee et al. [57] document systematic departures from median convergence in U.S. House elections, while Levitt [59] finds similar patterns in the U.S. Senate, and Bafumi and Herron [10] show that members of Congress tend to adopt positions more ideologically extreme than those of their constituencies.

One key difference between the benchmark Hotelling–Downs model and real-world elections is that the benchmark typically assumes *everyone votes*, whereas in practice, voters may abstain, which makes it increasingly important to incorporate non-participation into models of electoral competition [65]. That the two trends above are connected is not a new conjecture. Calibrating a spatial model with voter abstention to survey data from the U.S. presidential elections of 1980, 1984, 1988, 1996 and 2000, Adams, Merrill and Grofman report that in each of them the major-party candidates had electoral incentives to diverge under their estimated turnout model, with equilibrium divergence increasing in the policy

distance between the two parties’ partisans [4,3]. What has been missing is a characterization of *which* feature of the participation margin—the voters sitting at the edge of turning out—does the work.

In this paper, we extend the Hotelling–Downs framework by introducing costly voting. Inspired by the classic *Calculus of Voting* (see below), voters participate only when the relative utility gain from supporting their preferred candidate exceeds their individual cost of voting; otherwise, they abstain. Turnout therefore depends endogenously on the perceived policy difference between the candidates, and candidates must account not only for ideological proximity but also for the turnout incentives induced by their platform choices. Our model accommodates arbitrary distributions of voter ideal points (density v) and voting costs (CDF K , density k), and adds two one-parameter “dials” to the participation side and the candidate side of the game: voter *alienation* $\alpha \geq 0$, under which participation also declines with a voter’s distance from her own candidate, and candidate *competitiveness* $\beta \in [0, 1]$, which interpolates between maximizing one’s vote share ($\beta = 0$) and maximizing one’s margin of victory ($\beta = 1$). Costly voting is not itself a dial: it is the always-on baseline, carried by the entire cost distribution rather than by a single parameter, and both dials are read against it.

1.1 Our Contribution

We first derive necessary conditions that every pure Nash equilibrium must satisfy. A single *balance principle* asserts that at every equilibrium the marginal voters the candidates gain toward the center and those they lose on their flanks are traded off at a constant rate, one fixed by the candidates’ objective alone and not by the electorate or the cost distribution (Lemmas 7 and 33). Equilibrium takes a particularly simple form when either voter ideal points or voting costs are uniformly distributed: under uniform costs every equilibrium traps a share of the electorate between the platforms that again depends only on that objective, never on the electorate itself (Theorem 9 and Proposition 37). These conditions are not sufficient in general, and we show by example that profiles solving them may fail to be equilibria. We therefore complement them with sufficiency results: for uniformly distributed voters, a single monotonicity condition on the cost distribution, satisfied by every log-concave cost density, under which the equilibrium exists, is unique and is completely characterized, for *every* competitiveness $\beta < 1$ and alienation $\alpha \geq 0$; and for uniformly distributed costs, density-ratio bounds under which the profiles at fixed quantiles of a symmetric electorate (the quartiles in the baseline) are equilibria (Corollaries 11 and 38).

Our central message concerns polarization (Section 3.2). Costly voting alone breaks median convergence: under uniform costs the *quartile* profile of a symmetric electorate (density ratio at most four) is an equilibrium, so *mass polarization* (of the electorate) transmits to *elite polarization* (here, of the candidates) one for one. Fixing the electorate at uniform, we prove a comparative-statics theorem in the reverse-hazard-rate stochastic order: cost distributions that are higher in

this order induce weakly more polarized equilibria, under the monotonicity condition above. We further show this is exactly the right order: the equilibrium correspondence is invariant to rescaling K , hence depends on the cost distribution *only* through its reverse hazard rate k/K (the proportional responsiveness of turnout to the stakes), and in particular the *level* of turnout has no causal role: distributions exist with stochastically higher costs, lower turnout, and strictly less polarization. What polarizes candidates is an electorate whose participation is *responsive* to the stakes, not one whose turnout is merely low.

We then turn each dial. Under alienation (Section 4) we prove that stronger alienation strictly polarizes for a uniform electorate and every log-concave cost density, with the equilibrium separation Δ^* increasing in α and tending to full separation, and that this holds at an unchanged level of turnout, so the effect is not a by-product of lower participation (Corollary 31); that the cost channel survives alienation, so that costlier voting in the reverse-hazard-rate order still polarizes at every α ; and that turnout becomes non-monotone in the platform gap, so the reverse causal channel (extreme candidates depressing participation) reappears. Under competitiveness (Section 5) we prove that competition for the margin is the model’s one de-polarizing force, with a closed-form equilibrium path $\Delta^*(\alpha, \beta)$ in the doubly-uniform benchmark, but that under alienation the margin limit is a knife-edge: for a uniform electorate the $\beta \uparrow 1$ limit selects the *most* polarized point of the margin game’s equilibrium band rather than the median; under alienation and uniform costs the median is a strict local equilibrium of the margin game (no small deviation pays) when the electorate is centrally peaked ($v''(\frac{1}{2}) < 0$) and fails when it is hollow ($v''(\frac{1}{2}) > 0$), a dichotomy settled at fourth order because the second-order term vanishes; and for hollow electorates pure equilibria can fail to exist at all. A computer-assisted certificate proves non-existence at $\alpha = 1$ with uniform costs and an arcsine electorate ($v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$, the extreme hollow case), for every $\beta \geq 0.9168$, while at $\alpha = 0$, under uniform costs, we prove existence for every $\beta < 1$ for symmetric electorates satisfying explicit density bounds, the arcsine electorate among them (Theorem 94). Table 3 collects the resulting map from primitives to polarization.

Methodologically, the non-existence results rest on a certificate scheme for continuous-strategy spatial games: a Lipschitz bound on the payoff, exact quadrature at the grid points, and adaptive refinement of the cells that remain undecided, which together turn a finite computation into a statement about the continuum. The scheme is not specific to this model and may be of use wherever pure equilibria of a continuous game must be ruled out.

A released verification suite recomputes the paper’s numerical claims from the model’s definitions, including both computer-assisted theorems (Theorems 75 and 97): the grid certificates are reproduced to the digits reported, and the exact payoff evaluation they rest on is cross-checked against independent adaptive quadrature.¹ Each figure states in its source how its data were generated.

¹ Available at <https://github.com/ANONYMIZED/costly-voting-hd>; see its **README** for what is and is not covered.

1.2 Related Work

Hotelling–Downs and variants Due to the simplicity and interpretability of the Hotelling–Downs spatial model, a large literature has developed extensions and variations, often with the aim of getting more ‘natural’ equilibria where candidates diverge; Osborne [70] surveys it, and Grofman [41] catalogues the Downsian assumptions—full participation among them—whose relaxation produces divergence. Most extensions retain the assumption that voters prefer the closer candidate but vary the details of the model, e.g., to consider voters’ uncertainty about candidates [52], policy-motivated candidates [84,18], candidates’ uncertainty about voters [78,22], relaxing equilibrium requirements [83], introduce asymmetries between candidate favorability [8] and so on. A separate axis, less explored in this line of work, is the candidates’ *objective*: vote share, margin of victory, and probability of winning coincide under full participation, so the classical model cannot distinguish them [9,30], whereas with abstention they come apart. That is not incidental: the equivalence result of [73] holds precisely when abstention is ruled out, and variable participation is the setting in which Hinich and Ordeshook [48] separated plurality from vote maximization. Section 5 shows how consequential the choice is.

Abstention Voting costs and abstention have received considerable attention since the outset of formal political economy, beginning with the Calculus of Voting framework [77] and the pivotal-voter tradition it launched [71,72,33], continued in the costly-voting models of Börgers [13], Krasa and Polborn [53], Krishna and Morgan [54] and Herrera, Morelli and Palfrey [43], in which, as here, each voter draws a private cost from a distribution, but the alternatives on the ballot are fixed and the questions are turnout, welfare and information aggregation, as well as later “lazy bias” models [25,31]. In all of these, the fundamental assumption is that voters are more likely to participate when *the outcome matters more* and when *they are more pivotal*, and the analysis focuses on the implications for equilibrium outcomes. This assumption is relaxed in [66], where voters consider various heuristics for their pivotality. Other papers, such as [21,67], examine the effects of arbitrary or random participation on electoral outcomes under the Median rule.² However, none of the above models endogenize candidate responses to abstention; instead, candidate positions are taken as fixed. That the median result is fragile once participation is endogenous is an old theme: Hinich, Ledyard and Ordeshook [46] study existence of majority-rule equilibrium when voters may abstain, and Hinich [45] shows that in the probabilistic-voting version the median is an equilibrium only under a knife-edge symmetry: the convergence prediction is, in his phrase, an artifact. The sharpest foil is Ledyard [56], in which platforms and participation are determined together and voters, like ours, carry a private voting cost; but they weigh the platform difference by their probability of being pivotal, and the outcome is convergence. Both candidates adopt the

² In some simulations in [21] the authors sampled more voters closer to the edges of the interval (i.e., assuming ‘extreme’ voters are more active), but without any underlying utility model or formal justification.

same welfare-maximizing platform and, the platforms being identical, nobody votes. Our voters act as if decisive (Section 2.2), and Section 3 shows that convergence then fails, beginning with Lemma 6, which rules out tied equilibria. A classical and distinct driver of abstention is *alienation*—non-participation out of disaffection with even one’s preferred option—introduced into spatial models by Hinich and Ordeshook [47], and since distinguished from abstention out of *indifference* between the candidates [16,6,2]; our parameter α embeds this channel alongside voting costs, and Section 4 shows it is the model’s strongest polarizing force. That alienation-driven abstention can break convergence is known: Callander and Wilson [17] obtain divergent positions in a spatial entry model with alienation, and Oprea, Martin and Brennan [69] argue that compulsory voting de-polarizes precisely by removing extreme voters’ ability to threaten abstention. What is new here is that the threat is priced rather than assumed: the cost distribution K is a primitive, and Section 4 identifies which of its features the equilibrium responds to.

Closest to the present paper is Jones, Sirianni and Fu [51], who examine how a relative cost of voting that deters indifferent voters, third-party entry, and a bimodal electorate each undermine the median voter theorem. Their treatment of the first channel is numerical (candidate positions follow gradient dynamics on a two-Gaussian or empirically estimated electorate, and the relative cost of voting is a single scalar common to all voters), whereas ours is analytic and holds for arbitrary v and K , which is what lets Section 3.2 isolate the reverse hazard rate as the only feature of the cost distribution the equilibrium responds to.

Some variations of the Hotelling–Downs model explicitly incorporate the possibility of voter abstention. In [34,82] voters abstain whenever all candidates lie outside a fixed *attraction interval*, so participation does not decline gradually as candidates become less attractive; Cohen and Peleg [20] instead draw each voter’s tolerance range at random from a density, so that the share of voters who still turn out falls gradually with the distance to the nearest candidate, the nearest-candidate analogue of the cost draw used here. Coughlin [22] considers probabilistic attraction models with a finite set of voters, in which each voter may abstain according to an independent probability function. The attraction-interval models condition participation on the distance to the nearest candidate alone, which is natural in *economic competition*, where a consumer’s decision to buy depends only on the best offer available, but not in *political competition*, where a voter’s stake is the difference between the platforms. Models in which participation responds to that difference do exist: the indifference-based abstention models cited above [6,2], the pivotal model of Ledyard [56], the numerical study of Jones et al. [51], and a recent workshop paper whose voters are prospect-theoretic and whose focus is the computational complexity of optimizing a candidate’s position [19]. Two papers in political economy come closer still. In Llavador [62] voters weigh both platforms against a cost of voting that grows with the distance to the platform they support (the alienation channel of Section 4), but his parties are policy-motivated, so there the alienation cost is a moderating force that keeps them from radicalizing, whereas here it acts on

vote-share maximizers who would otherwise converge, and pushes them apart. In Herrera, Levine and Martinelli [42] turnout is likewise endogenous to the platforms, but through the parties' mobilization spending rather than through any voter's own cost-benefit comparison. What distinguishes the present model is the combination: participation responds to the platform difference; the threshold is drawn from a full cost distribution K that enters the analysis as a primitive; voters weigh the platform difference directly against their cost, with no pivot-probability discount (Section 2.2); and the analysis is carried out for arbitrary v and K rather than a fixed electorate.

Candidate polarization The distinction between *mass* and *elite* polarization drawn above is a live one in political science; one prominent debate is on which type of polarization precedes, or even causes, the other [44,85,1,35,32].

Neither trend with which this paper opened is universal: four-decade trends across OECD democracies are mixed, with increases in some countries and decreases in others [14]. Two recent studies speak directly to the channel studied here. Across eleven Western European countries (1977–2016), Dreyer and Bauer [29] find that parties adopt more extreme positions as their electorates polarize, and that the response runs through abstention: it is strongest where turnout is low and fades as turnout rises. Al Yusef [5] finds, across Dutch municipalities, that the vote is more polarized where turnout is lower, consistent with centrist voters abstaining disproportionately. Both studies measure the abstention margin by the turnout rate; Section 3.2 shows that the model separates the rate from the responsiveness of participation to the stakes, and that only the latter moves the equilibrium.

Prior theoretical work in computational social choice has modeled polarization in a variety of settings, such as committee selection [27], which sometimes require complex definitions to measure. In the context of 2-candidate spatial competition, measuring polarization is trivial, and so we can isolate and test the effect of each component of the model: the voters' distribution, the voting costs, alienation, and the candidates' objective.

Core versus swing The distributive-politics literature asks when a party should reward its core and when it should woo the center, with transfers rather than platforms as the instrument [24,60,26,40]; there a group is “core” because a party can deliver to it more effectively, a driver absent from our model, whose candidates hold one public, non-discriminating instrument. Within this literature the nearest to our question is Cox [23], who adds turnout to the targeting problem: the most valuable voter supports you whenever she votes, and her participation responds to what you offer. Both features describe the marginal voters on a candidate's outer flank, and his comparative static is the crossing of Corollary 17: the more mobilization and the less persuasion a party believes possible, the more it concentrates on its core. Our balance condition prices the trade-off between the two flanks at a rate that rises with competitiveness and falls with alienation (Lemma 33), a price no full-participation model can carry (Remark 5).

The two dials Each extension answers one of the gaps above: the α -dial lets participation fall with the distance to a voter’s *own* candidate, the difference between economic and political competition noted above, and the β -dial makes the candidates’ objective a parameter rather than a convention. Neither ingredient is new on its own; what the present model adds is that both are read against an arbitrary cost distribution, so the effect of each can be separated from the effect of costly voting itself.

Road map Section 2 sets out the model; Sections 3 to 5 analyze the baseline and the two extensions, each section moving from characterizations to their consequences for polarization, which in Sections 4 and 5 come in two steps: first at a fixed value of the parameter, then as it varies. Section 5 closes with the margin game $\beta = 1$, a special case; Section 6 concludes. Proofs are deferred to Appendix A and the subsequent appendices. A reader with limited time can take Sections 3.2, 4.3 and 5.3, the subsections on what each force does to polarization, together with Table 3; the subsections preceding each supply what they rest on.

2 Model

Section 2.1 states the classical Hotelling–Downs game and the median prediction it delivers, and isolates the assumption our model gives up. Section 2.2 gives it up: participation becomes a cost–benefit decision taken voter by voter, which is the paper’s one substantive departure and is in force everywhere in it. Sections 2.3 and 2.4 fix the candidates’ objective, the equilibrium notions and a running example; the game assembled by then is the one analyzed in Section 3. Section 2.5 adds the two extensions—alienation among voters, competitiveness among candidates—each governed by its own parameter, each switched off in the model presented first, and each taken up in a later section (Sections 4 and 5). Table 1 summarizes the notation.

2.1 The Hotelling–Downs Benchmark

In the classical model [50,28], every voter votes for the candidate nearest to her ideal point, and each candidate chooses a platform to maximize the number of votes she receives; if both candidates choose the same platform, they split the electorate evenly. With voters uniformly distributed on $[0, 1]$, the unique equilibrium famously places both candidates at the median, $\frac{1}{2}$. This is a special case of a more general statement. Let V be the CDF of an arbitrary continuous distribution of voter ideal points on $[0, 1]$, and let candidate $i \in \{1, 2\}$ choose a platform $c_i \in [0, 1]$, so that a *profile* is a pair (c_1, c_2) .

Claim 1 (See [12,28]). *The only equilibria of the Hotelling–Downs game are the profiles (c_1, c_2) with $V(c_1) = V(c_2) = \frac{1}{2}$, i.e., both candidates at a median of V .*

Table 1. Notation of the model. The last block belongs to the extensions of Section 2.5; the model of Sections 2.2 and 2.3 sets $\alpha = \beta = 0$. Shorthands introduced later, in the analysis, are collected separately in Table 2.

<i>Primitives and parameters</i>	
$V, v; K, k$	CDF/density of voter positions; of voting costs κ (on $[0, 1]$)
ρ_v	density ratio $\sup v / \inf v$ (electorate evenness)
<i>Profiles and induced behavior</i>	
$c_i; d_i(x) := x - c_i $	platform of candidate $i \in \{1, 2\}$; distance to voter x
$c_1 \leq c_2; \Delta; m$	labeling; separation $c_2 - c_1$; midpoint $\frac{c_1 + c_2}{2}$
$B_i(x); p_i(x)$	benefit of voting for i : $d_{-i} - d_i$, Eq. (1); share of the voters at x who vote for i , Eq. (2)
$u_i; T$	vote share $\int_0^1 p_i v dx$; turnout $u_1 + u_2$
$\bar{O}_i; \bar{I}_i$	votes from i 's flank; from her half of the contested middle, Eq. (3)
Δ^*	equilibrium separation
<i>Decorations</i>	
$^*, \sim$	equilibrium value; the comparison primitive of a stochastic order
<i>Extension parameters (Section 2.5)</i>	
$\alpha \geq 0$	alienation: benefit becomes $d_{-i} - (1 + \alpha)d_i$, Eq. (4)
$\beta \in [0, 1]; w_i$	competitiveness; margin $u_i - u_{-i}$
$U_i = u_i - \beta u_{-i}$	candidate i 's objective, Eq. (5)
$\Delta^*(\alpha); \Delta^*(\alpha, \beta)$	equilibrium separation, when the dependence matters

Claim 1 is robust to how a voter's preference over the two platforms is modeled, so long as she votes. Let her payoff depend on *both* platforms through the gain $d_{-i} - d_i$ introduced in Section 2.2 below, rather than on the nearer one alone: with full participation she still votes for the candidate nearer to her, the induced vote shares are the classical ones, and the equilibrium set is unchanged. What breaks the claim is therefore not the utility model but the decision of whether to vote at all. The remainder of this section replaces the assumption of full participation with an explicit cost–benefit decision by each voter.

2.2 Voters, Costs, and Participation

Positions and costs There is a continuum of voters with positions $x \in [0, 1]$, distributed according to a continuous distribution with CDF V and density v . In addition to her position, each voter draws an idiosyncratic voting cost $\kappa \in [0, 1]$, i.i.d. across voters according to a continuous distribution with CDF K and density k , independently of her position.³ The following mild conditions

³ Independence reflects the interpretation of κ as the logistics of voting (registration, travel, forgone time), which are plausibly unrelated to ideology. Restricting the sup-

are assumed throughout the paper; we note explicitly whenever one of them can be relaxed.⁴

Assumption 2 (Standing assumptions). *The densities v and k are continuous and strictly positive on $(0, 1)$, $K(0) = 0$, and $K(1) \in (0, 1]$. A defect $1 - K(1) > 0$ is the mass of voters whose cost exceeds every attainable benefit (the never-voters of Proposition 15), so K need not be a proper distribution on $[0, 1]$.*

The participation rule Given a profile of platforms (c_1, c_2) , write $d_i(x) := |x - c_i|$ for the distance between a voter at x and candidate i . The voter’s *benefit* from voting for candidate i is her preferred candidate’s distance advantage, how much better off she is if that candidate wins rather than the opponent:

$$B_i(x) := d_{-i}(x) - d_i(x), \quad (1)$$

and she votes for candidate i if $B_i(x) \geq \kappa$, abstaining if neither candidate clears her threshold. Since $B_i(x) > 0$ requires x to be strictly closer to c_i than to c_{-i} , at most one candidate can offer a positive benefit, so the rule assigns (almost) every voter at most one vote.

We emphasize that the benefit (1) compares the platforms directly, rather than weighting the comparison by the probability of being pivotal as in the *Calculus of Voting* [77] and the pivotal-voter models that formalize it [72]. Voters thus behave as if decisive, an assumption in the spirit of expressive [15] or boundedly rational voting, appropriate when voters do not know the positions of other voters or cannot assess their own pivotality [66].

The departure is narrower than it looks. A common pivot probability ϖ that does not depend on the platforms is absorbed into the cost distribution: voting when $\varpi B_i(x) \geq \kappa$ is voting when $B_i(x) \geq \kappa/\varpi$, which is the same game with costs $t \mapsto K(\varpi t)$. Proposition 28 shows what that reparametrization does: nothing at all when K is a power law, uniform costs included, and otherwise, for a uniform electorate, a move whose direction is set by the elasticity of K . What the model genuinely assumes away is a pivot probability that responds to the platforms themselves.

Participation shares Aggregating over costs, the share of voters at position x who vote for candidate i is

$$p_i(x) = \begin{cases} K(B_i(x)), & \text{if } B_i(x) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

port to $[0, 1]$ is essentially without loss of generality: all distances lie in $[0, 1]$, so the benefit from voting defined below never exceeds 1, voters with $\kappa > 1$ never participate, and only the restriction of K to $[0, 1]$ matters.

⁴ We reserve the term *regularity* for its usual meaning, monotonicity of the virtual cost (6); that is a hypothesis of Theorem 13(b), not a standing assumption.

Fig. 1 illustrates the mechanics. On candidate 1’s side of the midpoint, B_1 is a piecewise-linear “tent”: its *outer arm*, facing away from the opponent, is flat at the full gain $\Delta := c_2 - c_1$ —every voter on the far side of c_1 enjoys the same advantage—and its *inner arm*, facing the opponent, falls at slope 2 from the peak at c_1 to the midpoint $m := \frac{c_1+c_2}{2}$, where it vanishes; the mirror statement holds for candidate 2. Letting voting costs tend to zero recovers the vote shares of the classical model for any profile with $c_1 \neq c_2$.

Flank and contested middle The arms are pieces of the benefit function; the voters standing under them are worth naming separately, because every first-order condition in the paper is a statement about those two groups. Since $B_1 < 0$ beyond the midpoint, candidate 1 draws her votes entirely from $[0, m]$, which her own platform splits in two: her *flank* $[0, c_1]$, the voters standing behind her, and her half $[c_1, m]$ of the *contested middle* $[c_1, c_2]$, the voters lying between the two platforms. Write

$$\bar{O}_1 := \int_0^{c_1} K(B_1(x))v(x) dx = K(\Delta)V(c_1), \quad \bar{I}_1 := \int_{c_1}^m K(B_1(x))v(x) dx \quad (3)$$

for the votes each group yields, so that $u_1 = \bar{O}_1 + \bar{I}_1$; the closed form for $\bar{O}_1$ is available because the outer arm is flat, so every voter on the flank faces the same benefit Δ . The mirror definitions serve candidate 2. Note that B_1 is continuous at c_1 , taking the value Δ from both sides: nothing distinguishes the outermost voter of the middle from the innermost voter of the flank. That is what makes the split pure bookkeeping, and it is what Lemma 7 turns on.

The tie-breaking convention, however, differs from the classical one: at $c_1 = c_2$ every voter’s benefit is 0, so by $K(0) = 0$ (Assumption 2) (almost) every voter abstains and both candidates receive zero votes, whereas the classical model splits the electorate evenly. This discrepancy is immaterial for our analysis: as we show below (Lemma 6), tied profiles are never equilibria of the model just described. The one exception arises in the margin limit of the second extension, where they survive as zero-turnout, zero-margin equilibria (Proposition 46).

2.3 Candidates and Objectives

Two candidates, indexed by $i \in \{1, 2\}$, simultaneously choose platforms $c_i \in [0, 1]$ in a one-shot game. Candidates know the distributions V and K (but not the realized cost of any individual voter). Candidate i ’s *vote share* and the resulting *turnout* are

$$u_i(c_1, c_2) = \int_0^1 p_i(x) v(x) dx, \quad T = u_1 + u_2 \leq 1.$$

Each candidate maximizes her vote share u_i . This is the natural objective when raw support matters in itself (think of public funding, perceived legitimacy, or proportional systems), and it is also the objective of the classical candidate of Section 2.1, so the only thing that has changed so far is that voters may now stay home.

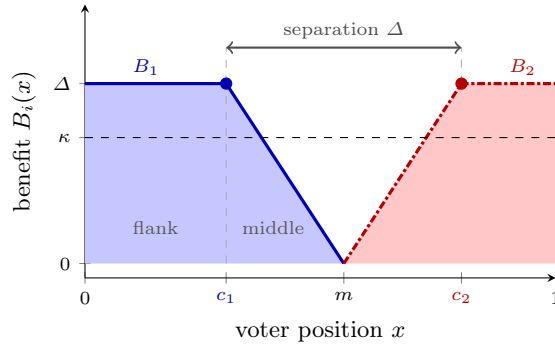


Fig. 1. The benefit of voting at the profile $(c_1, c_2) = (0.3, 0.8)$. A voter at x whose cost is κ votes for candidate i when $B_i(x)$ rises above the dashed line; aggregating over costs, a share $p_i(x) = K(B_i(x)^+)$ of the voters at x vote for i . The separation $\Delta = c_2 - c_1$ appears twice over, as the gap between the platforms and as the height of each tent, because a voter standing behind her own candidate gains exactly Δ by voting. The dashed vertical marks the partition of (3): candidate 1’s own platform splits the voters she draws from into her flank and her half of the contested middle, and candidate 2 mirrors it. Under uniform costs p_i is the positive part of B_i , so with uniform voters the shaded areas are the vote shares, here $(u_1, u_2) = (0.2125, 0.1625)$, and the two labelled pieces of the blue area are $\bar{O}_1$ and $\bar{I}_1$.

2.4 Equilibrium and a Running Example

We study pure Nash equilibria of the simultaneous-move game with payoffs (u_1, u_2) . The game is symmetric, so equilibria come in mirror pairs; when describing a profile we label the candidates so that $c_1 \leq c_2$, and write $\Delta = c_2 - c_1$ for their *separation* (our measure of candidate polarization) and $m = \frac{c_1 + c_2}{2}$ for the midpoint.

Three nested notions of a profile recur throughout, and the gaps between them carry real content, so we fix them here.

Definition 3 (Stationary, local, and Nash). *Fix a profile with $0 < c_1 < c_2 < 1$ and write π_i for candidate i ’s payoff, which is u_i here and U_i once Section 2.5 is in play. The profile is stationary if $\partial \pi_i / \partial c_i = 0$ for both candidates; a local equilibrium if each c_i is a local maximum of $\pi_i(\cdot, c_{-i})$, and a strict one if that maximum is strict; and a Nash equilibrium if each c_i is a global maximum of $\pi_i(\cdot, c_{-i})$ on $[0, 1]$.*

Every interior Nash equilibrium is a local equilibrium, and, the payoffs being differentiable at interior profiles (Lemma 53), every interior local equilibrium is stationary. Neither converse holds, and the paper exhibits both failures: a stationary profile can be a local *minimum* of the deviator’s payoff (Remark 71), and a local equilibrium can lose to a distant deviation to a different local maximum (Appendix F). That is why every characterization in this paper comes in two

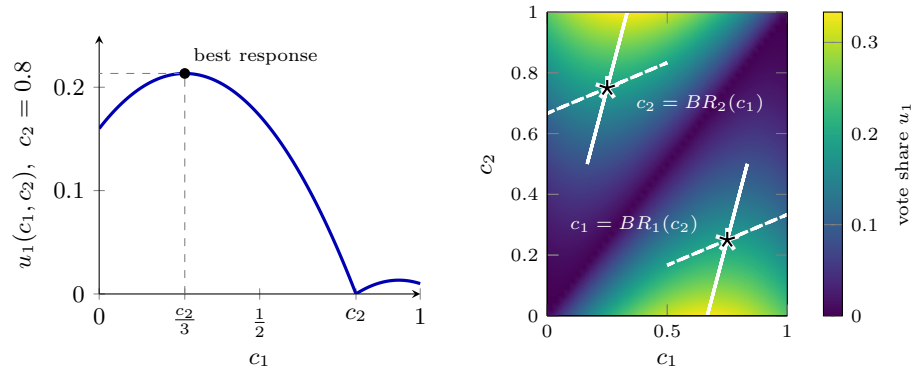


Fig. 2. The candidates' problem in the doubly-uniform environment ($x, \kappa \sim U(0, 1)$). *Left:* candidate 1's vote share against $c_2 = 0.8$; the best response $c_1 = c_2/3$ balances proximity to more voters against the turnout gained from separation (the kink is at $c_1 = c_2$). *Right:* heatmap of u_1 over all profiles, with the best-response curves of candidate 1 (solid) and candidate 2 (dashed); their intersections (stars) are the two equilibria $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{3}{4}, \frac{1}{4})$.

parts: conditions every equilibrium must satisfy, and distributional conditions under which a profile meeting them is an equilibrium.

As a running illustration we use the *doubly-uniform environment*, in which voter positions and voting costs are both uniform on $[0, 1]$: $x, \kappa \sim U(0, 1)$, that is $v \equiv 1$ and $K(t) = t$. It returns in every section that follows, where we also call it the doubly-uniform *benchmark*. Fig. 1 depicts the profile $(c_1, c_2) = (0.3, 0.8)$, under which $u_1 = 0.2125$ and $u_2 = 0.1625$. Fixing $c_2 = 0.8$, the left panel of Fig. 2 plots candidate 1's vote share as a function of her own platform: her best response $c_1 = c_2/3 \simeq 0.267$ trades off a larger flank of nearby voters (pushing her toward the opponent) against the higher turnout generated by a wider separation (pushing her away). The right panel shows the full strategy space: the two best-response curves intersect exactly twice, at $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{3}{4}, \frac{1}{4})$, which we later show (Corollary 14) to be the only equilibria in this environment. Intuitively, when the candidates are too close to each other, many voters abstain, driving the candidates apart, and when they are too far the mass of voters between them becomes worth chasing.

2.5 Extensions: Alienation and Competitiveness

The model above—costly voting, and nothing else—is the model analyzed in Section 3. We now add two further ingredients. *Alienation*, governed by a parameter $\alpha \geq 0$, changes the voters: participation may decline not only with a small benefit of voting, but also with the voter's distance from her *own* candidate; Section 4 switches it on. *Competitiveness*, governed by a parameter $\beta \in [0, 1]$, changes the candidates: they value their own votes but may also discount votes cast for the

opponent, interpolating between pure vote-share maximization ($\beta = 0$) and pure margin maximization ($\beta = 1$); Section 5 switches it on. Setting $(\alpha, \beta) = (0, 0)$ returns the model of Sections 2.2 and 2.3 verbatim.

Alienation Alienation is a classic driver of abstention in spatial models [47]: voters who are ideologically distant from all candidates are less inclined to turn out, even when their gain is unchanged. We capture it by penalizing the distance to a voter’s own candidate inside the benefit, so that Eq. (1) becomes

$$B_i(x) := d_{-i}(x) - (1 + \alpha) d_i(x), \quad \alpha \geq 0, \quad (4)$$

which is Eq. (1) at $\alpha = 0$. The definition is a convex combination of the two natural ways to value the outcome of an election. One is *relative*: a voter cares how much better one candidate is than the other, $d_{-i} - d_i$, the gain of Eq. (1), which makes her abstain when the candidates are too alike for the choice to matter. The other is *absolute*: she cares about the platform she actually helps elect, $-d_i$, which makes distance from her own candidate a reason to stay home—the alienation of [47]. A voter who puts weight $1 - \zeta$ on the first view and $\zeta \in [0, 1)$ on the second has

$$(1 - \zeta)(d_{-i} - d_i) + \zeta(-d_i) = (1 - \zeta)\left(d_{-i} - \frac{1}{1 - \zeta} d_i\right) = (1 - \zeta) B_i$$

for $\alpha = \zeta/(1 - \zeta)$, and the positive factor $1 - \zeta$ is absorbed into the cost distribution exactly as the pivot probability is in Section 2.2. So α is not an ad hoc penalty. It indexes the mix between the two views, from the purely relative voter of Section 3 at $\alpha = 0$ toward the purely absolute one as $\alpha \rightarrow \infty$; the weight is written ζ because a convex weight lies in $[0, 1)$ while α ranges over $[0, \infty)$. The endpoint $\zeta = 1$ is excluded, and necessarily so: there $B_i = -d_i \leq 0$, so with non-negative costs nobody ever votes and the model is empty. Every $\alpha \geq 0$ therefore keeps some weight on the comparison between the candidates.

Everything else in Section 2.2 is stated in terms of B_i and therefore holds verbatim for every $\alpha \geq 0$: the voting rule $B_i(x) \geq \kappa$, the argument that at most one candidate can offer a positive benefit, and the participation shares (2). The definition also has a compact reading: $B_i(x) \geq \kappa$ iff $d_{-i}(x) - d_i(x) \geq \kappa + \alpha d_i(x)$, so alienation inflates the effective cost of voting by a fraction of the distance to one’s own candidate.

Geometrically, alienation tilts the benefit tent of Fig. 1. On candidate 1’s side of the midpoint, B_1 now rises at slope α toward the platform, still peaks at height Δ at c_1 , and falls at slope $2 + \alpha$ to the midpoint m , where it equals $-\alpha\Delta/2$; the mirror statement holds for candidate 2. The outer arm is therefore no longer flat, so participation falls fastest among voters far from their own candidate, and a “dead zone” of full abstention opens around the midpoint. Fig. 3 places the two cases side by side. The tie-breaking convention of Section 2.2 is unaffected: at $c_1 = c_2$ the benefit is $-\alpha d_i(x) \leq 0$, so (almost) every voter abstains a fortiori.

Remark 4. Under full participation the parameter α would be irrelevant. A voter prefers candidate i exactly when $B_i > B_{-i}$, that is when $d_{-i} - (1 + \alpha)d_i >$

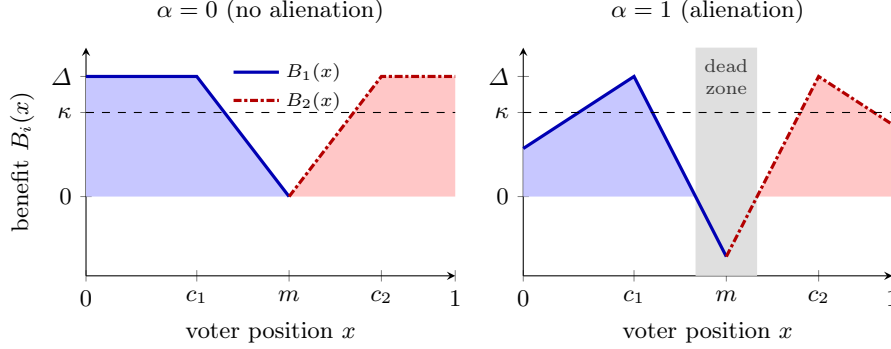


Fig. 3. Alienation reshapes the benefit of voting; the profile is $(c_1, c_2) = (0.3, 0.8)$, as in Fig. 1. *Left:* $\alpha = 0$, the model of Section 2.2, reproduced here for comparison. *Right:* $\alpha = 1$. Alienation tilts the outer arms (slope α instead of 0), steepens the inner arms (slope $2 + \alpha$ instead of 2), and opens a dead zone of abstention around the midpoint m . Under uniform costs and uniform voters the shaded areas are the vote shares: $(u_1, u_2) = (0.2125, 0.1625)$ on the left and $(0.147, 0.122)$ on the right.

$d_i - (1 + \alpha)d_{-i}$, i.e. when $(2 + \alpha)d_{-i} > (2 + \alpha)d_i$: the common factor cancels and she votes for the nearer candidate at every $\alpha \geq 0$. If $T \equiv 1$ the induced vote shares are therefore the classical ones and Claim 1 is untouched, median equilibrium included. Alienation is in this respect exactly like competitiveness (Remark 5): it has no bite of its own, and acts only through the participation decision. Everything in Section 4 is accordingly attributable to abstention.

Competitiveness Vote share is not the only objective a candidate might have: the *margin* $w_i := u_i - u_{-i}$ is the natural objective when only victory and the size of the mandate matter. We span both with a single competitiveness parameter $\beta \in [0, 1]$: candidate i values her own votes at 1 and the opponent's at $-\beta$,

$$U_i = u_i - \beta u_{-i} = (1 - \beta)u_i + \beta w_i = \frac{1 - \beta}{2}T + \frac{1 + \beta}{2}w_i. \quad (5)$$

The middle expression is the most direct reading of the parameter: U_i is the convex combination of vote share and margin that puts weight β on the margin. The one on the right shows that U_i is equally a weighted combination of turnout, which the candidates share, and margin, over which they are in pure conflict. At $\beta = 0$ we have $U_i = u_i$ and we are back in the model of Section 2.3; at $\beta = 1$ the common turnout term drops out, the game is zero-sum, and candidates are margin maximizers. With either extension in play, the equilibrium notion of Section 2.4 is applied to the game with payoffs (U_1, U_2) , which we call the β -game; its two endpoints are the *vote-share game* ($\beta = 0$) and the *margin game* ($\beta = 1$). We write $\Delta^*(\alpha)$ or $\Delta^*(\alpha, \beta)$ when the dependence of the equilibrium separation on α and β matters.

The reader can see how different parameters affect the turnout along the voter axis using the artifact: <https://claude.ai/artifact/2Ggrf3obhnSywKHsyEs2BE>

Remark 5. Under full participation the parameter β would be irrelevant: if $T \equiv 1$ then $U_i = (1 + \beta)u_i - \beta$ is a positive affine transformation of u_i , so all $\beta \in [0, 1]$ induce the same game. In particular, vote-share and margin maximization coincide in the classical model. It is precisely abstention, which makes turnout $T(c_1, c_2)$ respond to the platforms, that separates the objectives; how much they diverge is the subject of Section 5.

3 Costly Voting

We begin with the baseline model: voters weigh only the gain from voting against its cost, and candidates maximize vote share. Both extensions of Section 2.5 are off throughout this section ($\alpha = \beta = 0$, so that $B_i = d_{-i} - d_i$ and $U_i = u_i$), and every statement below is to be read under that convention, without further mention. This section establishes the pattern that the two extensions will repeat. Section 3.1 characterizes equilibria, starting from a single balance principle, stated here in its baseline form and extended in turn by each parameter, and specializing along a ladder of increasingly explicit cases: uniform costs, uniform voters, and both at once, which is the doubly-uniform environment of Section 2.4 that runs through the paper as its simplest worked example. Section 3.2 then extracts the paper’s central message: costly voting alone breaks median convergence, and it does so through two separate channels—the shape of the electorate and the shape of the cost distribution—while the *level* of turnout plays no role at all. Table 2 collects the shorthands used from here on.

3.1 Equilibrium Characterization

We first record that equilibria can involve neither ties nor boundary positions.

Lemma 6 (Interiority). *Under Assumption 2, every pure Nash equilibrium (c_1, c_2) satisfies $c_1 \neq c_2$ and, labeling the candidates so that $c_1 < c_2$, we have $0 < c_1 < c_2 < 1$.*

Proof in Appendix B.

One balance principle All our characterizations flow from a single first-order condition, which we state here in its baseline form and sharpen once in each of the two sections that follow. It is what the partition (3) gives on differentiation. Define the *marginal masses* I_i and O_i on the two arms: for candidate 1,

$$I_1 := \int_{c_1}^m k(B_1(x)) v(x) dx, \quad O_1 := \int_0^{c_1} k(B_1(x)) v(x) dx = k(\Delta)V(c_1),$$

with the mirror-image definitions for candidate 2, and $I := I_1 + I_2$, $O := O_1 + O_2$. The voter at x contributes $k(B_i(x))v(x)$: the density of voters at x who are exactly indifferent between voting and abstaining, and whom a marginal platform move therefore gains or loses.

Table 2. Shorthands introduced in the analysis, collected for reference; each is defined where it first appears. The second block is used only from Section 4 on. The symbols of the model itself are in Table 1.

<i>Analysis shorthands</i>	
$r := k/K$; $G(t) := \int_0^t K$	reverse hazard rate of K (Corollary 17); integrated CDF
ψ ; ψ_β	virtual cost $t + K/k$, Eq. (6); its β -weighting $t + \frac{1+\beta}{1-\beta} K/k$, Theorem 35
I_i, O_i	marginal voter mass on i 's inner/outer arm, Lemma 7
σ ; $\eta(u) = u k(u)/K(u)$	electorate spread; elasticity of K , Proposition 28
$\bar{\Delta}(\alpha)$	turnout-maximizing separation, Proposition 29
c_{med} ; $w(c, c')$	median $V^{-1}(\frac{1}{2})$; margin of a candidate at c against one at c' , Theorem 45 and Proposition 49
<i>Geometry of the benefit tent under alienation (Fig. 6)</i>	
$\lambda := 2 + \alpha$	inner-arm slope (outer arms have slope α)
x_i^{in} ; x_i^{out}	inner and outer zero of candidate i 's tent, where her benefit reaches 0
b, b_1, b_2	boundary benefits: B_i at the nearer edge of $[0, 1]$, Eq. (9)
$\Delta_0 := \alpha/\lambda$	maximal separation inside the margin band $E(\alpha)$, Proposition 46
$E(\alpha)$	margin band: the set of pure equilibria at $\beta = 1$, Proposition 46

Moving c_1 acts on $\bar{O}_1$ and $\bar{I}_1$ in two ways at once. The benefits inside each region change: on the flank $B_1 \equiv \Delta$ falls at unit rate, costing O_1 , while in the middle $B_1 = c_1 + c_2 - 2x$ rises at unit rate, earning I_1 . And the boundary between the regions moves with her, so voters just behind c_1 pass from the middle into the flank. That second effect is the one that cancels. Each transferred voter carries the same benefit Δ on either side of the boundary, so she adds $K(\Delta)v(c_1)$ to the flank and removes exactly as much from the middle:

$$\frac{\partial \bar{O}_1}{\partial c_1} = K(\Delta)v(c_1) - O_1, \quad \frac{\partial \bar{I}_1}{\partial c_1} = I_1 - K(\Delta)v(c_1), \quad \text{so} \quad \frac{\partial u_1}{\partial c_1} = I_1 - O_1.$$

Neither I_1 nor O_1 is by itself the derivative of the region it names; each is the part of that derivative due to benefits changing, and the parts due to the partition shifting are equal and opposite. Setting the sum to zero gives the lemma.

Lemma 7 (Balance conditions). *Let v, k satisfy Assumption 2. Every interior stationary profile (c_1, c_2) (Definition 3) satisfies, for each candidate i ,*

$$I_i = O_i,$$

and in aggregate $I = O$.

Proof: the baseline case of Proposition 82.

In aggregate, exactly half of the total marginal mass sits on the inner arms, for every v and k . Neither extension changes what is being balanced, only the rate at which the two arms exchange: alienation reshapes the arms themselves (Proposition 22) and competitiveness re-prices them, and once both are in play the two effects combine into a single statement, Lemma 33. The rest of this subsection specializes this balance one distribution at a time: first uniform costs against an arbitrary electorate, then a uniform electorate against arbitrary costs, then both at once.⁵

Remark 8 (Existence). Payoffs are continuous on $[0, 1]^2$ (including at ties, where vote shares vanish), so a possibly-mixed equilibrium always exists by Glicksberg’s fixed-point theorem [39]. Pure existence is subtler. It holds in the benchmark settings of this section, and in each extension:

- uniform voters, with K regular (Theorem 13);
- uniform costs, with a symmetric electorate of density ratio at most four (Corollary 11);
- a non-increasing cost density with $\rho_k := k(0^+)/k(1^-) < \infty$ and a symmetric electorate whose density ratio ρ_v (Table 1) satisfies $\rho_v \rho_k \leq 4$ (Proposition 59);
- under alienation (Theorem 24 and Corollary 58) and under competitiveness (Theorems 35 and 94), on the analogous classes.

Between the two lies a third statement, weaker than pure existence and available far more widely: a *stationary* profile—one solving the first-order conditions—exists by an intermediate-value argument along the symmetric diagonal, under no sufficiency hypothesis at all. For uniform voters and every K this is the boundary behaviour of Ψ in the proof of Theorem 24, which uses no super-regularity, so Eq. (12) always has a root. For uniform costs and a symmetric electorate, Proposition 89 gives the same conclusion for every $\alpha > 0$ and every $\beta < 1$: the own-side derivative on the diagonal is $(1 + \alpha + \beta)V(1/\lambda) > 0$ at $c_1 = 0^+$ and negative just left of $c_1 = \frac{1}{2}$. What these arguments do not deliver is that the root is a *best response*, and closing that gap is exactly what the density-ratio umbrellas above do. We conjecture that a pure equilibrium always exists in the baseline ($\alpha = \beta = 0$) under Assumption 2. The case the list leaves conspicuously open is uniform costs with $\rho_v > 4$, where we have neither a proof nor a counterexample: the four-cluster electorate of Appendix F has $\rho_v = 120$ and does possess equilibria, even though not every profile solving the conditions is one. We expect existence to survive alienation as well, at $\beta = 0$, but on thinner evidence, since the range of density ratios covered by Corollary 58 shrinks to nothing as α grows. Section 5 shows that pure existence can fail once candidates maximize the margin instead.

⁵ Each specialization gives conditions that are necessary but not sufficient, in the sense of Definition 3; Appendices C and F supply the two examples promised there.

Uniform costs Assume first that costs are uniform while voters are arbitrary. With $k \equiv 1$, the balance $I_i = O_i$ turns into a statement about voter mass alone.

Theorem 9 (Uniform costs: every equilibrium traps half the electorate). *Let $\kappa \sim U(0, 1)$ and let v satisfy Assumption 2. Then every Nash equilibrium (c_1, c_2) satisfies*

$$V(c_1) = V(c_2) - \frac{1}{2} = \frac{1}{2} V(m) :$$

the candidates trap exactly half of the electorate between them, and each candidate's flank holds exactly half the mass on her side of the midpoint.

Proof in Appendix B.

The one half is specific to vote-share maximization, but the law is not. Under the competitive objective $U_i = u_i - \beta u_{-i}$ of Eq. (5), with alienation still off, the same computation traps exactly $\frac{1-\beta}{2}$ of the electorate, again whatever the electorate (Proposition 37).

These conditions are not sufficient in general, but they are whenever the electorate is not too uneven. Let $\rho_v := \sup_{(0,1)} v / \inf_{(0,1)} v$ denote the density ratio.

Proposition 10 (Sufficiency under uniform costs). *Let $\kappa \sim U(0, 1)$ and $\rho_v \leq 4$. Then each candidate's payoff is concave in her own position on each side of the opponent, so a profile $0 < c_1 < c_2 < 1$ is a Nash equilibrium if and only if it satisfies Theorem 9 and neither candidate gains by a cross-over deviation, a move to the far side of her opponent. The cross-over condition is one inequality per candidate, comparing her equilibrium payoff with the best she could get on the far side.*

Proof in Appendix B.

Corollary 11 (Quartile equilibria). *Let $\kappa \sim U(0, 1)$ and $\rho_v \leq 4$. Any profile with $V(c_1) = \frac{1}{4}$, $V(c_2) = \frac{3}{4}$ and $V(m) = \frac{1}{2}$ is a Nash equilibrium. If v is symmetric about $\frac{1}{2}$, the quartile profile $(V^{-1}(\frac{1}{4}), V^{-1}(\frac{3}{4}))$ satisfies all three conditions automatically (so an equilibrium exists) and is the only profile satisfying them.*

Proof in Appendix B.

Competitiveness generalizes the quartiles to quantiles: Corollary 38 puts the candidates at $V^{-1}(\frac{1+\beta}{4})$ and $V^{-1}(\frac{3-\beta}{4})$, which are the quartiles at $\beta = 0$ and collapse to the median as $\beta \uparrow 1$.

For skewed electorates the equilibrium detaches from any fixed quantiles: for $v = \text{Beta}(1.5, 4)$ the conditions of Theorem 9 pin down $(c_1, c_2) \approx (0.149, 0.413)$, an asymmetric profile with $V(c_1) \approx 0.29$ and $V(c_2) \approx 0.79$, which we verified numerically to be an equilibrium; a reminder, too, that the bound $\rho_v \leq 4$ is sufficient but far from necessary (this density violates it). Nor need the solution be unique: Appendix E exhibits a *symmetric* electorate with three distinct equilibria, two of them asymmetric mirror images, with $\rho_v = 3$: inside the sufficiency range $\rho_v \leq 4$ of Proposition 10 the quartile profile is an equilibrium, but not necessarily the only one.

Uniform voters Now let voters be uniform while costs are arbitrary. This is our main result, and the characterization here is complete: a single monotonicity condition on the cost distribution delivers existence *and* uniqueness, and pins the equilibrium separation to the root of one scalar equation. That condition is monotonicity of the *virtual cost*

$$\psi(t) := t + \frac{K(t)}{k(t)}, \quad t \in (0, 1), \quad (6)$$

the cost-side counterpart of Myerson’s virtual value [68].

Definition 12 (Regularity). *A distribution whose virtual cost is strictly increasing is called regular.*

Regularity of cost distribution is a standard hypothesis in economic settings [61]. Several results below ask for more: that the second term of Eq. (6) not decrease on its own. Call such a K *super-regular*. Since the first term increases, super-regularity implies regularity. Moreover, super-regularity is implied by basic properties shared by most commonly used distributions [11]:

$$k \text{ non-increasing, or } k \text{ log-concave} \implies K \text{ super-regular} \implies K \text{ regular,}$$

where neither implication reverses. In particular, Power costs $K(t) = t^p$ with $p < 1$ are super-regular although k is log-convex (Example 20), and Footnote 7 exhibits a regular K that is not super-regular.

Theorem 13 (Uniform voters: every equilibrium, and when it is unique). *Let $x \sim U(0, 1)$ and let k satisfy Assumption 2.*

(a) *Every Nash equilibrium is symmetric, $c_1 = 1 - c_2$, with gap $\Delta \in (0, 1)$ satisfying*

$$K(\Delta) = (1 - \Delta) k(\Delta), \quad \text{equivalently} \quad \psi(\Delta) = 1. \quad (7)$$

(b) *If K is regular, then Eq. (7) has a unique root Δ^* , and $(\frac{1-\Delta^*}{2}, \frac{1+\Delta^*}{2})$ is the unique Nash equilibrium up to relabeling.*

Proof in Appendix B.

Without regularity, Eq. (7) may have several roots and, *not all of them are equilibria*: Appendix C gives a cost density with three roots of which only the outer two are equilibria. Part (a) therefore cannot be reversed without an extra condition. A root of Eq. (7) exists for every k satisfying Assumption 2, as does a solution of the conditions of Theorem 9 for every v (Appendix C); what is open (Remark 8) is only whether some solution is an equilibrium.

Corollary 14 (The doubly-uniform benchmark). *If $x, \kappa \sim U(0, 1)$ then $\psi(t) = 2t$, $\Delta^* = \frac{1}{2}$, and $(\frac{1}{4}, \frac{3}{4})$ is the unique Nash equilibrium up to relabeling; equilibrium turnout is $\frac{3}{8}$.*

Proof. With $K(t) = t$ the cost density $k \equiv 1$ satisfies Assumption 2 and $\psi(t) = t + K(t)/k(t) = 2t$ is strictly increasing, so Theorem 13(b) applies; Eq. (7) reads $\Delta = 1 - \Delta$, giving $\Delta^* = \frac{1}{2}$ and the profile $(\frac{1}{4}, \frac{3}{4})$. There, by Eq. (2), candidate 1's share is the area under the positive part of her benefit tent (Fig. 1): the flank $[0, \frac{1}{4}]$ votes with probability $K(\Delta^*) = \frac{1}{2}$, contributing $\frac{1}{8}$, and the inner arm contributes $\int_{1/4}^{1/2} (1 - 2x) dx = \frac{1}{16}$. Hence $u_1 = u_2 = \frac{3}{16}$ and $T = \frac{3}{8}$. $\square$

Corollary 14 is the benchmark every extension will be measured against, and it is the equilibrium depicted in Fig. 2.

Beyond the two benchmarks Sufficiency does not stop at the uniform cases. A stationary profile is an equilibrium once each candidate's payoff is single-peaked on her own side of the opponent, so that her stationary point is the best position available there, and neither candidate gains by crossing to the far side. Both hold when the two distributions are jointly not too uneven: for a non-increasing cost density,

$$\rho_v \rho_k \leq 4$$

suffices, and stationarity together with the two cross-over checks then characterizes equilibrium exactly as it does under uniform costs (Proposition 59). The two ratios multiply, so the criterion degrades gracefully: it is $\rho_v \leq 4$ when costs are uniform (Proposition 10), and for a uniform electorate it asks only that k be non-increasing, one of the cases in which K is regular (Theorem 13). For a symmetric electorate the cross-over checks come free: own-side optimality alone makes a mirror profile an equilibrium, whatever the cost distribution (Lemma 54). What is not covered is a cost density that rises and falls, where the second derivative picks up terms in k' of indefinite sign and no bound in ρ_v and ρ_k alone can absorb them (Remark 88); for a uniform electorate that case is settled anyway, by Theorem 13.

3.2 Polarization

Costly voting opens two routes from primitives to candidate polarization: who the voters are, and what voting costs look like. We take them in turn.

Mass polarization drives elite polarization Under uniform costs, Corollary 11 makes the quartile profile of any symmetric electorate with $\rho_v \leq 4$ an equilibrium, so at that equilibrium the separation *equals the interquartile range* of the voter distribution. Mass polarization therefore transmits to the candidates one for one: a *hollow* electorate, with more mass near the extremes than at the center, has a wider interquartile range and a correspondingly wider equilibrium, while a *peaked* one contracts both (Fig. 4). Under Claim 1 the shape of the electorate is irrelevant and both candidates sit at the median, so the whole dependence is an artifact of costly participation. Dreyer and Bauer [29] document this direction across eleven Western European party systems (1977–2016): parties' positions grow more extreme as their electorates polarize, and the effect runs through the abstention margin.

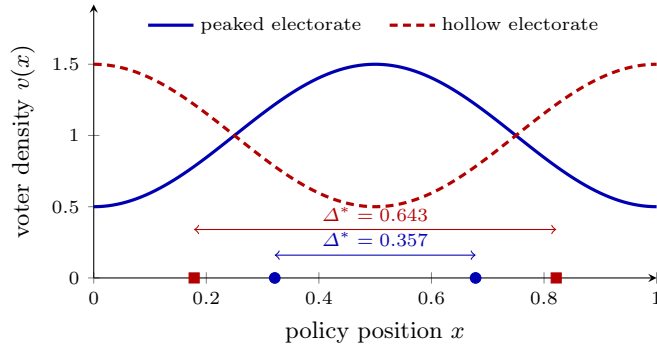


Fig. 4. Mass polarization drives elite polarization under uniform costs (Corollary 11). Two symmetric electorates with the same density ratio $\rho_v = 3$: $v(x) = 1 \mp \frac{1}{2} \cos(2\pi x)$. The quartile profile (marks) is an equilibrium (Corollary 11), so the hollow electorate—more mass near the extremes, wider interquartile range—supports an equilibrium nearly twice as polarized, even though nothing else in the environment changed.

Elite polarization without mass polarization The second channel is the one absent from the classical model altogether: fix the electorate (uniform, so entirely unpolarized, with its median at $\frac{1}{2}$) and vary only the cost distribution. Before comparing cost distributions it is worth asking which features of K the model can respond to at all. The answer is sharp, and it is not the level of turnout. Throughout write $r(t) := k(t)/K(t)$ for the *reverse hazard rate* of K and $G(t) := \int_0^t K$, so that $\psi = t + 1/r$ and super-regularity says exactly that r is non-increasing.⁶

Proposition 15 (Scale invariance). *Fix any voter distribution V , and let $z > 0$ be such that zK is again an admissible cost c.d.f. on $[0, 1]$ (i.e. $zK(1) \leq 1$; the residual mass sits above 1 and consists of voters who never participate). Then the games with cost distributions K and zK have identical best-response correspondences, hence identical sets of pure and mixed equilibria. All vote shares—and therefore turnout—are multiplied by z .*

The argument uses neither α nor β , so the statement survives both extensions unchanged; Proposition 32 records it in that generality.

Proof. By Eq. (2), $u_i(c_1, c_2) = \int_0^1 K(B_i(x)^+) v(x) dx$ is linear in K , so replacing K by zK multiplies u_1, u_2 , and hence each candidate’s payoff, by the constant $z > 0$. Positive affine transformations of payoffs leave $\arg \max$ unchanged. $\square$

Corollary 16. *The equilibrium set depends on K only through its reverse hazard rate r . Indeed $\log K(t) = \log K(1) - \int_t^1 r$, so two cost distributions with the same reverse hazard rate are positive multiples of one another, and by Proposition 15*

⁶ Throughout, the *reverse hazard rate* means k/K , not the hazard rate $k/(1 - K)$ of the monotone-hazard-rate condition familiar from mechanism design; every monotonicity hypothesis below concerns k/K (equivalently K/k).

they induce the same equilibria. By Proposition 32 the same is true at every (α, β) .

Corollary 16 explains *why* the reverse-hazard-rate order [80] is the right order for the comparative static of Proposition 18 below (we call it the rhr order): it is precisely the natural order on the equivalence classes of cost distributions that the model can distinguish. It is the stricter of the two orders one might reach for: if $\tilde{k}/\tilde{K} \geq k/K$ pointwise and $\tilde{K}(1) \leq K(1)$, which holds in particular when both are proper distributions on $[0, 1]$, then $\tilde{K} \leq K$ pointwise, so rhr-higher costs are stochastically higher too. The converse fails, and Remark 19 is the counterexample. It also fixes the form any polarization claim here can take: a true one must be a statement about the shape of r , never about the level of costs or of turnout, since both of those can be moved without moving the equilibrium at all.

What the reverse hazard rate measures When the benefit at stake is t , a fraction $K(t)$ of the electorate already votes, and

$$r(t) = \frac{k(t)}{K(t)} = \frac{d}{dt} \log K(t)$$

is the share of those voters who are roughly indifferent: raise the stake by ε and the number who turn out rises by a factor of about $r(t)\varepsilon$. So r is the *proportional responsiveness of turnout to the stakes*. A large r means an electorate poised at the margin; a small r a committed one, whose participation barely notices what is at stake.

This is exactly the quantity a candidate can trade on. Take uniform voters and consider candidate 1 widening the gap by moving out. She excites her own flank: every voter behind her faces the same benefit Δ , so the mass $\frac{1-\Delta}{2}$ behind her yields extra turnout at rate $k(\Delta)$. She also cedes the *contested middle* $[c_1, c_2]$ (the voters lying between the candidates), losing turnout $K(\Delta)/2$. Balancing the two gives Eq. (7), which rearranges into a transparent form.

Corollary 17 (Equilibrium as a crossing). *Let $x \sim U(0, 1)$ and let k satisfy Assumption 2. A gap Δ solves Eq. (7), equivalently $\psi(\Delta) = 1$, if and only if*

$$r(\Delta) = \frac{1}{1 - \Delta}. \quad (8)$$

If K is super-regular the left side falls while the right side rises, so this is the unique equilibrium.

The two sides of Eq. (8) move in opposite directions, and that is what makes both the crossing and the comparative static easy to see. As the gap widens, the flank that stands to gain shrinks to mass $\frac{1-\Delta}{2}$, so the right side rises: each further step out has to be paid for by a larger proportional response. The left side is what the cost distribution actually delivers at that gap. An electorate whose r

lies above another’s meets the rising requirement further out, so its crossing sits further out too (Fig. 5), which is the next proposition.

That hypothesis is super-regularity, and it is strictly stronger than what Theorem 13(b) needs: regularity permits K/k to fall, provided it falls more slowly than t rises—equivalently $r' < r^2$, so r may rise as long as it rises slowly.⁷ This is why we keep ψ alongside r : the picture needs a falling left-hand side, but Theorem 13(b) does not, and the sharp hypothesis is the one stated in terms of ψ .

Proposition 18 (More responsive participation polarizes). *Let $x \sim U(0, 1)$ and let $k, \tilde{k}$ both satisfy the hypotheses of Theorem 13(b), with equilibrium gaps $\Delta^*, \tilde{\Delta}^*$. If $\tilde{k}/\tilde{K} \geq k/K$ pointwise on $(0, 1)$, then $\tilde{\Delta}^* \geq \Delta^*$.*

Proof in Appendix B.

The hypothesis is far stronger than the conclusion needs. Since $\tilde{\psi}$ is strictly increasing, $\tilde{\Delta}^* \geq \Delta^*$ holds *exactly* when $\tilde{r}(\Delta^*) \geq r(\Delta^*) = 1/(1 - \Delta^*)$: only the two electorates’ responsiveness at the incumbent gap matters (Fig. 5, dashed rule). It is therefore enough for the rhr order to hold near Δ^* , and the pointwise order on all of $(0, 1)$ is merely the cleanest sufficient condition—strictly stronger than the conclusion needs.⁸

The logic behind Proposition 18 is not that costly elections have fewer voters, but that they have *more responsive* ones. When costs are high in the reverse-hazard-rate sense, the voters who still turn out are concentrated just above their indifference point, and a candidate who moves outward converts many of them; the flank is worth fighting for, and she goes to get it. When costs are low, her flank votes for her regardless, moving out buys almost nothing, and she stays near the center to contest the middle. What polarizes candidates is a flank that is *nearly indifferent*, not a flank that is small. Theorem 27 carries the comparison to every $\alpha > 0$, so this is the $\alpha = 0$ case of a channel that alienation does not close.

It also settles a claim about turnout itself that would be natural to make, and that is false.

Remark 19 (Turnout per se is not the driver). By Proposition 15, taking $z < 1$ (which adds a mass $1 - z$ of *never-voters*, whose cost exceeds the largest benefit

⁷ For an explicit witness, prescribe $\psi(t) = 2t + \frac{3}{4\pi} \sin(2\pi t)$, the doubly-uniform $\psi(t) = 2t$ perturbed, and recover K from $r = 1/(\psi(t) - t)$. Then $\psi' = 2 + \frac{3}{2} \cos(2\pi t) \geq \frac{1}{2}$, so K is regular and satisfies Assumption 2, while r increases wherever $\cos(2\pi t) < -\frac{2}{3}$. Here $\psi(\frac{1}{2}) = 1$, so the equilibrium gap is exactly $\frac{1}{2}$.

⁸ A witness, against the uniform baseline $K(t) = t$, where $\psi(t) = 2t$ and $\Delta^* = \frac{1}{2}$. Prescribe $\tilde{\psi}(t) = 2t - 3t(t - \frac{3}{10})(\frac{7}{10} - t)$ and recover $\tilde{K}$ from $\tilde{r} = 1/(\tilde{\psi}(t) - t)$; since $\psi = t + 1/r$, the rhr order is exactly $\tilde{\psi} \leq \psi$, so the cubic makes it hold on $(\frac{3}{10}, \frac{7}{10})$ and fail outside. Assumption 2 holds and $\tilde{\psi}' \geq 1.63$, so $\tilde{K}$ is regular; $\tilde{r}(\frac{1}{2}) = \frac{25}{11} > 2$, and the equilibrium gap moves out to 0.5311. The conclusion of Proposition 18 holds although its hypothesis fails on 60% of $(0, 1)$.

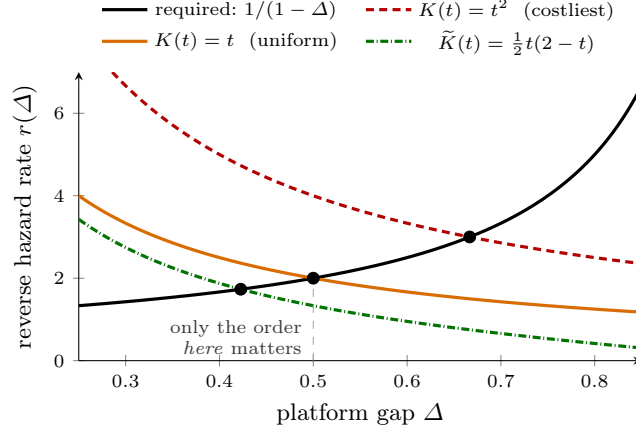


Fig. 5. Equilibrium as a crossing (Corollary 17). The rising black curve is the responsiveness a candidate *needs* in order to justify a gap Δ ; the falling curves are the responsiveness each cost distribution *supplies*. Equilibrium is where they meet (dots), at $\Delta^* = \frac{2}{3}, \frac{1}{2}$ and $1 - \frac{1}{\sqrt{3}}$. Raising the whole supply curve—costs higher in the reverse-hazard-rate order—slides the crossing to the right: Proposition 18. The three curves here are ordered along the whole axis, but that is more than the comparison needs: it is settled at the incumbent crossing alone (dashed rule), which is why the pointwise hypothesis of Proposition 18 can be weakened to a neighbourhood. Note $\tilde{K}$ lies *below* $K(t) = t$ here even though it puts costs stochastically higher (Remark 19).

any profile can offer) drives turnout arbitrarily low while leaving the platforms exactly where they were. So “lower turnout $\Rightarrow$ more polarization” cannot hold in this model. Nor does the first-order stochastic order suffice, and regularity does not rescue it: the two distributions below are both regular, with $\psi(t) = 2t$ and $\tilde{\psi}' \geq 2$ throughout. Let $x \sim U(0, 1)$ and compare $K(t) = t$ with

$$\tilde{K}(t) = \frac{1}{2} t(2 - t), \quad \tilde{k}(t) = 1 - t$$

(half of the electorate never votes and the rest have decreasing cost density). Then $\tilde{K} \leq K$ pointwise, so costs are stochastically higher (the mean cost rises from $\frac{1}{2}$ to at least $\frac{2}{3}$, the never-voters’ costs lying above every attainable benefit), and equilibrium turnout falls from 0.375 to 0.269; yet the equilibrium gap *shrinks*, from $\Delta^* = \frac{1}{2}$ to $\Delta^* = 1 - \frac{1}{\sqrt{3}} \approx 0.4226$. Higher costs, lower turnout, *less* polarization. There is no contradiction with the comparative static ($\tilde{K}$ is *lower* than K in the reverse-hazard-rate order), but the logic must be described as a statement about the shape of K , not about the level of participation.

The two distributions in Remark 19 sit on either side of $K(t) = t$ in Fig. 5 in the way the logic predicts: $\tilde{K}$ has a *decreasing* density, so its remaining voters are the committed ones rather than the marginal ones, its supply curve lies below, and its candidates converge, even though fewer people vote. Turnout and

polarization are both outputs of the model; neither drives the other, and it is the shape of K that drives both.

Example 20 (Power costs: separation equals the mean cost). Let $x \sim U(0, 1)$, as throughout this subsection. For $K(t) = t^p$ with $p > 0$ the reverse hazard rate $r(t) = p/t$ is increasing in p , and Eq. (7) solves to

$$\Delta^* = \frac{p}{1+p} = \mathbb{E}[\kappa] :$$

the equilibrium separation equals the mean voting cost *exactly*, for every p . Raising p raises r pointwise, so Proposition 18 applies; on this family its conclusion can be read straight off the formula.

Equilibrium turnout on this family is $T = (1 - \Delta^*)K(\Delta^*) + \int_0^{\Delta^*} K = (\Delta^*)^p \frac{1+2p}{(1+p)^2}$, which falls as p rises: $(\Delta^*, T) = (\frac{1}{3}, 0.513)$ at $p = \frac{1}{2}$, $(\frac{1}{2}, \frac{3}{8})$ at $p = 1$, $(\frac{2}{3}, \frac{20}{81})$ at $p = 2$ and $(\frac{3}{4}, 0.185)$ at $p = 3$. A single shift in the cost distribution therefore reproduces both trends of Section 1 at once—rising polarization alongside falling participation—without either one causing the other.

Corollary 11 put an equilibrium at the quartiles. Within the power family the same construction reaches *any* symmetric pair of quantiles.

Remark 21 (Prescribing the equilibrium profile). Let $x \sim U(0, 1)$ and fix a target profile $(q, 1 - q)$ with $q \in (0, \frac{1}{2})$. Take the power costs

$$K_q(t) = t^{p_q}, \quad p_q := \frac{1}{2q} - 1 = \frac{1-2q}{2q} \in (0, \infty).$$

Then $\psi(t) = t + K_q(t)/k_q(t) = (1 + \frac{1}{p_q})t$ is linear and strictly increasing, so Theorem 13(b) applies: the game has a *unique* equilibrium, and by Example 20 its gap is $\Delta^* = \frac{p_q}{1+p_q} = 1 - 2q$. The candidates therefore sit at exactly

$$c_1^* = q, \quad c_2^* = 1 - q,$$

and the mean voting cost is $1 - 2q$, the separation itself.

So every interior symmetric profile is the unique equilibrium of exactly one member of the family, and p_q is strictly decreasing in q : the quartiles of Corollary 11 are $q = \frac{1}{4}$, where $p_q = 1$ and costs are uniform; $q \uparrow \frac{1}{2}$ sends $p_q \downarrow 0$ and the platforms to the median; $q \downarrow 0$ sends $p_q \rightarrow \infty$ and the platforms to the end-points. Those two limits are approached but never attained, since neither $p = 0$ nor $p = \infty$ is an admissible cost distribution, which is the sense in which costly voting breaks median convergence for *every* member of the family.

Two caveats delimit the statement. The distribution-freeness of Corollary 11 (a quartile equilibrium for every symmetric electorate with $\rho_v \leq 4$) is special to $p = 1$, because uniform costs make k constant and the balance condition of Lemma 7 then compares voter mass alone; for $p \neq 1$ the marginal masses are k -weighted and the equilibrium quantile depends on the shape of v . And the family one reaches for first, scaling costs up and down ($K_z = zK$), cannot do this at all: rescaling leaves the equilibrium exactly where it was (Proposition 15), so any family that moves the quantile must move the reverse hazard rate.

Power costs against a *non-uniform* electorate fall outside every sufficiency result above, yet the profile is an equilibrium in each case we checked, and the quantile stays close to the uniform-voter value; Appendix D records what the computation shows and what is left to prove.

Which way does causality run? The feedback is one-directional. Higher costs (in the sense that the model can detect, Corollary 16) polarize the candidates; conversely, a wider gap raises every voter’s benefit from participating, so turnout is *increasing* in the imposed separation: polarization raises turnout, never the reverse. Abstention is therefore a cause of candidate divergence here, not a consequence of it. This one-directionality is special to the baseline: under alienation the turnout curve bends back (Proposition 29), and the reverse channel (extreme candidates depressing participation) reappears.

4 Alienated Voters

We now turn the first dial. Throughout this section $\alpha > 0$ while candidates still maximize vote share, so the convention is $\beta = 0$. Alienation is a second route to divergence—not an independent one: without costly voting it would change nothing at all, since the factor $2 + \alpha$ cancels out of the comparison between the candidates and the median equilibrium survives (Remark 4). What it changes is *who* turns out, and its logic can be stated in one sentence: on the contested middle $[c_1, c_2]$ the two distances sum to the constant $d_1(x) + d_2(x) \equiv \Delta$, so a penalty on the distance to one’s *own* candidate bites hardest exactly at the midpoint—and pushes the candidates apart. The tent geometry (outer arms of slope α , inner arms of slope $2 + \alpha$, a dead zone of abstention around m) is laid out in Fig. 6.

The section has three parts. Section 4.1 characterizes the equilibrium at a fixed α : the balance principle survives in a general form (Proposition 22), and for uniform voters the equilibrium is unique and symmetric (Theorem 24). Section 4.2 carries the comparative statics of Section 3.2 to a fixed α : the cost channel and the transmission of mass polarization both survive (Theorem 27 and Proposition 28). Section 4.3 contains the section’s point: *alienation polarizes*, with the gap strictly increasing in α and tending to full separation (Theorem 30), and it does so at a fixed level of participation (Corollary 31), so this is not the cost channel of Section 3 renamed. Two observations of Section 3 do not survive alienation, and each is flagged where it falls: the distribution-free quartile equilibrium (Remark 26) and the one-directional link from polarization to turnout (Proposition 29), the second of which is what lets the model speak to the empirical literature.

4.1 Equilibrium at a Fixed α

Alienation leaves the shape of the characterization intact: one first-order condition for an arbitrary electorate (Proposition 22), and, for a uniform one, a unique symmetric equilibrium (Theorem 24).

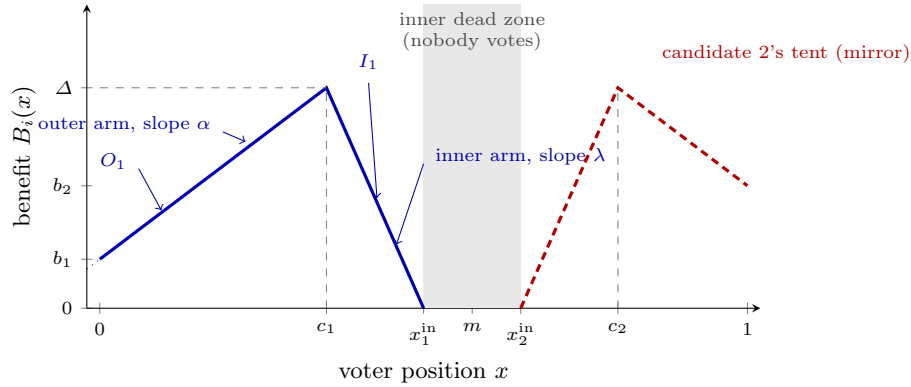


Fig. 6. Anatomy of the benefit tents at the profile $(c_1, c_2) = (0.35, 0.8)$ with $\alpha = 1$. Candidate 1's tent (solid) peaks at height Δ over c_1 ; its *inner arm* falls at slope $\lambda = 2 + \alpha$ to the inner zero $x_1^{\text{in}} = c_1 + \Delta/\lambda$, and its *outer arm* at slope α toward the outer zero $x_1^{\text{out}} = c_1 - \Delta/\alpha$, here off the policy space, so the arm is truncated at the boundary with *boundary benefit* $b_1 = B_1(0)$. Candidate 2's tent (dashed) is the mirror image, with inner zero x_2^{in} and boundary benefit $b_2 = B_2(1)$. Between the inner zeros lies the shaded *inner dead zone* $(x_1^{\text{in}}, x_2^{\text{in}})$, where both benefits are negative and nobody votes; here $x_1^{\text{out}} < 0$, so there is no outer dead zone to draw. The marginal masses of Proposition 22 weight each point of an arm by $k(B_1(x))v(x)$: O_1 on the outer arm, I_1 on the inner. The baseline of Section 3 is the case $\alpha = 0$, in which the outer arms are flat at height Δ , the inner zeros merge at m , and the dead zone vanishes.

Here and in Section 5 write

$$\lambda := 2 + \alpha, \quad b_1 := B_1(0) = c_2 - (1 + \alpha)c_1, \quad b_2 := B_2(1) = (1 + \alpha)c_2 - c_1 - \alpha$$

for the *boundary benefits*, the height of each tent's outer arm where it meets the edge of the policy space (see Fig. 6). At a symmetric profile $c_1 = \frac{1-\Delta}{2}$ they coincide, $b_1 = b_2 = b$, with

$$b = \frac{\lambda\Delta - \alpha}{2}, \quad \text{so} \quad \Delta - b = \frac{\alpha(1 - \Delta)}{2}, \quad (9)$$

and $b \geq 0$ exactly when $\Delta \geq \alpha/\lambda$.

Alienation opens two regions in which nobody votes, and only one of them is a case the analysis has to track. The *inner dead zone* between the tents' inner zeros is present at every profile with $c_1 < c_2$ and $\alpha > 0$. An *outer dead zone* behind a candidate, where her most distant supporters abstain, appears only when her tent's outer zero falls inside the policy space, and it is that case distinction the first-order conditions must carry. From here on “dead zone” unqualified means the outer one (Fig. 6). Lemma 6 survives alienation: every equilibrium is untied and interior for every $\alpha \geq 0$ and every (v, k) (Lemma 52).

The master condition, in general form One first-order condition covers alienation for every electorate and every cost distribution, whether or not each candidate's

tent reaches the edge of the policy space. One might hope that at equilibrium each tent reaches the boundary: for candidate 1, that the outer zero $x_1^{\text{out}} = c_1 - \Delta/\alpha$ of her tent (Fig. 6) satisfies $x_1^{\text{out}} \leq 0$, i.e. $(1 + \alpha)c_1 \leq c_2$, so that there is no outer dead zone, and its mirror for candidate 2. For concentrated electorates this fails: the equilibrium tents *do* detach from the boundary (Remark 23). The first-order condition valid in both regimes is the following.

Proposition 22 (Master first-order condition). *Under alienation, every interior best response of candidate 1 satisfies*

$$\int_{c_1}^{x_1^{\text{in}}} k(B_1(x))v(x) dx = \int_{x_1^{\text{out}} \vee 0}^{c_1} k(B_1(x))v(x) dx, \quad (10)$$

$$x_1^{\text{in}} = c_1 + \frac{\Delta}{\lambda}, \quad x_1^{\text{out}} = c_1 - \frac{\Delta}{\alpha},$$

with the mirror condition for candidate 2; here $a \vee b$ and $a \wedge b$ are the larger and the smaller of a and b , so $x_1^{\text{out}} \vee 0$ is the outer zero clipped to the policy space. For uniform costs this reads

$$V(x_1^{\text{in}}) + V(x_1^{\text{out}} \vee 0) = 2V(c_1). \quad (11)$$

When $x_1^{\text{out}} \leq 0$ (no dead zone) $V(x_1^{\text{out}} \vee 0) = 0$ and Eq. (11) reduces to $V(x_1^{\text{in}}) = 2V(c_1)$; when $x_1^{\text{out}} > 0$ the extra term is genuinely present.

This is Lemma 7 again. The left side of Eq. (10) is I_1 and the right side is O_1 , with each arm cut at the tent's own zero instead of at the midpoint and the boundary; alienation moves the cuts, not the quantities being balanced. The limits collapse correctly at $\alpha = 0$, where $x_1^{\text{in}} = c_1 + \Delta/2 = m$ and $x_1^{\text{out}} \rightarrow -\infty$, so $x_1^{\text{out}} \vee 0 = 0$. Lemma 33 records the same identification once competitiveness is in play as well.

Proof in Appendix A.

Alienation also creates a case the baseline did not have—the outer zero can fall inside the policy space—and the first claim of this section needs that case to be harmless where the characterizations are proved.

Remark 23 (Dead-zone equilibria exist). Under uniform voters dead zones never arise at a best response: if $x_1^{\text{out}} \geq 0$ the tent lies entirely inside $[0, 1]$ and $u_1 = (\frac{1}{\alpha} + \frac{1}{\lambda})G(\Delta)$ (recall $G(t) = \int_0^t K$) depends on c_1 only through Δ , so u_1 is strictly decreasing in c_1 on the dead-zone region $c_1 \in [\frac{c_2}{1+\alpha}, c_2)$ and the candidate strictly prefers to move out of it. For non-uniform v this argument fails and dead-zone equilibria do occur. The supplementary analysis sharpens both halves: Proposition 61 shows a stationary dead zone forces $\rho_v \geq \frac{2+\alpha}{\alpha}$ for any K , that uniform voters never have dead-zone equilibria for any α and any K , and, under uniform costs, gives the criterion $V(\frac{2(1+\alpha)}{\lambda^2}) > 2V(\frac{1}{\lambda})$ under which a symmetric v admits a stationary profile with a dead zone. For $v = \text{Beta}(8, 8)$ and uniform costs the symmetric stationary profile is unique, $(0.3812, 0.6188)$ at $\alpha = 2$ and $(0.3748, 0.6252)$ at $\alpha = 4$, and is an equilibrium (mutual global best responses,

verified numerically; Remark 62); in both $x_1^{\text{out}} > 0$ (respectively 0.26 and 0.31), the condition $V(x_1^{\text{in}}) = 2V(c_1)$ fails by $2.3 \cdot 10^{-2}$ and $6.2 \cdot 10^{-2}$, and Eq. (11) holds to 10^{-9} . The same happens for truncated normals with small variance and for Beta(20, 20).

The dead-zone case being real, sufficiency also changes shape. Call the *umbrella* of a setting the range of density ratios ρ_v within which the first-order conditions are known to characterize equilibria. The baseline umbrella $\rho_v \leq 4$ (Proposition 10) deforms to $\rho_v \leq \min \left\{ \frac{2\lambda}{1+\alpha}, \frac{\lambda}{\alpha} \right\}$ —the first branch controlling concavity, the second the dead zone—with existence for symmetric electorates below the bound (Theorem 56 and Corollary 58). Both branches *decrease* in α : alienation shrinks the umbrella rather than widening it, foreshadowing a contrast with competitiveness (Table 4).

Uniform voters: a one-line characterization For a uniform electorate the characterization collapses to a single equation in the reverse hazard rate, Eq. (12) below, whose unique root is the equilibrium gap of Theorem 24. Under uniform voters every equilibrium is untied and interior (Lemma 52) and, by Remark 23, has no dead zone, so specializing Eq. (10) to $v \equiv 1$ the two first-order conditions read $K(b_i) = \frac{2}{\lambda} K(\Delta)$, $i = 1, 2$. Hence $K(b_1) = K(b_2)$, so $b_1 = b_2$ (K is strictly increasing on $[0, 1]$), and since $b_2 - b_1 = \alpha(c_1 + c_2 - 1)$, for $\alpha > 0$ every equilibrium is symmetric (this is the necessity step of Theorem 64 at $\beta = 0$). Its gap therefore solves

$$\lambda K(b) = 2K(\Delta), \quad \text{equivalently} \quad \int_b^\Delta r(t) dt = \log \frac{\lambda}{2} = \log \left(1 + \frac{\alpha}{2} \right), \quad (12)$$

with b as in Eq. (9). The second form makes Corollary 16 visible: only r enters. At $\alpha = 0$ both sides vanish; dividing by α and letting $\alpha \downarrow 0$ recovers $K(\Delta) = (1 - \Delta)k(\Delta)$, the baseline condition (7), as it must. The theorem below is therefore stated for $\alpha > 0$; at $\alpha = 0$ the model is the baseline, and under super-regularity its unique equilibrium is the one of Theorem 13(b) (Corollary 17).⁹

Theorem 24 (Uniform voters: the equilibrium at each α). *Let $x \sim U(0, 1)$ and let k satisfy Assumption 2 with K super-regular. Then for every $\alpha > 0$, Eq. (12) has a unique root $\Delta^*(\alpha) \in (\frac{\alpha}{\lambda}, 1)$, and the profile $(\frac{1-\Delta^*}{2}, \frac{1+\Delta^*}{2})$ is the unique Nash equilibrium up to relabeling.*

Proof in Appendix A.

Super-regularity is used only for uniqueness and for the equilibrium property: the boundary values of Ψ in the proof use no monotonicity, so Eq. (12) has a root for every K .

Corollary 25. *If $x, \kappa \sim U(0, 1)$ then $\Delta^*(\alpha) = \frac{2+\alpha}{4+\alpha}$, attained at the symmetric profile $(\frac{1}{4+\alpha}, \frac{3+\alpha}{4+\alpha})$, with equilibrium turnout $\frac{2(3+\alpha)}{(4+\alpha)^2}$.*

⁹ A reader who simply substitutes $\alpha = 0$ into Eq. (12) gets the identity $2K(\Delta) = 2K(\Delta)$ and no information; the content is at first order in α .

At $\alpha = 0$ this is Corollary 14: the gap is $\frac{1}{2}$, the profile is $(\frac{1}{4}, \frac{3}{4})$, and turnout is $\frac{3}{8}$. The benchmark is the left endpoint of a one-parameter family, and every quantity in it moves monotonically from there.

Remark 26 (Uniform costs: no fixed-quantile analogue). [numerical observation] The quartile profile of Corollary 11 is the same for every symmetric electorate, and alienation removes the reason. Under uniform costs Eq. (11) reads $V(x_1^{\text{in}}) + V(x_1^{\text{out}} \vee 0) = 2V(c_1)$. At $\alpha = 0$ the inner zero is the midpoint and the outer zero lies off the policy space, so for a symmetric electorate the condition is $\frac{1}{2} = 2V(c_1)$: the distribution enters only through $V(m) = \frac{1}{2}$, and every symmetric electorate lands on the quartiles. For $\alpha > 0$ the inner zero moves inward to $x_1^{\text{in}} = c_1 + \Delta/\lambda$, which moves with c_1 , so V enters in its own right and the equilibrium quantile becomes a property of the electorate. Across uniform, Beta(2, 2), Beta(5, 5), arcsine and cosine-peaked electorates, $V(c_1^*)$ agrees to 10^{-12} at $\alpha = 0$ and then separates: it spans $[0.194, 0.207]$ at $\alpha = 1$ and $[0.125, 0.153]$ at $\alpha = 4$, against $1/(4 + \alpha)$ for uniform voters (Corollary 25). Each of those profiles is still an equilibrium—we verified mutual global best responses—so what alienation costs is the distribution-freeness of the quantile, not the equilibrium.

4.2 Costs and Polarization at a Fixed α

At a fixed α the comparative statics of Section 3.2 carry over—the cost channel (Theorem 27) and the transmission of mass polarization (Proposition 28)—while the one-directional link from polarization to turnout does not (Proposition 29).

The cost channel of Section 3.2 survives alienation intact: at every α the two forces push the same way.

Theorem 27 (The cost channel survives alienation). *Let $x \sim U(0, 1)$ and let $k, \tilde{k}$ both satisfy the hypotheses of Theorem 24. If $\tilde{r} \geq r$ pointwise on $(0, 1)$ —i.e. $\tilde{K}$ is higher than K in the reverse-hazard-rate order—then $\tilde{\Delta}^*(\alpha) \geq \Delta^*(\alpha)$ for every $\alpha > 0$, and for $\alpha = 0$ by Proposition 18. As there, the comparison is also one of stochastically higher costs whenever $\tilde{K}(1) \leq K(1)$.*

Proof. $\tilde{\Psi}(\Delta) = \int_{b(\Delta)}^{\Delta} \tilde{r} - \log(\lambda/2) \geq \Psi(\Delta)$ for every Δ , and both are strictly decreasing with a unique root (proof of Theorem 24). Hence $\tilde{\Psi}(\tilde{\Delta}^*) \geq \Psi(\tilde{\Delta}^*) = 0 = \tilde{\Psi}(\Delta^*)$ and therefore $\tilde{\Delta}^* \geq \Delta^*$. $\square$

At $\alpha = 0$ this is exactly the baseline comparative static; the point is that neither force switches the other off: at any fixed α , higher costs still polarize, and at any fixed K , stronger alienation still polarizes. (The theorem orders levels, not increments: along the power family $K(t) = t^p$ the gain in Δ^* from raising α is smaller at larger p .)

Mass polarization under alienation Remark 26 leaves this section unable to say where the equilibrium sits for a general electorate. It can still say exactly how

that equilibrium *moves* when the electorate itself polarizes—how much elite polarization a given amount of mass polarization buys, and what fixes the exchange rate. This is a question about V , not about α : the answer below holds at every α , the baseline included.

Proposition 28 (Mass polarization scales elite polarization). *Let V_σ be the law of $\frac{1}{2} + \sigma(a - \frac{1}{2})$ where $a \sim V_0$ and $\sigma \in (0, 1]$, so that σ measures the spread of the electorate (for $\sigma < 1$ the support of V_σ is $[\frac{1-\sigma}{2}, \frac{1+\sigma}{2}]$, outside which Assumption 2 fails; the proof first shows that no best response lies there). Then the game (V_σ, K, α) is isomorphic to (V_0, K_σ, α) under $c = \frac{1}{2} + \sigma(a - \frac{1}{2})$, where $K_\sigma(t) := K(\sigma t)$. Consequently:*

- (a) (power costs) for $K(t) = t^p$ (Example 20), $K_\sigma = \sigma^p K$, so by Proposition 32 the equilibrium positions in a -space do not move and $\Delta^*(\sigma) = \sigma \cdot \Delta^*(1)$ for every $\alpha \geq 0$ and every V_0 : elite polarization is exactly proportional to mass polarization (uniform costs are $p = 1$);
- (b) (general costs; $a \sim U(0, 1)$ and K super-regular, as in Theorem 24) the reverse hazard rate of K_σ is $r_\sigma(t) = \eta(\sigma t)/t$, where $\eta(u) = u k(u)/K(u)$ is the elasticity of K . If η is non-decreasing then $\sigma \mapsto r_\sigma$ increases pointwise, so by Theorem 27 the equilibrium gap $\Delta_a(\sigma)$ of (V_0, K_σ, α) increases and $\Delta^*(\sigma) = \sigma \Delta_a(\sigma)$ grows super-proportionally: a more dispersed electorate is amplified. If η is non-increasing, it is damped. Constant elasticity—the power family of part (a)—is the exact knife-edge.

So the transmission from mass to elite polarization has a rate, and the cost distribution sets it: proportional under power costs, amplified when the elasticity of K rises and damped when it falls. It says nothing about whether α polarizes, and holds equally at $\alpha = 0$.

Proof in Appendix A.

In words: *stretching a uniform electorate always produces more polarized candidates*, at every level of alienation, and the cost distribution decides whether the transmission is one-for-one (constant elasticity, e.g. power-law costs), amplified or damped. An increasing cost elasticity (participation that grows relatively more responsive as the stakes rise) amplifies mass polarization; a decreasing one damps it; power costs transmit it exactly. Relative to the interquartile comparison of Section 3.2, which needed uniform costs, symmetry and $\rho_v \leq 4$, part (a) keeps only the power shape and holds at every $\alpha \geq 0$, while part (b) drops that too, at the price of a uniform electorate; both compare affine stretches of a fixed electorate rather than arbitrary pairs of symmetric ones. At $\alpha = 0$ the transmission has a one-line closed form: applying Corollary 17 to $(V_0, K_\sigma, 0)$, the equilibrium gap solves

$$r(\Delta^*) = \frac{1}{\sigma - \Delta^*},$$

so widening the electorate relaxes the demand side of the crossing, and Δ^* is strictly increasing in σ whenever r is non-increasing. The elasticity of K governs only the *rate* of transmission, never its direction. (Uniform costs: $\Delta^* = \sigma/2$ exactly.)

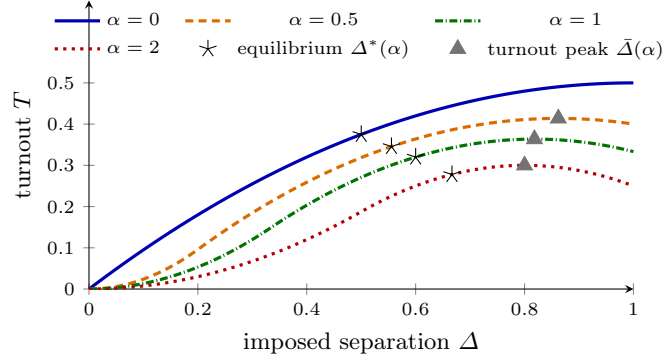


Fig. 7. Turnout against an *imposed* separation in the doubly-uniform benchmark (Proposition 29). Stars mark the equilibrium $\Delta^*(\alpha)$, triangles the turnout peak $\bar{\Delta}(\alpha)$. The equilibrium always lies on the rising branch, so locally polarization raises turnout; but once $\alpha > 0$ the curve turns over, and candidates pushed beyond $\bar{\Delta}$ by forces outside the model depress participation. At $\alpha = 0$ turnout is increasing throughout.

Turnout: the reverse channel reappears The one-directional causality of Section 3.2, in which polarization raises turnout and never the reverse, is an artifact of $\alpha = 0$.

Proposition 29 (Non-monotone turnout under alienation). *Let $x \sim U(0, 1)$, let $\alpha > 0$, and let $T(\Delta) = u_1 + u_2$ be turnout at the symmetric profile with gap $\Delta \geq \alpha/\lambda$. Then*

$$T'(\Delta) = \frac{2}{\alpha} \left[\frac{2(1+\alpha)}{\lambda} K(\Delta) - \frac{\lambda}{2} K(b) \right], \quad \text{so} \quad T'(\Delta) < 0 \iff \frac{K(b)}{K(\Delta)} > \frac{4(1+\alpha)}{\lambda^2}.$$

Consequently:

- (a) *at any equilibrium, Eq. (12) gives $K(b)/K(\Delta) = 2/\lambda$ and hence $T'(\Delta^*) = \frac{2}{\lambda} K(\Delta^*) > 0$: locally, more polarized candidates still raise turnout;*
- (b) *but for $\alpha > 0$ turnout is not globally increasing: for every K , $T'(1) = -\frac{\alpha}{\lambda} K(1) < 0$, so the turnout curve turns over somewhere between the equilibrium and full separation. In the doubly-uniform benchmark T is single-peaked with peak at $\bar{\Delta}(\alpha) = \frac{(2+\alpha)^2}{\alpha^2+6\alpha+4} < 1$, and strictly decreasing on $(\bar{\Delta}, 1]$. E.g. $\bar{\Delta}(1) = \frac{9}{11} \approx 0.818$ while $\Delta^*(1) = 0.6$.*

Proof in Appendix A.

Proposition 29 is the one result here whose implication for polarization is indirect, so it is worth stating plainly. It does *not* say that alienation lowers equilibrium polarization—Theorem 30 says the opposite. It says that the *feedback* from polarization to turnout, which is unambiguously positive in the baseline, changes sign once the gap grows beyond $\bar{\Delta}(\alpha)$ (Fig. 7). Equilibrium candidates never get there on their own. But a model in which some outside force (primaries,

activists, party discipline) pushes candidates past $\bar{\Delta}$ predicts falling turnout, and in that regime polarization and participation move in opposite directions: the pattern Rogowski [79] estimates from joint measures of voter preferences and candidate platforms in U.S. House and Senate races, where greater ideological conflict between the candidates lowers turnout. Alienation together with voting costs can thus explain how candidate polarization can both increase and decrease participation. Both directions come out of the same two parameters, so neither is an artifact of a separate modeling choice.

4.3 Higher α Polarizes

The main result of this section is that higher alienation polarizes more:

Theorem 30 (Alienation drives divergence, for every super-regular cost distribution). *In the setting of Theorem 24, the equilibrium gap $\Delta^*(\alpha)$ is strictly increasing in α , with $\Delta^*(\alpha) \rightarrow 1$ as $\alpha \rightarrow \infty$.*

Proof in Appendix A.

Theorem 30 is this section’s headline, and the general form of the doubly-uniform benchmark $\Delta^*(\alpha) = \frac{2+\alpha}{4+\alpha}$ of Corollary 25: *stronger alienation strictly polarizes*, from the baseline gap at $\alpha = 0$ all the way to full separation as $\alpha \rightarrow \infty$, for uniform voters and every super-regular cost distribution (Fig. 8). Alienation simultaneously depresses turnout—in the doubly-uniform benchmark, from $\frac{3}{8}$ to 0—so the model produces the empirically familiar pairing of rising elite polarization with falling participation from a single voter-side parameter. For a uniform electorate the direction is unambiguous and needs no condition on the cost distribution beyond super-regularity: *alienation increases polarization, always, and without bound*. It is the strongest single force in the paper. The cost channel moves Δ^* within $(0, 1)$ according to the shape of K , whereas alienation drives it to 1 regardless of K . The uniform-electorate hypothesis is not decorative: for the concentrated electorate of Remark 23 the equilibrium gap also grows with α ($\Delta^* = 0.238$ at $\alpha = 2$ and 0.250 at $\alpha = 4$ for Beta(8, 8) and uniform costs), but far more slowly, and we have no general-electorate theorem.

Why alienation pushes candidates apart The logic is worth isolating, because it is different from the cost channel. At $\alpha = 0$ a candidate’s flank is *captive and uniform*: every voter behind c_1 faces exactly the same benefit Δ , whatever their position. Alienation destroys that. For $x < c_1$,

$$B_1(x) = \Delta - \alpha(c_1 - x),$$

so a candidate’s own extremists—the voters furthest behind her—are the ones with the least at stake and the first to abstain. She now has a second, quite separate reason to move outward: not only to widen Δ for the whole flank, but to *go out and get her own flank*, whose benefit rises at rate α as she approaches it. The tension in Corollary 17 is thereby shifted in favor of separation at every cost distribution, and Eq. (12) records the new balance: the boundary voter’s

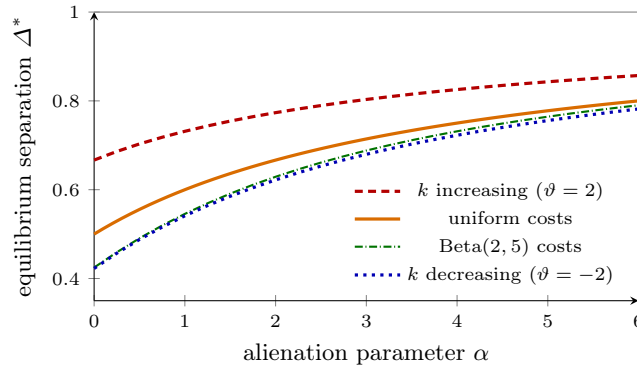


Fig. 8. Equilibrium separation under alienation for uniform voters and four cost distributions (Theorem 30). Here $k_\vartheta(t) = \vartheta t + (1 - \frac{\vartheta}{2})$, $\vartheta \in [-2, 2]$, is the linear cost family of Appendix G. Every curve is increasing in α and tends to 1. The three linear-family curves are ordered as Theorem 27 predicts, r_ϑ being pointwise increasing in ϑ ; Beta(2,5) is not comparable with them in the reverse-hazard-rate order (its rate crosses the uniform one), so its position is not an instance of the theorem. The $\alpha = 0$ intercepts are $\frac{2}{3}$, $\frac{1}{2}$, 0.425 and $1 - \frac{1}{\sqrt{3}}$.

turnout rate must be held at the fixed fraction $K(b)/K(\Delta) = 2/(2 + \alpha)$ of the flank's, a fraction that falls as α grows and can only be restored by moving out. This is Hinich and Ordeshook's alienation channel [47] recovered inside a costly-voting model. It is a distinct channel—one changes *who* is at the margin, the other *how many*—but the two are not independent: both branches of the sufficiency umbrella shrink as α grows, so alienation makes the cost channel harder to certify, not merely different from it.

One reading of Theorem 30 has to be ruled out. Raising α lowers every voter's benefit and therefore lowers turnout, so the result might be the cost channel of Section 3 under another name. It is not, and Remark 19 already says why: by Proposition 15 the *level* of participation can be set anywhere without moving the equilibrium at all, so “turnout fell, hence the candidates diverged” is not a mechanism this model has. The constructive version is immediate.

Corollary 31 (Polarization at unchanged turnout). *In the setting of Theorem 30, fix $\alpha > 0$ and write $T^*(\alpha)$ for equilibrium turnout. Then there are positive multiples K_0 and K_α of K , both admissible, such that the games $(V, K_0, 0)$ and (V, K_α, α) have the same equilibrium turnout while their equilibrium gaps are $\Delta^*(0)$ and $\Delta^*(\alpha) > \Delta^*(0)$.*

Proof. By Proposition 15, replacing K by zK for admissible $z > 0$ leaves the equilibrium set unchanged and multiplies turnout by z . If $T^*(\alpha) \leq T^*(0)$, take $K_\alpha = K$ and $K_0 = zK$ with $z = T^*(\alpha)/T^*(0)$; otherwise take $K_0 = K$ and $K_\alpha = zK$ with $z = T^*(0)/T^*(\alpha)$. Either way $z \leq 1$, so $zK(1) \leq K(1) \leq 1$ and zK is a cost distribution, and the two games have equal equilibrium turnout

by construction. Neither scaling moves its own equilibrium, so the gaps are still $\Delta^*(0)$ and $\Delta^*(\alpha)$, and the inequality is Theorem 30. $\square$

Scaling the *larger* of the two turnouts down, rather than the smaller up, is what keeps $z \leq 1$; it costs nothing and avoids assuming that T^* falls in α , which we have not proved (Proposition 29 concerns T as a function of Δ at fixed α). In the doubly-uniform benchmark the rescaling is explicit: by Corollary 25, $T^*(\alpha) = \frac{2(3+\alpha)}{(4+\alpha)^2}$ falls from $\frac{3}{8}$, so $z(\alpha) = \frac{16(3+\alpha)}{3(4+\alpha)^2}$, and at $\alpha = 1$ both games turn out $\frac{8}{25}$ of the electorate while the gap is $\frac{3}{5}$ under alienation against $\frac{1}{2}$ without it. The gap between the matched games can be much wider: for $K(t) = t^{1/2}$ at $\alpha = 4$ both turn out ≈ 0.313 , with gaps ≈ 0.692 and $\frac{1}{3}$.

Telling the two channels apart Higher costs and stronger alienation both raise polarization and both depress turnout, so the two are observationally similar in aggregate data. They differ cross-sectionally: alienation reshapes *who* abstains—participation falls fastest among voters far from both candidates, opening the inner dead zone around m visible in Fig. 3, whereas a cost shift moves participation for ideologues and centrists alike. The model thus offers a concrete empirical signature separating “voting became harder” from “voters became disaffected”. The signature is a shape: since $p_i = K(B_i^+)$ and B_i is a tent peaked at c_i , alienation concentrates the active electorate into two humps centred on the candidates, falling away on both sides and vanishing outside $[x_1^{\text{out}} \vee 0, x_1^{\text{in}}]$. At $\alpha = 0$ there is no outer edge to fall away from: the flank is flat at the full gain Δ , so the active electorate simply tracks v there. A cost shift scales that picture without changing its shape; alienation makes it bimodal.

5 Competitive Candidates

The second extension changes the candidates rather than the voters: $U_i = u_i - \beta u_{-i}$, spanning vote-share maximization ($\beta = 0$) to pure margin maximization ($\beta = 1$). Remark 5 showed that under full participation β would be irrelevant; everything in this section is therefore attributable to abstention.

The message is that the separation between the two objectives is *de*-polarizing, that it runs against both channels of Sections 3 and 4, and that it is continuous in β up to a single knife-edge at $\beta = 1$. Accordingly we take $\beta \in [0, 1)$ throughout the first three parts, which follow those of Section 4 result for result, and treat the margin game $\beta = 1$ as a special case at the end. Section 5.1 characterizes the equilibrium at a fixed β : it is unique and symmetric (Theorem 35), and under uniform costs the mass-gap law locates it for any electorate (Proposition 37 and Corollary 38). Section 5.2 carries the comparative statics of Section 3.2 to a fixed β : the cost channel and the transmission of mass polarization survive (Propositions 39 and 40), and the turnout curve is the same at every β (Proposition 41). Section 5.3 contains the section’s point: the gap falls in β at every level of alienation (Proposition 42), and does so at a fixed level of participation (Corollary 43). At $\beta = 1$ (Section 5.4) this picture breaks: the

equilibrium set becomes a band rather than a point (Proposition 46), the $\beta \uparrow 1$ limit settles on its *most polarized* profile rather than the median (Theorem 47), the median fails as soon as the electorate is hollow at the centre (Theorem 48), and for some electorates no pure equilibrium survives at all (Proposition 49).

With α and β both in play, the scale invariance of Section 3 can be stated in full generality.

Proposition 32 (Scale invariance, general form). *Proposition 15 holds verbatim for every $\alpha \geq 0$ and every $\beta \in [0, 1]$.*

Proof. Neither parameter enters the proof of Proposition 15. Alienation acts only through B_i in Eq. (4), which does not involve K and is therefore unchanged; and for any β , multiplying u_1 and u_2 by the same constant $z > 0$ multiplies $U_i = u_i - \beta u_{-i}$ by z as well. Positive affine transformations of payoffs leave $\arg \max$ unchanged. $\square$

5.1 Equilibrium at a Fixed β

The logic is visible already in Lemma 7. A candidate has two ways to add to her tally: convert an abstainer on her own flank, or take a voter away from her opponent in the contested middle. The distributive-politics literature has long framed this choice as a targeting problem [24,60,23] and which Glaeser, Ponzetto and Shapiro [38] make the engine of candidate divergence. The two differ in their effect on the opponent. Converting an abstainer on one's own flank means moving *out*, which widens the separation and so raises the benefit of voting—and with it the turnout—on the opponent's flank as well: the first strategy actually *helps* the opponent. Taking a voter from the contested middle costs the opponent a vote outright. To a candidate who weighs the opponent's votes negatively the two are therefore not merely unequal but opposite in sign. That is exactly the asymmetry the balance condition prices, and it can now be stated for every α and β .

Lemma 33 (Balance conditions, general form). *Let v, k satisfy Assumption 2, and let $\alpha \geq 0$, $\beta \in [0, 1]$. Every interior stationary profile (c_1, c_2) of the game (Definition 3) satisfies, for each candidate i ,*

$$(1 + \alpha)(I_i - O_i) + \beta(I_{-i} + O_{-i}) = 0.$$

Summing and differencing the two conditions gives a ratio law and an asymmetry law,

$$(1 + \alpha + \beta)I = (1 + \alpha - \beta)O, \quad (1 + \alpha - \beta)(I_1 - I_2) = (1 + \alpha + \beta)(O_1 - O_2) :$$

the aggregate inner share of marginal voters is the universal constant $\frac{1+\alpha-\beta}{2(1+\alpha)}$, for every v and k . At $(\alpha, \beta) = (0, 0)$ this is Lemma 7; at $\beta = 0$ it is Proposition 22, summed over the two candidates.

Proof: Proposition 82.

The restriction to interior profiles matters only in the joint case: ties and boundary positions are excluded at $\beta = 0$ for every α , at $\alpha = 0$ for every $\beta < 1$, and for uniform voters at every (α, β) with $\beta < 1$ (Lemma 52); with $\alpha, \beta > 0$ and a non-uniform electorate they are not excluded in general. So the inner marginal mass carries weight $1 + \alpha + \beta$ and the outer mass weight $1 + \alpha - \beta$, and the equilibrium inner share falls from $\frac{1}{2}$ at $\beta = 0$ (Lemma 7) toward $\frac{\alpha}{2(1+\alpha)}$ as $\beta \uparrow 1$. As competitiveness rises the contested middle becomes worth more relative to the flank, and the candidates move in to contest it: *this is the paper's only de-polarizing force*. It also explains why β is inert under full participation (Remark 5): with nobody abstaining there is no flank to convert, every vote is taken from the opponent, and the distinction the parameter prices does not exist.

What the model adds to that choice is a price: persuasion is worth $\frac{1+\alpha+\beta}{1+\alpha-\beta}$ units of mobilization, for every v , every K and every $\beta < 1$ (Lemma 33). It also corrects the folk version of the claim: a flank is worth mobilizing because its turnout is *responsive*, not because it is low (Proposition 32 and Remark 19).

Both extensions, non-uniform electorates With both extensions on and a non-uniform electorate the characterization is partial, and we record what is known. Under uniform costs the balance conditions of Lemma 33 still characterize equilibria, up to the two cross-over checks, whenever $\rho_v \leq \min\{f_\beta, g_\beta\}$ with the thresholds of Table 4 (Theorem 85); competition moves the two branches in opposite directions, widening the concavity branch f_β and narrowing the dead-zone branch g_β . A symmetric stationary profile exists for every $\beta < 1$ (Proposition 89), and for a symmetric electorate strictly inside the vote-share umbrella, with a unique symmetric stationary profile there, the symmetric equilibrium of Corollary 58 continues into small $\beta > 0$ (Theorem 90); numerically it persists to moderate β , inside and outside the umbrella (Remark 91). Competition also makes alienation dead zones easier to sustain: the density ratio above which a stationary profile can have one falls from λ/α at $\beta = 0$ to g_β (Proposition 92). Beyond uniform costs the concavity argument meets an obstruction that we isolate rather than resolve (Remark 88).

Uniform voters For a uniform electorate the characterization again collapses to a single equation, Eq. (13) below, whose unique root is the equilibrium gap of Theorem 35.

Proposition 34 (The stationarity condition for $\beta < 1$). *Let $x \sim U(0, 1)$, $\alpha > 0$, $\beta \in [0, 1)$, and let K satisfy Assumption 2. Every symmetric interior stationary profile of the β -game satisfies*

$$(1 + \alpha - \beta) K(b) = (1 - \beta) \frac{2(1 + \alpha)}{2 + \alpha} K(\Delta). \quad (13)$$

Proof in Appendix A.

Proposition 34 characterizes stationary profiles; under super-regularity its root is in fact the game's unique equilibrium, so the characterization is not an artifact of restricting to symmetric or stationary profiles. Define $\psi_\beta(t) := t + \frac{1+\beta}{1-\beta} K(t)/k(t)$ —the baseline ψ with the term $K/k = 1/r$ weighted by how much a converted opponent voter is worth relative to a converted abstainer. The $\alpha \downarrow 0$ limit of Eq. (13) then reads $\psi_\beta(\Delta) = 1$, which is Eq. (7) with competition priced in. (As in Eq. (12), substituting $\alpha = 0$ outright yields an identity; the content is at first order in α .)

The next theorem is the continuity half of the section's message: below the knife-edge the equilibrium is unique, for every $\alpha \geq 0$.

Theorem 35 (Existence and uniqueness for every $\beta < 1$). *Let $x \sim U(0, 1)$, $\beta \in [0, 1)$, and let K be super-regular.*

- (a) *If $\alpha = 0$, the game has a unique Nash equilibrium: symmetric, with gap the unique root of $\psi_\beta(\Delta) = 1$.*
- (b) *If $\alpha > 0$, the game has a unique Nash equilibrium: symmetric, with no dead zone, with gap the unique root of Eq. (13), and strictly more polarized than every symmetric margin-game profile: $\Delta^*(\alpha, \beta) > \frac{\alpha}{2+\alpha}$ (Proposition 46).*

Proof: Theorems 63 and 64.

Corollary 36. *If $x, \kappa \sim U(0, 1)$ then*

$$\Delta^*(\alpha, \beta) = \frac{\alpha(2+\alpha)(1+\alpha-\beta)}{(1+\alpha-\beta)(2+\alpha)^2 - 4(1-\beta)(1+\alpha)}, \quad \Delta^*(0, \beta) = \frac{1-\beta}{2}, \quad (14)$$

the second entry being the $\alpha \downarrow 0$ limit of the first, equivalently the root of $\psi_\beta(\Delta) = 1$ for uniform costs.

Proof. Put $K = \text{id}$ and $b = (\lambda\Delta - \alpha)/2$ in Eq. (13) and solve the resulting linear equation in Δ . At $\beta = 0$ this returns $\frac{2+\alpha}{4+\alpha}$ (Corollary 25), and as $\beta \rightarrow 1$ it tends to $\frac{\alpha}{2+\alpha}$, the top of the band of Section 5.4. $\square$

Uniform costs, arbitrary electorates The quartile structure of Corollary 11 also generalizes, giving the distribution-free face of the same de-polarization.

Proposition 37 (Mass-gap law). *Let $\alpha = 0$, $\kappa \sim U(0, 1)$, $\beta \in [0, 1)$, and let v satisfy Assumption 2. Every Nash equilibrium satisfies*

$$V(c_2) - V(c_1) = \frac{1-\beta}{2}, \quad 2V(c_1) = V(m) + \beta(1 - V(m)) :$$

the candidates trap voter mass exactly $\frac{1-\beta}{2}$, independently of v . If moreover $\rho_v \leq \frac{4}{1-\beta}$, each candidate's payoff is concave on her side, and equilibria are exactly the solutions passing the two cross-over checks.

Proof: Proposition 78.

Corollary 38 (Quantile equilibria). *In the setting of Proposition 37 with v symmetric about $\frac{1}{2}$, the quantile profile (c_1^*, c_2^*) with $V(c_1^*) = \frac{1+\beta}{4}$, $V(c_2^*) = \frac{3-\beta}{4}$ is a Nash equilibrium under explicit bounds on v (Theorem 94); for convex symmetric electorates the single condition $v(c_2^*)/v(\frac{1}{2}) \leq \frac{4}{1-\beta}$ suffices. In particular, for the arcsine electorate $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$,*

$$c_2^* = \sin^2\left(\frac{\pi(3-\beta)}{8}\right), \quad \frac{v(c_2^*)}{v(\frac{1}{2})} = \frac{1}{\sin\left(\frac{\pi(3-\beta)}{4}\right)} \in (1, \sqrt{2}],$$

so a pure equilibrium exists for every $\beta \in [0, 1)$.

Note how the two conditions of Proposition 37 interpolate the whole paper’s geometry: at $\beta = 0$ they are Theorem 9; as $\beta \uparrow 1$ the trapped mass vanishes and the quantiles collapse onto the median of Theorem 45, moving from the quartiles to the median along an explicit path of quantile pairs.

5.2 Costs and Polarization at a Fixed β

At a fixed β the comparative statics of Section 3.2 carry over as they do at a fixed α —the cost channel (Proposition 39) and the transmission of mass polarization (Proposition 40)—and the turnout curve of Proposition 29 does not depend on β at all (Proposition 41).

Proposition 39 (The cost channel survives competition). *In the setting of Theorem 35, let $\tilde{K}$ be a second super-regular cost distribution with $\tilde{r} \geq r$ pointwise on $(0, 1)$. Then $\tilde{\Delta}^*(\alpha, \beta) \geq \Delta^*(\alpha, \beta)$ for every $\alpha \geq 0$ and every $\beta \in [0, 1)$.*

Proof: Proposition 67.

At $\beta = 0$ this is Theorem 27, and at $(\alpha, \beta) = (0, 0)$ it is Proposition 18: more responsive electorates sustain weakly more polarization, uniformly in competitiveness.

Mass polarization under competition The transmission of mass polarization is unchanged as well.

Proposition 40 (Mass polarization scales elite polarization, general form). *Proposition 28 holds verbatim for every $\beta \in [0, 1)$: the game $(V_\sigma, K, \alpha, \beta)$ is isomorphic to $(V_0, K_\sigma, \alpha, \beta)$, and in part (b) the setting is that of Theorem 35.*

Proof. Under the affine map both benefits scale, $B_i = \sigma B_i^0$, so both vote shares are those of (V_0, K_σ, α) , and hence so is $U_i = u_i - \beta u_{-i}$; the argument that no best response lies outside $\text{supp } V_\sigma$ is already made for every $\beta \geq 0$ in the proof of Proposition 28. Part (a) then follows from Proposition 32, and part (b) from Proposition 39 in place of Theorem 27, since K_σ is super-regular whenever K is. $\square$

So under power costs elite polarization is exactly proportional to mass polarization at every level of competition, and otherwise the elasticity of K decides whether it is amplified or damped. At the margin endpoint mass polarization acts differently, only through the shape of the electorate at the centre (Theorem 48).

Turnout: competition lowers it Turnout depends on the platforms but not on β , which enters only the candidates' objectives; so the turnout curve of Proposition 29 is the same at every level of competition, and what competition changes is where on it the equilibrium sits.

Proposition 41 (Turnout under competition). *In the setting of Theorem 35, let $\alpha > 0$. The formula for $T'(\Delta)$ and part (b) of Proposition 29 hold unchanged for every $\beta \in [0, 1)$, and at the equilibrium*

$$T'(\Delta^*) = \frac{2(1+\alpha)(1+\beta)}{\lambda(1+\alpha-\beta)} K(\Delta^*) > 0.$$

Equilibrium turnout $T(\Delta^(\alpha, \beta))$ is strictly decreasing in β .*

Proof. T is the sum of the two vote shares at the symmetric profile, and neither involves β , so the formula for $T'(\Delta)$ and part (b) carry over. At the equilibrium there is no dead zone (Theorem 35), and Eq. (13) gives $K(b)/K(\Delta^*) = \frac{2(1+\alpha)(1-\beta)}{\lambda(1+\alpha-\beta)}$; substituting into $T'(\Delta)$ gives the display, whose sign is that of $2(1+\alpha-\beta) - \lambda(1-\beta) = \alpha(1+\beta) > 0$. Finally, the root $\Delta^*(\alpha, \beta)$ of Eq. (13) is unique, hence continuous in β , and strictly decreasing in β (Proposition 42); as β rises it therefore sweeps an interval at every point of which $T' > 0$, and $T(\Delta^*)$ falls. $\square$

At $\beta = 0$ the display is part (a) of Proposition 29. Locally, more polarized candidates raise turnout at every level of competition; and since competition narrows the gap, it lowers turnout too.

5.3 Higher β De-polarizes

Under the hypotheses that delivered uniqueness, the equilibrium gap is monotone in α and β at once: competition for the margin pulls the candidates together at every level of alienation, and alienation pushes them apart at every level of competition.

Proposition 42. *In the setting of Theorem 35, the equilibrium gap $\Delta^*(\alpha, \beta)$ is strictly decreasing in β on $[0, 1)$ and (for $\alpha > 0$) strictly increasing in α .*

Proof: Proposition 66.

Competition for the margin thus pulls candidates together exactly as costly voting and alienation push them apart; in the doubly-uniform benchmark the closed form (14) prices the trade-off between α and β exactly.

Under alienation, competition lowers turnout as well as polarization (Proposition 41), so the reading ruled out for alienation must be ruled out here too: that the candidates converge because fewer people vote. The argument of Corollary 31 applies unchanged.

Corollary 43 (De-polarization at unchanged turnout). *In the setting of Theorem 35, fix $\beta \in (0, 1)$. Then there are positive multiples K_0 and K_β of K , both admissible, such that the games $(V, K_0, \alpha, 0)$ and $(V, K_\beta, \alpha, \beta)$ have the same equilibrium turnout while their equilibrium gaps are $\Delta^*(\alpha, 0)$ and $\Delta^*(\alpha, \beta) < \Delta^*(\alpha, 0)$.*

Proof. As for Corollary 31, with Proposition 32 and Proposition 42 in place of Proposition 15 and Theorem 30. $\square$

Super-regularity is a genuine hypothesis, not a convenience:

Example 44 (A fold: competition can polarize). There is an explicit cost distribution with piecewise-linear K/k (Example 70) whose vote-share game has *three* equilibria, with gaps $\frac{1}{3}$, $\frac{19}{50}$ and $\frac{11}{20}$. Along the middle branch $d\Delta^*/d\beta > 0$: more competitive candidates, *more* polarization. At $\beta^* = \frac{3}{47} \approx 0.064$ the middle and upper branches collide and annihilate in a fold, and the equilibrium set jumps discontinuously from three points to one. (We verified the middle branch is a genuine global equilibrium, to deviation gain $2 \cdot 10^{-12}$.)

The example is not a curiosity about multiplicity; it locates precisely where the de-polarizing logic can break. Monotonicity of K/k is what guarantees that the “supply” curve of Corollary 17 meets the “demand” curve once and from above. When it does not, an increase in β can slide the crossing the wrong way along a branch where supply cuts demand from below—and such a branch is fragile, which is why it is the one that dies in the fold. Two lessons for the polarization story: comparative statics in β are safe exactly under the same hypothesis that gives uniqueness, and a small increase in competitiveness can produce a *discontinuous* collapse in polarization when the fold is reached: at $\beta^* = \frac{3}{47}$ the upper branch, down from $\frac{11}{20}$ at $\beta = 0$ to $\frac{1}{2}$, disappears and the equilibrium gap drops to $\frac{11}{36} \approx 0.31$.

Where the decline ends as $\beta \uparrow 1$ —not at the median—is taken up in Section 5.4.

5.4 The Margin Game ($\beta = 1$)

At the endpoint $\beta = 1$ the game is zero-sum, and without alienation the classical Hotelling–Downs prediction of Claim 1 returns in full force—costs and all.

Theorem 45 (Without alienation, the margin game selects the median). *Let $\alpha = 0$, let v, k satisfy Assumption 2, and let $c_{\text{med}} = V^{-1}(\frac{1}{2})$. Then $(c_{\text{med}}, c_{\text{med}})$ is the unique Nash equilibrium of the margin game, and every unilateral deviation strictly lowers the deviator’s margin.*

Proof in Appendix B.

Together with Section 3.1, this cleanly separates the two objectives: costly participation moves *vote-share* maximizers away from the median for every distribution satisfying Assumption 2, yet leaves *margin* maximizers exactly at the median, for all distributions. With full alienation, however, the margin game degenerates:

Proposition 46 (The margin band under alienation). *Suppose $\beta = 1$. Let $x \sim U(0, 1)$ and let K satisfy Assumption 2. For every $\alpha \geq 0$, the pure Nash equilibria of the margin game are exactly*

$$E(\alpha) = \left[\frac{1}{2+\alpha}, \frac{1+\alpha}{2+\alpha} \right]^2,$$

each with margin 0: both platforms may sit anywhere in a band of width $\Delta_0 := \frac{\alpha}{2+\alpha}$ around the median, independent of the cost distribution.

Proof in Appendix B.

The band contains a continuum of profiles, from the tied median profile to the maximally separated $(\frac{1}{\lambda}, \frac{1+\alpha}{\lambda})$ —and even tied profiles $c_1 = c_2$ in which (almost) nobody votes and both candidates, caring only about the margin, are content. Which of these predictions should one believe? Theorem 47 below answers the question, and the answer is not the median.

Both results are knife-edges—Theorem 45 in the objective, Proposition 46 in the voter density. Theorem 47 shows which point of the band survives a small vote-share motive, and Theorem 48 what happens to the median when the electorate is not uniform.

The decline in β of Section 5.3 ends at a specific point of the band.

Theorem 47 (Selection in the band). *In the setting of Proposition 34, as $\beta \uparrow 1$ the right-hand side of Eq. (13) vanishes, forcing $K(b) \rightarrow 0$ and hence $b \rightarrow 0$: for every cost distribution K and every $\alpha > 0$ the selected profile converges to*

$$(c_1, c_2) \rightarrow \left(\frac{1}{2+\alpha}, \frac{1+\alpha}{2+\alpha} \right), \quad \Delta \rightarrow \frac{\alpha}{2+\alpha},$$

which is the most polarized profile of the band $E(\alpha)$ —and also its turnout-maximizing profile, since inside the band both tents are interior and $T = 2(\frac{1}{\alpha} + \frac{1}{\lambda})G(\Delta)$ increases in Δ .

Proof. In Eq. (13) the factor $1 + \alpha - \beta$ tends to $\alpha > 0$ while the right-hand side tends to 0, so $K(b) \rightarrow 0$; as K is strictly increasing with $K(0) = 0$, $b \rightarrow 0$, and $b = (\lambda\Delta - \alpha)/2$ from Eq. (9) gives $\Delta \rightarrow \alpha/\lambda$. The profile is symmetric, so $c_1 = \frac{1-\Delta}{2} \rightarrow \frac{1}{\lambda}$, the lower edge of $E(\alpha)$ in Proposition 46. $\square$

Thus the indeterminacy of the band is resolved in a specific direction: an arbitrarily small vote-share motive added to a margin-maximizing candidate selects maximal polarization, not the median (Fig. 9). The median is the *least*

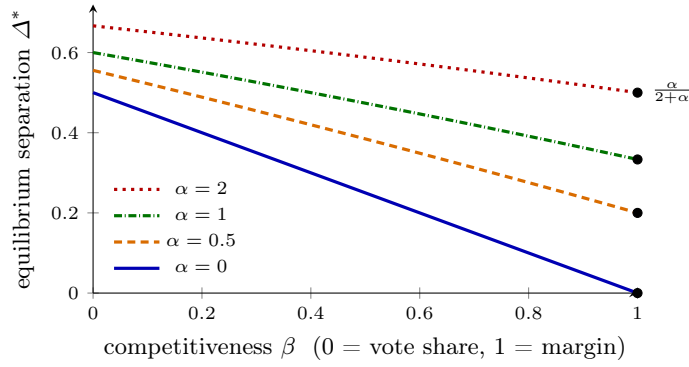


Fig. 9. Equilibrium separation as competitiveness rises, doubly uniform (Theorem 47). Increasing β pulls the candidates together, but for $\alpha > 0$ the limit at $\beta = 1$ is $\frac{\alpha}{2+\alpha}$ —the top of the margin band $E(\alpha)$, not the median. Only at $\alpha = 0$ does the limit reach the median, and there the collapse is linear.

robust point of the band. At $\alpha = 0$ the band is a single point and the limit is the median; Eq. (14) shows the collapse is linear, $\Delta^* = (1 - \beta)/2$.

Under super-regularity the collapse of Theorem 47 has an exact rate: the approach to the band top is linear in $1 - \beta$ whenever $k(0^+) \in (0, \infty)$, and of order $(1 - \beta)^{1/p}$ when $K(t) \sim Ct^p$ near 0; at $\alpha = 0$ the rate $\frac{1-\beta}{2}$ of Eq. (14) is universal, $\Delta^* = \frac{1-\beta}{2} (1 + o(1))$ for every K with $k(0^+) \in (0, \infty)$ (Proposition 68).

Mass polarization breaks the median for margin maximizers With $\alpha > 0$ the median solves the margin first-order condition but need not be an equilibrium. The second-order condition at the median is degenerate (identically zero), so the question is settled at fourth order; the answer is a clean dichotomy in the curvature of the electorate at the center.

Theorem 48 (Median dichotomy). *Let $\kappa \sim U(0, 1)$, $\alpha > 0$, and let v be symmetric about $\frac{1}{2}$ and C^2 near $\frac{1}{2}$. Consider the margin obtained by a candidate deviating to $\frac{1}{2} - \delta$ against an opponent at the median. Then*

$$w_1(\delta) = C(\alpha) v''(\tfrac{1}{2}) \delta^4 + o(\delta^4), \quad C(\alpha) = \frac{1+\alpha}{2\alpha(2+\alpha)} + \frac{1}{6} \left(\frac{1}{\alpha^2} - \frac{1}{(2+\alpha)^2} \right) > 0.$$

Hence the median profile is a strict local equilibrium of the margin game if $v''(\frac{1}{2}) < 0$ (an electorate peaked at the center), and is not an equilibrium if $v''(\frac{1}{2}) > 0$ (a hollow, polarized electorate). Uniform voters, $v'' \equiv 0$, are exactly the knife-edge that produces the band $E(\alpha)$.

Proof in Appendix A.

The formula is sharp: for $\alpha = 1$, $\delta = 0.02$ it predicts $w_1 = -9.244 \cdot 10^{-7}$ for $v = \text{Beta}(2, 2)$ and $+1.962 \cdot 10^{-7}$ for $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$, against simulated values

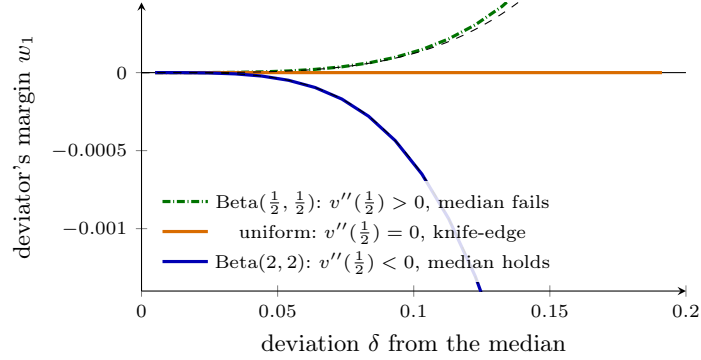


Fig. 10. The quartic law of Theorem 48 at $\alpha = 1$ with uniform costs: a candidate deviating to $\frac{1}{2} - \delta$ against an opponent at the median gains iff the electorate is hollow at the center. Thin dashed lines are the prediction $C(\alpha)v''(\frac{1}{2})\delta^4$; they are indistinguishable from the computed curves for small δ . For uniform voters the margin is zero to machine precision at every δ —the knife-edge that manufactures the band $E(\alpha)$.

$-9.244 \cdot 10^{-7}$ and $+1.967 \cdot 10^{-7}$; for uniform voters the simulated margin is zero to machine precision at every δ (Fig. 10). Uniform costs enter only through the constant. For a general K the same reflection argument gives

$$w_1(\delta) = v''(\tfrac{1}{2}) \left[\frac{\delta^2}{2} \left(\frac{1}{\alpha} + \frac{1}{\lambda} \right) \int_0^\delta K + \delta \left(\frac{1}{\alpha^2} - \frac{1}{\lambda^2} \right) \int_0^\delta K(t) (\delta - t) dt \right] + o\left(\delta^2 \int_0^\delta K\right),$$

whose bracket is positive because $\lambda > \alpha$: the dichotomy in the sign of $v''(\frac{1}{2})$ holds for every cost distribution satisfying Assumption 2, and at $K = \text{id}$ the bracket is $C(\alpha)\delta^4$. Theorem 48 is the margin-game counterpart of the mass-polarization result of Section 3.2, and it sharpens what “mass polarization” has to mean. In Section 3.2 the relevant statistic was a global one, the interquartile range. Here it is purely *local*: only the sign of v'' at the median matters, and the size of $C(\alpha)$ is irrelevant to the conclusion. An electorate can be enormously dispersed and still hold the candidates at the median against small deviations, provided it is peaked there; and it can be barely dispersed at all and lose the median, provided it is locally hollow. For margin maximizers, then, what drives elite polarization is not how spread out the voters are but whether the center is *empty*; the classical Downsian prediction fails as soon as it is, the moment $\alpha > 0$.

Existence can fail Where alienation meets a hollow electorate, the margin game can have no pure equilibrium at all, and a certificate rules one out for the arcsine electorate at $\alpha = 1$ and for every $\beta \geq 0.9168$.

Proposition 49 (Non-existence). *[computer-assisted] Let $\alpha = 1$ and $\kappa \sim U(0, 1)$. For $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$ and for $v = \text{Beta}(0.8, 0.8)$ the margin game has no pure Nash equilibrium (Theorem 75); for the former, the same holds for every $\beta \geq 0.9168$ (Theorem 97).*

Each statement is a finite computation at a fixed (α, v) . We expect, but do not prove, that non-existence persists for every sufficiently hollow v and every α above a v -dependent threshold.

Proof (Sketch; the full certificate is Theorems 75 and 97). The margin game is symmetric and zero-sum, so a pure equilibrium exists if and only if some position has security value zero, $\max_{c \in [0,1]} \min_{c' \in [0,1]} w(c, c') = 0$, writing $w(c, c')$ for the margin of a candidate at c against one at c' ; since $w(c, c) = 0$ and w is antisymmetric, the security value is always ≤ 0 . For $\alpha = 1$ and uniform costs it equals $-8.2244 \cdot 10^{-3}$ (at $c = 0.829$) for the arcsine density $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$ and $-3.0238 \cdot 10^{-3}$ for $\text{Beta}(0.8, 0.8)$, so *every* position can be strictly beaten (Fig. 11); for $\text{Beta}(2, 2)$ the grid security value attains 0 at the median and only there, as Theorem 48 leads one to expect. A Lipschitz bound on w turns the finite grid computation into an exact statement about the continuum, giving a computer-assisted proof. A mixed equilibrium always exists by Glicksberg’s fixed-point theorem, with value 0. $\square$

The certificate extends off the boundary $\beta = 1$: no pure equilibrium exists for any $\beta \geq 0.9168$ (Theorem 97), while a pure equilibrium is still found numerically at $\beta = 0.90$, placing the true threshold in $(0.90, 0.9168]$ with the numerics placing that threshold β_c between 0.908 and 0.909. The contrast with $\alpha = 0$ is stark: there, under uniform costs, the very arcsine electorate that admits no pure equilibrium at $\alpha = 1$ keeps one for *every* $\beta < 1$, along with every other convex symmetric electorate meeting the density bound of Corollary 38. Non-existence is therefore a phenomenon of alienation and competitiveness *together*, not of competitiveness alone.

Two remarks complete the picture. First, equilibria of nearly-zero-sum games converge onto margin-game equilibria (upper hemicontinuity, Theorem 72). Corollary 38 shows the statement is not vacuous at $\alpha = 0$, where the approximating equilibria exist all the way to the limit and select the median. Second, at $\alpha = 0$ we conjecture pure existence for *every* (v, k) and $\beta < 1$ (Conjecture 100); the certified failure at $\alpha = 1$ shows that no proof can survive a relaxation of $\alpha = 0$. Together with Theorem 48, this gives the picture for the margin game: the median survives alienation, at least locally, when the electorate is centrally peaked; when it is hollow, the median fails and, at least for the densities examined here, no pure equilibrium takes its place.

6 Discussion and Further Research

What drives polarization Table 3 collects the answer the paper was built to give. Three forces raise equilibrium polarization—a more dispersed electorate, costlier voting in the reverse-hazard-rate sense, and stronger alienation; a hollow center removes the median as a margin-game equilibrium; one force lowers polarization—competition for the margin; and one candidate that every intuition nominates, the level of turnout, provably does nothing at all. The two voter-side channels do not interfere with each other (Theorem 27): each still

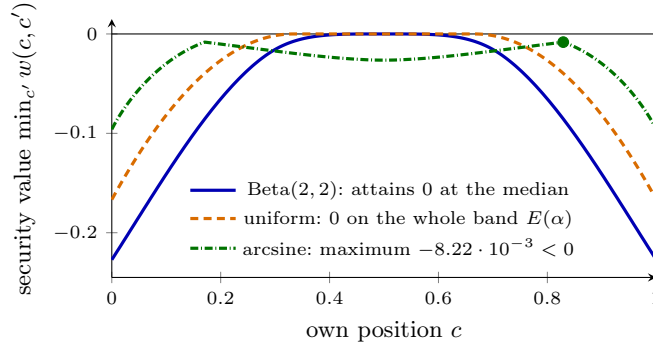


Fig. 11. The certificate behind Proposition 49. The margin game is symmetric zero-sum, so a pure equilibrium exists iff some position has security value 0. For a centrally peaked electorate the median achieves it; for uniform voters every point of the band does; for the arcsine (hollow) electorate the maximum is strictly negative at $c = 0.829$, so *every* position can be strictly beaten and no pure equilibrium exists. Computing this curve requires quadrature that respects the endpoint singularity of v : a naive grid rule returns -0.387 for the arcsine maximum, a factor of 47 too negative (Remark 99).

polarizes at every level of the other, and both survive the presence of competitiveness (Propositions 39 and 42). The one de-polarizing force is also the only one whose comparative static is shown to reverse sign when super-regularity fails (Example 44), an asymmetry worth keeping in mind when reading the model empirically.

The baseline and the two dials together Table 4 collects the doubly-uniform benchmarks. Reading it as a story: costly voting moves vote-share maximizers from the median to the quartiles; alienation pushes them further out, without bound; competitiveness pulls them back in, linearly to the median when voters are not alienated, but only to the top of the margin band when they are. The classical Downsian prediction survives costly voting only under the margin objective, and even then only when voters are unalienated or, locally, when the electorate is centrally peaked; for alienated voters facing a hollow electorate, nothing need survive—not even equilibrium existence. Polarization in this model is thus neither an inevitability nor an accident: it is the resultant of two voter-side forces—costly voting and alienation—and one candidate-side force, each with an explicit comparative static.

Open questions Several questions remain open. On sufficiency: sharp thresholds for the density-ratio umbrellas of Table 4, and a sufficiency theory beyond uniform costs (Proposition 59 covers non-increasing k at $\beta = 0$; the obstruction at $\beta > 0$ is isolated in Remark 88). On alienation: a general-electorate version of Theorem 30, whose conclusion holds numerically for the concentrated electorate of Remark 23 but is proved only for uniform voters. On existence: under Assumption 2 we conjecture pure existence for every (v, k) in the baseline and, at

Table 3. Every force in the model, and which way it moves equilibrium polarization Δ^* . The last row is not a gap in the analysis but a theorem: the level of participation is an output of the model, never an input to it.

Increase in . . .	Δ^*	Holds when	Where
electorate spread (IQR)	$\uparrow$	unif. costs, v symmetric, $\rho_v \leq 4$	Corollary 11
electorate scale σ	$\uparrow$	unif. voters, K super-reg.; exact for power costs (any v)	Proposition 28
voting costs (rhr order)	$\uparrow$	unif. voters, K super-reg., any (α, β)	Propositions 18 and 39 and Theorem 27
alienation α	$\uparrow$	unif. voters, K super-reg., $\alpha > 0$; $\Delta^* \rightarrow 1$	Theorem 30
competitiveness β	$\downarrow$	unif. voters, monotone rhr <i>only</i>	Proposition 42 and Example 44
hollow center, $v''(\frac{1}{2}) > 0$ breaks median		margin game, unif. costs, $\alpha > 0$, v symmetric	Theorem 48
cost level, $K \mapsto zK$	—	always: no effect	Proposition 32 and Remark 19

$\alpha = 0$, for every $\beta < 1$ (Remark 8 and Conjecture 100), the first of these open already for uniform costs once $\rho_v > 4$; closing the certified gap $(0.90, 0.9168]$ of Theorem 97; and characterizing the mixed equilibria of the margin game where pure ones fail. On the margin game with alienation: the exact non-existence region in (α, β, v) , and whether the quantile path of Corollary 38 has an $\alpha > 0$ analogue. The certificate technique behind Theorem 97 may be of independent interest for other spatial games with continuous strategy spaces.

Empirically, where political polarization has risen [74,14] alongside declining turnout [49,37], the model offers a channel complementary to media- and network-based accounts [75,55], with a sharpened prediction: the association should run through the *composition* of the abstaining margin rather than the participation rate, and the model separates “voting became harder” (uniform shifts in participation) from “voters became disaffected” (abstention concentrated among the ideologically distant)—distinctions the turnout literature is equipped to test [58,36]. A first piece of evidence on the composition side exists: across Dutch municipalities, lower turnout goes with a more polarized distribution of the votes cast, the parties furthest from the center gaining most, consistent with centrist voters being the ones who abstain [5]—the abstaining margin of the baseline model, where abstention concentrates between the candidates.

That test has a confound the present model assumes away. Costs here are drawn i.i.d. and independently of position (Section 2), and the assumption does

Table 4. The baseline and the two dials at a glance. Second column: the equilibrium in the doubly-uniform benchmark $(x, \kappa \sim U(0, 1))$, with $\lambda = 2 + \alpha$. Third column: sufficient density-ratio bound on v for the first-order conditions to characterize equilibria under uniform costs, where $f_\beta = \frac{2\lambda(1+\alpha)+\beta}{(1+\alpha)^2}$ (concavity) and $g_\beta = \frac{(1+\alpha)\lambda}{\alpha(1+\alpha+\beta)+\beta\lambda}$ (dead zone), from Theorem 85; the bounds come from different arguments and are not claimed tight. The row ‘both’ is stated for $\alpha > 0$ and does not reduce to the row above it as $\alpha \downarrow 0$: $f_\beta \rightarrow 4 + \beta \leq \frac{4}{1-\beta}$, because the $\alpha = 0$ bound uses an exact cancellation at the midpoint that the worst-case argument of Theorem 85 forgoes (Remark 86), and $g_\beta \rightarrow 1/\beta$ bounds a dead-zone piece of the strategy space that vanishes at $\alpha = 0$.

Setting	Δ^* (doubly unif.)	Umbrella $\rho_v \leq$	Where
baseline $(0, 0)$	$\frac{1}{2}$, at $(\frac{1}{4}, \frac{3}{4})$	4	Corollary 14 and Proposition 10
alienation $(\alpha, 0)$	$\frac{2+\alpha}{4+\alpha}$	$\min\{\frac{2\lambda}{1+\alpha}, \frac{\lambda}{\alpha}\}$	Corollary 25 and Theorem 56
competition $(0, \beta)$	$\frac{1-\beta}{2}$ (quantile profile)	$\frac{4}{1-\beta}$	Corollary 36 and Proposition 37
both (α, β)	Eq. (14)	$\min\{f_\beta, g_\beta\}$	Corollary 36 and Theorem 85
margin limit $(\alpha, 1)$	band: $\Delta \in [0, \frac{\alpha}{\lambda}]$	knife-edge	Proposition 46 and Theorem 48

real work: written out for the baseline, the balance condition of Lemma 7 has an outer term that factorizes as $k(\Delta)V(c_1)$, one cost density times one voter mass, because every voter on the flank both faces the same benefit Δ and draws from the same K . Replace K by a family K_x and that product becomes $\int_0^{c_1} k_x(\Delta)v(x) dx$, which does not split; Proposition 15 and the balance principle itself survive, but Corollary 16 weakens from “one scalar function summarizes the cost side” to a statement about a field $\{r_x\}$. Note that this is not the position-dependence the model already has. Alienation acts through the benefit, via d_i , so its bite moves when the candidates move; a cost–position correlation is pinned to the voter and does not move at all. The paper treats endogenous position-dependence thoroughly and exogenous position-dependence not at all, and no choice of α substitutes for the latter. The direction is signed, moreover, and the two channels disagree: alienation predicts that a candidate’s own extremists are the first to abstain, whereas if the ideologically committed are simply cheaper to mobilize, then k_x is shifted toward low costs at extreme x , the flank is more responsive than Proposition 18 assumes, and the model *understates* polarization. Which of the two dominates is an empirical question, and it is the same question as whether a composition shift in the abstaining margin identifies the strategic channel or merely reflects who finds voting cheap.

Finally, the costly-turnout framework introduced here can be naturally combined with additional dynamics. One extension is to allow for non-linear or probabilistic voter utilities, as in [22], which may better capture how voters evaluate platforms. Another is to embed the model in a dynamic environment with re-

peated elections, building on iterative voting models [64]; such a setting could generate endogenous status quo effects that shape participation incentives, in the spirit of [81]. Further natural directions include more than two candidates and endogenous entry, where the alienation dead zones suggest room for centrist or extremist entrants; asymmetric candidates (valence or cost advantages), for which our asymmetry law (Lemma 33) already prices the trade-off; and finite electorates with pivotality considerations in the spirit of the Calculus of Voting [77], connecting the present continuum model back to the classical turnout literature.

A Deferred Proofs

This appendix collects the proofs deferred from Sections 3 to 5. Proofs of Lemma 6, Theorems 9, 13 and 45, Propositions 10, 18 and 46, and Corollary 11 are in Appendix B. The remaining proofs, together with the supplementary results, worked examples and certificates they draw on, occupy Appendices I to Q.

Proof of Proposition 22. $u_1 = \int_{x_1^{\text{out}} \vee 0}^{x_1^{\text{in}}} K(B_1)v$. The Leibniz terms at both endpoints vanish because $B_1 = 0$ there and $K(0) = 0$ (at a truncated lower limit $x_1^{\text{out}} < 0$ the limit is the constant 0 and contributes nothing either). On $[x_1^{\text{out}} \vee 0, c_1]$ we have $\partial_{c_1} B_1 = -(1 + \alpha)$ and on $[c_1, x_1^{\text{in}}]$, $\partial_{c_1} B_1 = +(1 + \alpha)$, whence $\partial_{c_1} u_1 = (1 + \alpha) [\int_{c_1}^{x_1^{\text{in}}} k(B_1)v - \int_{x_1^{\text{out}} \vee 0}^{c_1} k(B_1)v]$. Setting this to zero gives Eq. (10). $\square$

Proof of Theorem 24. Write $\Psi(\Delta) := \int_{b(\Delta)}^{\Delta} r - \log(\lambda/2)$ on $[\frac{\alpha}{\lambda}, 1)$, so that Eq. (12) reads $\Psi(\Delta) = 0$.

Necessity. Every equilibrium is symmetric with gap a root of Eq. (12), as shown before the theorem.

Uniqueness of the root. By Eq. (9) the interval $[b(\Delta), \Delta]$ has length $\frac{\alpha}{2}(1 - \Delta)$, strictly decreasing in Δ , and its left endpoint $b(\Delta)$ is strictly increasing in Δ (slope $\lambda/2 > 0$). Hence as Δ grows the interval both shifts right and shrinks; since r is non-increasing and positive, $\int_b^{\Delta} r$ is strictly decreasing, so Ψ is strictly decreasing. At $\Delta = \alpha/\lambda$ we have $b = 0$ and $\int_0^{\Delta} r = +\infty$ (as $\log K(0^+) = -\infty$), so $\Psi > 0$ there; at $\Delta \uparrow 1$ we have $b \uparrow 1$, the interval degenerates, and $\Psi \rightarrow -\log(\lambda/2) < 0$. A unique root Δ^* follows.

The root is an equilibrium. Fix the opponent at c_2 and let candidate 1 choose $c_1 \in [0, c_2)$. In the no-dead-zone region $u_1 = A G(\Delta) - \frac{1}{\alpha} G(b_1)$ with $A = \frac{2(1+\alpha)}{\alpha\lambda}$, so

$$\frac{\partial u_1}{\partial c_1} = \frac{1 + \alpha}{\alpha} \left[K(b_1) - \frac{2}{\lambda} K(\Delta) \right], \quad \text{whose sign is that of } \log \frac{\lambda}{2} - \int_{b_1}^{\Delta} r.$$

Moving c_1 toward the opponent decreases Δ at unit rate and b_1 at rate $1 + \alpha$, so the interval $[b_1, \Delta]$ lengthens and moves left; with r non-increasing, $\int_{b_1}^{\Delta} r$ is

strictly increasing in c_1 . Hence $\partial_{c_1} u_1$ changes sign exactly once, from $+$ to $-$: $u_1(\cdot, c_2)$ is strictly single-peaked on the own side, and by Remark 23 the dead-zone region contains no maximizer. For a cross-over deviation note that, at a fixed distance t from the opponent, the payoff is $A G(t) - \frac{1}{\alpha} G((t - \alpha \varrho)^+)$, where ϱ is the room behind the deviator ($\varrho = c_2 - t$ on the left, $\varrho = 1 - c_2 - t$ on the right); this is increasing in ϱ , and at the symmetric profile $c_2 > \frac{1}{2}$ gives $c_2 - t > 1 - c_2 - t$. Hence every far-side deviation is dominated by its mirror image on the own side, and the tie yields 0. The profile is therefore a global mutual best response. $\square$

Proof of Proposition 28. No best response lies outside $\text{supp } V_\sigma$, so the strategy space may be taken to be that interval, on which the affine map is a bijection. Indeed, let $L := \min \text{supp } V_\sigma$, write $B_i(x; c_1, c_2)$ for the benefit at the profile (c_1, c_2) , and suppose $c_1 < L$, wherever c_2 lies. Every voter x in the support has $x > c_1$, so $\partial B_1(x)/\partial c_1 = 1 + \alpha > 0$ and $\partial B_2(x)/\partial c_1 = -1 < 0$ pointwise; as K is non-decreasing, u_1 is non-decreasing and u_2 non-increasing in c_1 on $[c_1, L]$, whence $U_1(c_1, c_2) \leq U_1(L, c_2)$ for every $\beta \geq 0$. Moreover c_1 is not a best response. If $c_2 \neq L$, the voter at L has $B_1(L; L, c_2) = |L - c_2| \in (0, 1)$, so by continuity $B_1(\cdot; L, c_2) \in (0, 1)$ on a set of positive V_σ -measure; there K is strictly increasing (Assumption 2) and $B_1(\cdot; c_1, c_2) = B_1(\cdot; L, c_2) - (1 + \alpha)(L - c_1)$ is pointwise smaller, hence $u_1(c_1, c_2) < u_1(L, c_2)$ and $U_1(c_1, c_2) < U_1(L, c_2)$. If $c_2 = L$, then $B_1(x; c_1, L) = (c_1 - L) - \alpha(x - c_1) < 0$ on the support, so $u_1(c_1, L) = 0$, while $B_2(L; c_1, L) = L - c_1 > 0$ gives $u_2(c_1, L) > 0$: for $\beta > 0$ the tied profile (L, L) , at which both shares vanish, is strictly better, and for $\beta = 0$ any c'_1 in the interior of the support, where $B_1(c'_1; c'_1, L) = c'_1 - L > 0$, earns a strictly positive share. The right-hand case is the mirror image.

Under the affine map $d_i(x) = \sigma|a - a_i|$, so $B_i(x) = \sigma B_i^0(a)$ and $u_i = \int K(\sigma B_i^0(a)) dV_0(a)$, which is the payoff of the game (V_0, K_σ, α) . Part (a) is Proposition 32 applied to $K_\sigma = \sigma^p K$ (legitimate since all benefits lie in $[0, 1]$ and $\sigma \leq 1$). Part (b) computes $r_\sigma(t) = \sigma k(\sigma t)/K(\sigma t) = \eta(\sigma t)/t$ and applies Theorem 27 to (V_0, K_σ, α) . Since $r_\sigma(t) = \sigma r(\sigma t)$ is non-increasing whenever r is, and $K_\sigma(1) = K(\sigma) \in (0, 1]$, every K_σ satisfies the hypotheses of that theorem. That $\Delta^*(\sigma)$ itself rises in σ at every $\alpha > 0$, whether the transmission is amplified or damped, follows from writing Eq. (12) for (V_0, K_σ, α) in x -space: substituting $u = \sigma t$ and using Eq. (9), Δ^* solves $\int_{\Delta^* - \frac{\alpha}{2}(\sigma - \Delta^*)}^{\Delta^*} r du = \log \frac{\lambda}{2}$, whose left side is strictly increasing in σ and, for r non-increasing, strictly decreasing in Δ^* . $\square$

Proof of Proposition 29. Differentiate $T = 2[A G(\Delta) - \frac{1}{\alpha} G(b)]$ using $db/d\Delta = \lambda/2$ and $G' = K$; part (a) substitutes Eq. (12). For (b), at $\Delta = 1$ we have $b = 1$ by Eq. (9), so the displayed derivative gives $T'(1) = \frac{2}{\alpha} K(1) \left[\frac{2(1+\alpha)}{\lambda} - \frac{\lambda}{2} \right] = -\frac{\alpha}{\lambda} K(1) < 0$, while $T'(\Delta^*) > 0$ by (a); since T' is continuous, T turns over in $(\Delta^*, 1)$. For the benchmark put $K = \text{id}$: $T' < 0$ iff $b/\Delta > 4(1 + \alpha)/\lambda^2$, and substituting $b = (\lambda\Delta - \alpha)/2$ and solving gives the stated $\bar{\Delta}$ (the ratio b/Δ is increasing in Δ , so T is single-peaked). That $\Delta^*(\alpha) = \frac{2+\alpha}{4+\alpha} < \bar{\Delta}(\alpha)$ is consistent with (a). $\square$

Proof of Theorem 30. Let Ψ be as in the proof of Theorem 24, strictly decreasing in Δ with root Δ^* . With $\lambda = 2 + \alpha$ and $\partial_\alpha b = -\frac{1-\Delta}{2}$, $\partial_\alpha \Psi = r(b) \frac{1-\Delta}{2} - \frac{1}{\lambda}$. At a root, super-regularity gives $\log \frac{\lambda}{2} = \int_b^\Delta r \leq r(b)(\Delta - b) = r(b) \frac{\alpha(1-\Delta)}{2}$, so $r(b) \frac{1-\Delta}{2} \geq \frac{1}{\alpha} \log \frac{\lambda}{2}$ and

$$\left. \frac{\partial \Psi}{\partial \alpha} \right|_{\Delta=\Delta^*} \geq \frac{1}{\alpha} \log \left(1 + \frac{\alpha}{2} \right) - \frac{1}{2 + \alpha} = \frac{\lambda \log(\lambda/2) - \alpha}{\alpha \lambda} > 0,$$

the last inequality being $(1 + s) \log(1 + s) > s$ for $s = \alpha/2 > 0$. Since Ψ is strictly decreasing in Δ , the implicit function theorem gives $d\Delta^*/d\alpha > 0$. Finally $\Delta^* > \alpha/\lambda \rightarrow 1$. $\square$

Proof of Proposition 34. Under uniform voters no untied symmetric stationary profile has a dead zone, for any $\alpha > 0$, any $\beta < 1$ and any K : if $b_1 = b_2 = b \leq 0$ then $u_1 = u_2 = A G(\Delta)$ with $A = \frac{2(1+\alpha)}{\alpha \lambda}$ (Remark 23), so $\partial U_1 / \partial c_1 = -(1 - \beta) A K(\Delta) < 0$ for $\Delta \in (0, 1)$ —the argument of Proposition 61(b), which uses only the first-order conditions. Hence $b_1, b_2 > 0$ and $u_1 = A G(\Delta) - \frac{1}{\alpha} G(b_1)$, $u_2 = A G(\Delta) - \frac{1}{\alpha} G(b_2)$. Since $\partial_{c_1} \Delta = -1$, $\partial_{c_1} b_1 = -(1 + \alpha)$ and $\partial_{c_1} b_2 = -1$,

$$\frac{\partial U_1}{\partial c_1} = -(1 - \beta) A K(\Delta) + \frac{1 + \alpha}{\alpha} K(b_1) - \frac{\beta}{\alpha} K(b_2).$$

At a symmetric profile $b_1 = b_2 = b$, and setting the derivative to zero yields Eq. (13). $\square$

Proof of Theorem 48. Write $q = \frac{1}{2}$, $c_1 = q - \delta$, $c_2 = q$, so $\Delta = \delta$ and $m = q - \delta/2$. The two benefit tents are reflections of one another about m : $B_2(x) = B_1(2m - x)$. With uniform costs $w_1 = \int B_1^+(x)[v(x) - v(2m - x)]dx$, and since v is symmetric about q , $v(2m - x) = v(1 - \delta - x) = v(x + \delta)$, so

$$w_1 = \int B_1^+(x)[v(x) - v(x + \delta)]dx = -\delta \int B_1^+ v' - \frac{\delta^2}{2} \int B_1^+ v'' + O(\delta^5).$$

B_1 is a tent with apex c_1 , height δ , left half-width δ/α and right half-width δ/λ , so $\int B_1^+ = \frac{\delta^2}{2} A$ with $A = \frac{1}{\alpha} + \frac{1}{\lambda}$, and $\int B_1^+(x - c_1) = \frac{\delta^3}{6} \left(\frac{1}{\lambda^2} - \frac{1}{\alpha^2} \right)$. Expanding $v'(x) = v'(c_1) + v''(c_1)(x - c_1) + O((x - c_1)^2)$ and using $v'(q) = 0$, $v'(c_1) = -\delta v''(q) + O(\delta^2)$,

$$\begin{aligned} -\delta \int B_1^+ v' &= \delta^4 v''(q) \left[\frac{A}{2} - \frac{1}{6} \left(\frac{1}{\lambda^2} - \frac{1}{\alpha^2} \right) \right] + O(\delta^5), \\ -\frac{\delta^2}{2} \int B_1^+ v'' &= -\frac{\delta^4}{4} v''(q) A + O(\delta^5). \end{aligned}$$

Adding gives $w_1 = \delta^4 v''(q) \left[\frac{A}{4} + \frac{1}{6} \left(\frac{1}{\alpha^2} - \frac{1}{\lambda^2} \right) \right] + O(\delta^5)$, which is the claim since $A/4 = \frac{1+\alpha}{2\alpha(2+\alpha)}$. $\square$

B Omitted Proofs: Baseline and Margin Game

This appendix proves the results of Sections 3 and 5 that are stated without proof in the main text; proofs of the remaining results are in Appendix A and in Appendices I to Q. For the reader's convenience we restate each result before its proof.

B.1 Preliminaries: Payoffs and Their Derivatives

Throughout, fix a profile (c_1, c_2) with $c_1 < c_2$ and write $\Delta := c_2 - c_1$ and $m := \frac{c_1 + c_2}{2}$. Voters on $[0, m)$ either vote for candidate 1 or abstain, and voters on $(m, 1]$ either vote for candidate 2 or abstain. Evaluating the gain $d_{-i}(x) - d_i(x)$ piecewise—it equals Δ outside $[c_1, c_2]$ and $|c_1 + c_2 - 2x|$ inside—we obtain

$$u_1(c_1, c_2) = K(\Delta) V(c_1) + \int_{c_1}^m K(c_1 + c_2 - 2x) v(x) dx, \quad (15)$$

$$u_2(c_1, c_2) = K(\Delta) (1 - V(c_2)) + \int_m^{c_2} K(2x - c_1 - c_2) v(x) dx. \quad (16)$$

Since v and k are continuous, u_1 is differentiable in c_1 on $(0, c_2)$. Differentiating Eq. (15) by the Leibniz rule, the term $K(\Delta)v(c_1)$ produced by the first summand cancels against the lower-limit term of the integral, and the upper-limit term vanishes because $K(0) = 0$; what remains is

$$\frac{\partial u_1}{\partial c_1}(c_1, c_2) = \int_{c_1}^m k(c_1 + c_2 - 2x) v(x) dx - k(\Delta) V(c_1), \quad (17)$$

and, symmetrically,

$$\frac{\partial u_2}{\partial c_2}(c_1, c_2) = k(\Delta) (1 - V(c_2)) - \int_m^{c_2} k(2x - c_1 - c_2) v(x) dx. \quad (18)$$

Both formulas remain valid as one-sided derivatives at $c_1 = 0$ and $c_2 = 1$.

Two specializations will be used repeatedly. First, for *uniform costs* ($K(t) = t$, $k \equiv 1$), Eqs. (17) to (18) become

$$\frac{\partial u_1}{\partial c_1} = V(m) - 2V(c_1), \quad \frac{\partial u_2}{\partial c_2} = 1 + V(m) - 2V(c_2), \quad (19)$$

and integrating Eq. (15) by parts gives the compact representation

$$u_1(c_1, c_2) = 2 \int_{c_1}^m V(x) dx, \quad u_2(c_1, c_2) = 2 \int_m^{c_2} (1 - V(x)) dx. \quad (20)$$

(The analogous formulas with the roles of the candidates reversed hold when $c_1 > c_2$; they follow from the reflection $x \mapsto 1 - x$, which maps V to the CDF $\tilde{V}(x) = 1 - V(1 - x)$, whose density $\tilde{v}(x) = v(1 - x)$ has the same bounds as v .)

Second, for *uniform voters* ($v \equiv 1$), substituting $y = c_1 + c_2 - 2x$ in Eq. (15) gives, with $G(t) := \int_0^t K(y) dy$,

$$u_1(c_1, c_2) = c_1 K(\Delta) + \frac{1}{2} G(\Delta), \quad u_2(c_1, c_2) = (1 - c_2) K(\Delta) + \frac{1}{2} G(\Delta), \quad (21)$$

so that

$$\frac{\partial u_1}{\partial c_1} = \frac{1}{2} K(\Delta) - c_1 k(\Delta), \quad \frac{\partial u_2}{\partial c_2} = (1 - c_2) k(\Delta) - \frac{1}{2} K(\Delta). \quad (22)$$

Lemma 6 (restated). *In the baseline model, under Assumption 2, every pure Nash equilibrium (c_1, c_2) satisfies $c_1 \neq c_2$ and, labeling the candidates so that $c_1 < c_2$, $0 < c_1 < c_2 < 1$.*

Proof. No ties. At a tied profile $c_1 = c_2 = c$ both candidates receive 0. If $c > 0$, let candidate 1 deviate to any $c'_1 \in (0, c)$; by Eq. (15), her payoff is at least $\int_{c'_1}^{m'} K(c'_1 + c - 2x) v(x) dx > 0$, since on the interior of that interval the argument of K lies in $(0, \Delta') \subseteq (0, 1)$, where $K > 0$, and $v > 0$. If $c = 0$, the mirror deviation to any $c'_1 \in (0, 1)$ works. Hence tied profiles are never equilibria.

No boundary positions. Suppose $c_1 = 0 < c_2$. By Eq. (17), the one-sided derivative at $c_1 = 0$ equals $\int_0^{c_2/2} k(c_2 - 2x) v(x) dx > 0$ (the argument of k ranges over $(0, c_2) \subseteq (0, 1)$, where $k > 0$, and $v > 0$), so a small move to the right strictly increases u_1 , and $c_1 = 0$ is not a best response. The mirror argument using Eq. (18) rules out $c_2 = 1$. $\square$

Theorem 9 (restated). *Let $\kappa \sim U(0, 1)$ and let v satisfy Assumption 2. Then every Nash equilibrium (c_1, c_2) satisfies*

$$V(c_1) = V(c_2) - \frac{1}{2} = \frac{1}{2} V\left(\frac{c_1 + c_2}{2}\right).$$

Proof. By Lemma 6 and Eq. (19), an equilibrium satisfies

$$V(m) = 2V(c_1) \quad \text{and} \quad V(m) = 2V(c_2) - 1. \quad (23)$$

Combining the two equalities yields $V(c_1) = V(c_2) - \frac{1}{2} = \frac{1}{2} V(m)$, as required. $\square$

Proposition 10 (restated). *Let $\kappa \sim U(0, 1)$ and suppose $\rho_v \leq 4$. Then each candidate's payoff is concave in her own position on each side of the opponent. Consequently, (c_1, c_2) with $0 < c_1 < c_2 < 1$ is a Nash equilibrium if and only if it satisfies the conditions of Theorem 9 and neither candidate gains by a cross-over deviation to the far side of her opponent.*

Proof. Fix c_2 and consider $c'_1 \in [0, c_2)$. By Eq. (19), $\partial u_1 / \partial c'_1 = V(m') - 2V(c'_1)$ with $m' = \frac{c'_1 + c_2}{2}$, whose derivative in c'_1 is

$$\frac{1}{2} v(m') - 2v(c'_1) \leq \frac{1}{2} \sup v - 2 \inf v \leq 0$$

whenever $\rho_v \leq 4$. Hence $u_1(\cdot, c_2)$ has a non-increasing derivative, i.e., is concave, on $[0, c_2]$. For $c'_1 \in (c_2, 1]$ the same argument applies after the reflection $x \mapsto 1-x$ (which preserves the density ratio), so $u_1(\cdot, c_2)$ is also concave on $[c_2, 1]$. The argument for candidate 2 is symmetric.

($\Rightarrow$) If (c_1, c_2) is an equilibrium then the balance conditions hold by Theorem 9, and the cross-over inequalities hold trivially, since they compare the equilibrium payoff with payoffs of feasible deviations.

($\Leftarrow$) Suppose the conditions of Theorem 9 hold at an interior profile. Then $\partial u_1 / \partial c_1 = V(m) - 2V(c_1) = 0$, so by concavity c_1 maximizes $u_1(\cdot, c_2)$ on $[0, c_2]$. Note also that $\rho_v \leq 4$ forces $\inf v \geq \frac{1}{4} \sup v \geq \frac{1}{4}$ (as $\sup v \geq \int_0^1 v = 1$), so $V(c_1) \geq c_1/4 > 0$ and, by Eq. (20), $u_1(c_1, c_2) = 2 \int_{c_1}^m V > 0$; in particular the tie $c'_1 = c_2$ (payoff 0) is not profitable. The cross-over inequality covers all $c'_1 \in [c_2, 1]$. Hence c_1 is a global best response, and symmetrically for c_2 , so the profile is an equilibrium. $\square$

Corollary 11 (restated). *Let $\kappa \sim U(0, 1)$ and suppose $\rho_v \leq 4$. Then any profile with $V(c_1) = \frac{1}{4}$, $V(c_2) = \frac{3}{4}$ and $V(\frac{c_1+c_2}{2}) = \frac{1}{2}$ is a Nash equilibrium.*

Proof. Write $s := \inf v$; as noted above, $\rho_v \leq 4$ implies $s \geq \frac{1}{4}$ and $\sup v \leq 4s$. The profile is interior ($0 < V(c_1) < V(c_2) < 1$ and V is strictly increasing) and satisfies Eq. (23): $V(m) = \frac{1}{2} = 2V(c_1)$ and $2V(c_2) - 1 = \frac{1}{2} = V(m)$. By Proposition 10 it therefore suffices to verify the two cross-over inequalities.

A lower bound on the candidates' payoffs. Since $V(x) \geq V(c_1) + s(x - c_1) = \frac{1}{4} + s(x - c_1)$, Eq. (20) gives

$$u_1(c_1, c_2) = 2 \int_{c_1}^m V(x) dx \geq 2 \left[\frac{1}{4} \cdot \frac{\Delta}{2} + \frac{s}{2} \left(\frac{\Delta}{2} \right)^2 \right] = \frac{\Delta}{4} + \frac{s\Delta^2}{4}.$$

Moreover $\frac{1}{2} = V(c_2) - V(c_1) = \int_{c_1}^{c_2} v \leq 4s\Delta$, i.e., $\Delta \geq \frac{1}{8s}$, whence

$$u_1(c_1, c_2) \geq \frac{1}{32s} + \frac{1}{256s} > \frac{1}{32s}. \quad (24)$$

The same bound holds for $u_2(c_1, c_2)$, using $1 - V(x) \geq \frac{1}{4} + s(c_2 - x)$ in Eq. (20).

An upper bound on cross-over payoffs. Consider a deviation of candidate 1 to $c'_1 = c_2 + t$ with $t \in (0, 1 - c_2]$. Every voter who votes for the deviator lies to the right of the new midpoint $m' = c_2 + \frac{t}{2}$, and votes with probability at most $K(t) = t$. Hence

$$u_1(c'_1, c_2) \leq t(1 - V(c_2 + \frac{t}{2})) \leq t \left(\frac{1}{4} - \frac{st}{2} \right) \leq \max_{\tau \geq 0} \tau \left(\frac{1}{4} - \frac{s\tau}{2} \right) = \frac{1}{32s},$$

where the second inequality uses $1 - V(c_2) = \frac{1}{4}$ and $V(c_2 + \frac{t}{2}) - V(c_2) \geq \frac{st}{2}$. Combining with Eq. (24), no cross-over deviation of candidate 1 is profitable. The bound for candidate 2 is the mirror image (using $V(c_1) = \frac{1}{4}$ and $V(c_1) - V(c_1 - \frac{t}{2}) \geq \frac{st}{2}$). By Proposition 10, the profile is a Nash equilibrium. $\square$

Remark 50. The bound $\rho_v \leq 4$ is tight for this argument in the following sense: at $\rho_v = 4$ the deviator’s payoff can be exactly flat over an interval of alternative best responses. For instance, for the symmetric step density $v = \frac{5}{8}$ on $[0, \frac{2}{5}) \cup (\frac{3}{5}, 1]$ and $v = \frac{5}{2}$ on $[\frac{2}{5}, \frac{3}{5}]$ (density ratio exactly 4), the quartile profile is $(\frac{2}{5}, \frac{3}{5})$, and one computes from Eq. (20) that $u_1(c'_1, \frac{3}{5}) = \frac{3}{40}$ for every $c'_1 \in [\frac{1}{5}, \frac{2}{5}]$: candidate 1 is indifferent across a whole interval of positions, and the quartile profile is a (non-strict) equilibrium.

Corollary 11, symmetric case (restated). *Let $\kappa \sim U(0, 1)$ and let v be symmetric around $\frac{1}{2}$ with $\rho_v \leq 4$. Then the quartile profile $(V^{-1}(\frac{1}{4}), V^{-1}(\frac{3}{4}))$ is a Nash equilibrium; in particular an equilibrium exists. Moreover, any profile satisfying the conditions of Corollary 11 must be symmetric, $c_2 = 1 - c_1$.*

Proof. Symmetry of v gives $V(x) + V(1 - x) = 1$ for all x , hence $V^{-1}(\frac{3}{4}) = 1 - V^{-1}(\frac{1}{4})$; the midpoint of the quartile profile is then $\frac{1}{2}$ and $V(\frac{1}{2}) = \frac{1}{2}$, so all three conditions of Corollary 11 hold and the profile is an equilibrium. Conversely, if $V(c_1) = \frac{1}{4}$ and $V(c_2) = \frac{3}{4}$, then $V(1 - c_1) = 1 - V(c_1) = \frac{3}{4} = V(c_2)$; since $\rho_v \leq 4$ forces $\inf v > 0$, V is strictly increasing, so $c_2 = 1 - c_1$. $\square$

Theorem 13 (restated). *Let $x \sim U(0, 1)$ and let k satisfy Assumption 2. (a) Every Nash equilibrium is symmetric, $c_1 = 1 - c_2$, with gap $\Delta \in (0, 1)$ solving $K(\Delta) = (1 - \Delta)k(\Delta)$. (b) If moreover $\psi(t) = t + K(t)/k(t)$ is strictly increasing on $(0, 1)$, then this equation has a unique solution $\Delta^* \in (0, 1)$, and $(\frac{1-\Delta^*}{2}, \frac{1+\Delta^*}{2})$ is the unique Nash equilibrium up to relabeling.*

Proof. (a) By Lemma 6, an equilibrium is interior with $0 < \Delta < 1$, so both derivatives in Eq. (22) vanish:

$$\frac{1}{2}K(\Delta) = c_1 k(\Delta) \quad \text{and} \quad \frac{1}{2}K(\Delta) = (1 - c_2)k(\Delta). \quad (25)$$

Since $\Delta \in (0, 1)$ and $k > 0$ on $(0, 1)$, we have $K(\Delta) > 0$; the first equation then forces $k(\Delta) > 0$, so comparing the two equations gives $c_1 = 1 - c_2$, whence $c_1 = \frac{1-\Delta}{2}$, and Eq. (25) becomes $K(\Delta) = (1 - \Delta)k(\Delta)$.

(b) On $(0, 1)$, where $k > 0$, the equation $K(\Delta) = (1 - \Delta)k(\Delta)$ is equivalent to $\psi(\Delta) = 1$.

Existence and uniqueness of the root. ψ is continuous on $(0, 1)$ and strictly increasing, so $L := \lim_{t \downarrow 0} \psi(t)$ exists. Suppose $L > 0$; then $K(t)/k(t) \geq L/2$ for all sufficiently small t , i.e., $(\log K)'(t) = k(t)/K(t) \leq 2/L$, which upon integrating from t to a fixed small t_0 gives $K(t) \geq K(t_0)e^{-2t_0/L} > 0$ as $t \downarrow 0$ —contradicting $K(0) = 0$ and the continuity of K . Hence $L = 0 < 1$. At the other end, ψ exceeds 1 somewhere on $(0, 1)$: otherwise $K(t)/k(t) \leq 1 - t$, i.e. $k(t)/K(t) \geq 1/(1 - t)$, for all $t \in (0, 1)$, and integrating from t to $s < 1$ gives $\log K(s) - \log K(t) = \int_t^s k/K \geq \log \frac{1-t}{1-s} \rightarrow \infty$ as $s \uparrow 1$, contradicting $K(s) \leq K(1) \leq 1$. (No bound on k is needed: Assumption 2 permits k unbounded at the endpoints, as in Example 20 with $p < 1$.) By the intermediate value theorem and strict monotonicity, $\psi(\Delta) = 1$ has a unique root $\Delta^* \in (0, 1)$.

The symmetric profile is an equilibrium. Let $c_1 = \frac{1-\Delta^*}{2}$, $c_2 = \frac{1+\Delta^*}{2}$, and consider deviations of candidate 1 (the case of candidate 2 is the mirror image). Parameterize a deviation on the *own* side of the opponent by its distance $t \in (0, c_2]$ from c_2 : by Eq. (21), the deviation payoff is $u_L(t) = (c_2 - t)K(t) + \frac{1}{2}G(t)$, with derivative

$$u'_L(t) = (c_2 - t)k(t) - \frac{1}{2}K(t) = k(t) \left[c_2 - \frac{t + \psi(t)}{2} \right].$$

Since $t \mapsto t + \psi(t)$ is strictly increasing and equals $2c_2 = 1 + \Delta^*$ exactly at $t = \Delta^*$ (because $\psi(\Delta^*) = 1$), the bracket is positive for $t < \Delta^*$ and negative for $t > \Delta^*$: u_L is strictly single-peaked with maximum at $t = \Delta^*$, i.e., at c_1 itself. A deviation *across* the opponent to $c'_1 = c_2 + t$, $t \in (0, 1 - c_2]$, yields by Eq. (21) the payoff $u_R(t) = (1 - c_2 - t)K(t) + \frac{1}{2}G(t) \leq u_L(t)$, since $1 - c_2 < c_2$; hence its supremum is at most $\max_t u_L(t) = u_1(c_1, c_2)$. Finally, the tie $c'_1 = c_2$ yields $0 < u_1(c_1, c_2)$. So no deviation is profitable, and the profile is an equilibrium.

Uniqueness. By part (a), any equilibrium is symmetric with gap solving $\psi(\Delta) = 1$, i.e., $\Delta = \Delta^*$. $\square$

Proposition 18 (restated). *Let $x \sim U(0, 1)$ and let $k, \tilde{k}$ both satisfy the hypotheses of Theorem 13(b), with equilibrium gaps $\Delta^*, \tilde{\Delta}^*$. If $\tilde{k}/\tilde{K} \geq k/K$ pointwise on $(0, 1)$, then $\tilde{\Delta}^* \geq \Delta^*$. If in addition $\tilde{K}(1) \leq K(1)$, then $\tilde{K} \leq K$ pointwise.*

Proof. First claim. Since $k, \tilde{k} > 0$ on $(0, 1)$, the hypothesis gives $\tilde{K}(t)/\tilde{k}(t) \leq K(t)/k(t)$, hence $\tilde{\psi}(t) = t + \tilde{K}(t)/\tilde{k}(t) \leq t + K(t)/k(t) = \psi(t)$ for all $t \in (0, 1)$, so

$$\tilde{\psi}(\Delta^*) \leq \psi(\Delta^*) = 1 = \tilde{\psi}(\tilde{\Delta}^*),$$

and since $\tilde{\psi}$ is strictly increasing, $\Delta^* \leq \tilde{\Delta}^*$. No condition on $\tilde{K}(1)$ is used.

Second claim. The reverse hazard rate is the derivative of $\log K$, and $\int_t^1 k/K < \infty$ for $t \in (0, 1)$ because $K \geq K(t) > 0$ on $[t, 1]$ and $\int_t^1 k = K(1) - K(t)$. Hence, for $t \in (0, 1)$, using $\tilde{K}(1) \leq K(1)$,

$$\log \tilde{K}(t) = \log \tilde{K}(1) - \int_t^1 \frac{\tilde{k}(s)}{\tilde{K}(s)} ds \leq \log K(1) - \int_t^1 \frac{k(s)}{K(s)} ds = \log K(t),$$

so $\tilde{K} \leq K$ pointwise, i.e., costs under $\tilde{K}$ are stochastically higher. $\square$

Proposition 46 (restated). *Under uniform voters, $k > 0$ on $(0, 1)$, and alienation with parameter $\alpha \geq 0$, the pure Nash equilibria of the margin game are exactly $E = \left[\frac{1}{2+\alpha}, \frac{1+\alpha}{2+\alpha} \right]^2$, each with margin 0, independently of K .*

Proof. We first treat $\alpha > 0$; the case $\alpha = 0$ is handled at the end. Take any profile with $c_1 \leq c_2$ and let $G(t) = \int_0^t K$. Candidate 1's benefit tent has arms

of slope α (outer) and $2 + \alpha$ (inner); changing variables along each arm, and subtracting the part of the outer arm that is truncated by the boundary 0, gives

$$u_1 = \left(\frac{1}{\alpha} + \frac{1}{2 + \alpha} \right) G(\Delta) - \frac{1}{\alpha} G(b_1), \quad b_1 := (B_1(0))^+ = (\Delta - \alpha c_1)^+,$$

and symmetrically for u_2 with $b_2 = (B_2(1))^+ = (\Delta - \alpha(1 - c_2))^+$ (throughout this proof, b_1, b_2 denote the boundary benefits of Section 4.1 truncated at zero). Hence the margin is

$$w_1 = u_1 - u_2 = \frac{1}{\alpha} (G(b_2) - G(b_1)). \quad (26)$$

Since $k > 0$ on $(0, 1)$, G is strictly increasing on $(0, 1]$, so w_1 and $b_2 - b_1$ have the same sign whenever $\max(b_1, b_2) > 0$, and $w_1 = 0$ iff $b_1 = b_2$. In words: a candidate leads exactly when her *opponent's* tent is more severely truncated by the boundary.

Profiles in E have margin 0. If $c_1 \geq \frac{1}{2+\alpha}$ and $c_2 \leq \frac{1+\alpha}{2+\alpha}$, then $\Delta - \alpha c_1 = c_2 - (1+\alpha)c_1 \leq \frac{1+\alpha}{2+\alpha} - \frac{1+\alpha}{2+\alpha} = 0$, and likewise $\Delta - \alpha(1 - c_2) = (1+\alpha)c_2 - c_1 - \alpha \leq 0$: both tents are interior, $b_1 = b_2 = 0$, and $w_1 = 0$.

When can candidate 1 secure a positive margin? By Eq. (26), $w_1 > 0$ at a profile (c'_1, c_2) with $c'_1 < c_2$ requires $b'_2 > b'_1$, i.e., either (i) $b'_1 = 0 < b'_2$, which reads $c'_1 \geq \frac{c_2}{1+\alpha}$ together with $c'_1 < c_2 - \alpha(1 - c_2)$, and admits a solution iff $\frac{c_2}{1+\alpha} < c_2 - \alpha(1 - c_2)$, i.e., iff $c_2 > \frac{1+\alpha}{2+\alpha}$; or (ii) $b'_2 > b'_1 > 0$, which reads $c'_1 > 1 - c_2$ together with $c'_1 < \min(\frac{c_2}{1+\alpha}, c_2 - \alpha(1 - c_2))$, and both $1 - c_2 < \frac{c_2}{1+\alpha}$ and $1 - c_2 < c_2 - \alpha(1 - c_2)$ are again equivalent to $c_2 > \frac{1+\alpha}{2+\alpha}$. Thus candidate 1 can achieve a strictly positive margin at some position left of c_2 iff $c_2 > \frac{1+\alpha}{2+\alpha}$; by the reflection $x \mapsto 1 - x$, she can achieve one at some position right of c_2 iff $c_2 < \frac{1}{2+\alpha}$. The mirror statements hold for candidate 2 with respect to c_1 .

Profiles in E are equilibria. Let $(c_1, c_2) \in E$. Then $c_2 \in [\frac{1}{2+\alpha}, \frac{1+\alpha}{2+\alpha}]$, so by the previous paragraph no deviation of candidate 1 achieves a positive margin; since $w_1 = 0$ at the profile, no deviation is profitable. Symmetrically for candidate 2. (Note that E includes tied profiles, which are thus zero-turnout, zero-margin equilibria of the margin game.)

Profiles outside E are not equilibria. Consider a profile violating the bounds; after relabeling, either $c_2 > \frac{1+\alpha}{2+\alpha}$ or $c_1 < \frac{1}{2+\alpha}$ (or both). Say $c_2 > \frac{1+\alpha}{2+\alpha}$ (the other case is the mirror image). If $w_1 \leq 0$ at the profile, candidate 1 deviates into the window of case (i) above and achieves $w'_1 > 0 \geq w_1$: profitable. If $w_1 > 0$, then $w_2 < 0$, and candidate 2 deviates to the mirror position $c'_2 = 1 - c_1$, where the two truncation levels coincide and hence $w'_2 = 0 > w_2$: profitable. In all cases some candidate has a profitable deviation, so the profile is not an equilibrium.

The case $\alpha = 0$. Here $B_1 \equiv \Delta$ to the left of c_1 , so by Eq. (21), $w_1 = (c_1 - (1 - c_2))K(\Delta)$, and $E = \{(\frac{1}{2}, \frac{1}{2})\}$. At $(\frac{1}{2}, \frac{1}{2})$, a deviation of candidate 1 to distance $\delta > 0$ from the opponent (on either side) yields margin $-\delta K(\delta) < 0$, so the median profile is an equilibrium and it is strict. Conversely, at any other profile: if $c_1 < c_2$ and $c_1 \neq 1 - c_2$, then $w_1 \neq 0$ and the trailing candidate

deviates to the mirror position to obtain margin 0; if $c_1 = 1 - c_2 < \frac{1}{2}$, candidate 1 deviates to $c'_1 \in (c_1, c_2)$ and obtains $w'_1 = (c'_1 - c_1)K(c_2 - c'_1) > 0$; and if $c_1 = c_2 = c \neq \frac{1}{2}$, say $c > \frac{1}{2}$, then candidate 1 deviates to $c'_1 \in (1 - c, c)$ and obtains $w'_1 = (c'_1 - (1 - c))K(c - c'_1) > 0$. Hence $(\frac{1}{2}, \frac{1}{2})$ is the unique equilibrium.

Finally, all the thresholds above are determined by the benefits B_i alone; K enters only through the strictly increasing G , so the set E is independent of K . $\square$

Theorem 45 (restated). *Let $\alpha = 0$, let v, k satisfy Assumption 2, and let $c_{\text{med}} = V^{-1}(\frac{1}{2})$. Then $(c_{\text{med}}, c_{\text{med}})$ is the unique Nash equilibrium of the margin game, and every unilateral deviation from it strictly lowers the deviator's margin.*

Proof. Write $q := c_{\text{med}}$, which is well-defined, unique, and lies in $(0, 1)$ since V is strictly increasing under Assumption 2.

Step 1: any deviation against the median loses strictly. Consider first a deviation of candidate 1 to $c'_1 < q$, and write $\Delta' = q - c'_1 \in (0, 1)$ and $m' = \frac{c'_1 + q}{2}$. The deviator collects the constant-gain region $[0, c'_1]$ (probability $K(\Delta')$ each) plus the inner arm on $[c'_1, m']$; the opponent collects the constant-gain region $[q, 1]$, of voter mass $1 - V(q) = \frac{1}{2}$, plus the inner arm on $[m', q]$. Substituting $s = m' - x$ and $s = x - m'$ in the two inner-arm integrals,

$$w_1(c'_1, q) = K(\Delta') \left(V(c'_1) - \frac{1}{2} \right) + \int_0^{\Delta'/2} K(2s) [v(m' - s) - v(m' + s)] ds.$$

Bounding the positive part of the integral using $K(2s) \leq K(\Delta')$ for $s \leq \Delta'/2$,

$$\int_0^{\Delta'/2} K(2s) v(m' - s) ds \leq K(\Delta') (V(m') - V(c'_1)),$$

we obtain

$$\begin{aligned} w_1(c'_1, q) &\leq K(\Delta') \left(V(m') - \frac{1}{2} \right) - \int_0^{\Delta'/2} K(2s) v(m' + s) ds \\ &\leq K(\Delta') \left(V(m') - \frac{1}{2} \right) < 0, \end{aligned}$$

where the final strict inequality holds because $K(\Delta') > 0$ (as $k > 0$ on $(0, 1)$ and $0 < \Delta' \leq q < 1$) and $V(m') < V(q) = \frac{1}{2}$ (as $m' < q$ and $v > 0$). For a deviation to $c'_1 > q$, the mirror computation gives $w_1(c'_1, q) \leq K(\Delta') (\frac{1}{2} - V(m')) < 0$, now with $m' = \frac{c'_1 + q}{2} > q$. Finally, $w_1(q, q) = 0$. Hence every unilateral deviation from (q, q) strictly lowers the deviator's margin, and (q, q) is an equilibrium.

Step 2: uniqueness. Suppose (c_1, c_2) is an equilibrium with, say, $c_1 \neq q$. Deviating to q is available to candidate 2, and by Step 1 (with the roles of the players reversed), $w_2(c_1, q) = -w_1(c_1, q) > 0$; equilibrium therefore requires $w_2(c_1, c_2) \geq w_2(c_1, q) > 0$, i.e., $w_1(c_1, c_2) < 0$. But candidate 1 could deviate to q : if $c_2 \neq q$ then $w_1(q, c_2) > 0$ by Step 1, and if $c_2 = q$ then $w_1(q, q) = 0$; either way candidate 1's margin strictly improves, contradicting equilibrium. Hence $c_1 = q$, and symmetrically $c_2 = q$. $\square$

C Existence and Uniqueness of Equilibria

This appendix collects the existence arguments and the (non-)uniqueness examples referenced in the main text. Recall the distinction stressed in Section 3.1: the balance conditions of Lemma 7 and Theorems 9 and 13 are *necessary* for equilibrium. Below we show that *solutions of the conditions* always exist in the two benchmark settings; whether these solutions are genuine equilibria is a separate question—answered affirmatively under the sufficient conditions of the main text, and negatively in general (see the example below and Appendix F).

Solutions of the conditions always exist under uniform costs. Under Assumption 2, V is strictly increasing, so its inverse V^{-1} is well-defined on $[0, 1]$. For any $c_1 \in [0, V^{-1}(\frac{1}{2})]$, define

$$c_2(c_1) := V^{-1}\left(V(c_1) + \frac{1}{2}\right),$$

which is well-defined since $V(c_1) \in [0, \frac{1}{2}]$ on this domain. Next, define

$$f(c_1) := V\left(\frac{c_1 + c_2(c_1)}{2}\right) - 2V(c_1).$$

By construction, f is continuous on $[0, V^{-1}(\frac{1}{2})]$. Moreover,

$$f(0) = V\left(\frac{V^{-1}(\frac{1}{2})}{2}\right) \geq 0,$$

and, letting $q := V^{-1}(\frac{1}{2})$, so that $c_2(q) = V^{-1}(1) = 1$,

$$f(q) = V\left(\frac{q+1}{2}\right) - 1 \leq 0.$$

Hence there exists $c_1^* \in [0, q]$ such that $f(c_1^*) = 0$. Setting $c_2^* = c_2(c_1^*)$ yields

$$V(c_2^*) = V(c_1^*) + \frac{1}{2} \quad \text{and} \quad V\left(\frac{c_1^* + c_2^*}{2}\right) = 2V(c_1^*),$$

so (c_1^*, c_2^*) satisfies the conditions of Theorem 9. Whether it is an equilibrium requires a separate verification—Appendix F shows it may fail; under the hypotheses of Corollary 11 (symmetric case), equilibrium existence is guaranteed.

Solutions of the conditions always exist under uniform voters. Define, on $(0, 1)$,

$$f(\Delta) := K(\Delta) - (1 - \Delta)k(\Delta),$$

which is continuous there by Assumption 2. Near 0, $f < 0$ at points arbitrarily close to 0: if $f \geq 0$ on some $(0, t_0)$ with $t_0 \leq \frac{1}{2}$, then $k/K \leq 1/(1 - t) \leq 2$ there, so $\log K(t_0) - \log K(t) = \int_t^{t_0} k/K \leq 2t_0$ for all $t \in (0, t_0)$ (note $K(t_0) > 0$

since $k > 0$ on $(0, 1)$; hence $K(t) \geq K(t_0)e^{-2t_0} > 0$ as $t \downarrow 0$, contradicting $K(0) = 0$ and the continuity of K . Near 1, $f > 0$ at points arbitrarily close to 1: if $f \leq 0$ on some $(t_1, 1)$ with $t_1 \in (0, 1)$, then $k/K \geq 1/(1-t)$ there, so $\log K(t) - \log K(t_1) = \int_{t_1}^t k/K \geq \log \frac{1-t_1}{1-t} \rightarrow \infty$ as $t \uparrow 1$, contradicting $K \leq 1$. By the intermediate value theorem, f has a root $\Delta^* \in (0, 1)$. Under the monotonicity condition of Theorem 13(b), this root is unique and yields the (unique) equilibrium. Without that condition, some roots may fail to be equilibria, as the following example shows. (When $k(0^+) = 0$, extending k by continuity to 0 makes $\Delta = 0$ a further solution of $K(\Delta) = (1-\Delta)k(\Delta)$; it is not an equilibrium gap, since Theorem 13(a) requires $\Delta \in (0, 1)$.)

Non-uniqueness (and non-sufficiency) under uniform voters. Consider the cost density $k(\Delta) = 8\Delta^2 - \frac{16}{3}\Delta + 1$ (a valid density: $K(1) = 1$ and $k \geq \frac{1}{9} > 0$, with minimum at $\Delta = \frac{1}{3}$). The equation $K(\Delta) = (1-\Delta)k(\Delta)$ admits three solutions, $\Delta \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ (the function ψ of Eq. (6) is not monotone for this k , so Theorem 13(b) does not apply). The three corresponding symmetric profiles behave very differently:

- $\Delta = \frac{1}{4}$, profile $(\frac{3}{8}, \frac{5}{8})$: a Nash equilibrium (verified by computing global best responses numerically), with $u_1 = u_2 = \frac{131}{2304} \approx 0.057$;
- $\Delta = \frac{3}{4}$, profile $(\frac{1}{8}, \frac{7}{8})$: a Nash equilibrium, with $u_1 = u_2 = \frac{27}{256} \approx 0.105$;
- $\Delta = \frac{1}{2}$, profile $(\frac{1}{4}, \frac{3}{4})$: **not** a Nash equilibrium. Here $u_1 = \frac{1}{4}K(\frac{1}{2}) + \frac{1}{2} \int_0^{1/2} K = \frac{5}{72} \approx 0.069$, but candidate 1's best response to $c_2 = \frac{3}{4}$ is $c'_1 \approx 0.473$, yielding a payoff of ≈ 0.073 . The profile solves both first-order conditions, yet it is not even a local equilibrium: Remark 71 shows this middle root to be a local *minimum* of the deviator's payoff, so deviating either way pays.

Thus equation (7) may have multiple roots; the set of equilibria may be a strict subset of these roots; and, in particular, the necessary condition of Theorem 13(a) cannot be reversed without an assumption such as the monotonicity of ψ . Whether at least one root is always an equilibrium (i.e., whether a pure equilibrium always exists under uniform voters) is open.

For comparison, the simple sufficient conditions for the monotonicity of ψ (hence for uniqueness *and* sufficiency) are the two that give super-regularity, k non-increasing or log-concave ((6) and Theorem 13(b)); Fig. 5 visualizes the resulting unique crossing for three such families.

Non-uniqueness under uniform costs. Uniqueness can also fail when the voter density (rather than the cost density) is non-uniform; Appendix E works out a symmetric voter density that admits three distinct equilibria, two of which are asymmetric.

Mixed equilibria. Finally, we note that although the existence of *pure* equilibria is open in general, the payoffs u_i are continuous on the compact strategy space

$[0, 1]^2$ (continuity at tied profiles holds because vote shares vanish as the candidates approach each other), so a possibly-mixed Nash equilibrium always exists by Glicksberg’s fixed-point theorem [39].

D Power Costs against a Non-Uniform Electorate

This appendix records a case that no sufficiency result in the main text reaches, and where the conditions nonetheless appear to deliver equilibria.

Remark 51 (Power costs against a non-uniform electorate). [numerical observation] Every sufficiency result above misses this family. For $K(t) = t^p$ the cost density is $k(t) = p t^{p-1}$, which is increasing when $p > 1$; and when $p < 1$ it is non-increasing, but $\rho_k := k(0^+)/k(1^-)$ is infinite. So the criterion $\rho_v \rho_k \leq 4$ of Proposition 59 reaches the family only at $p = 1$, where it is Corollary 11, and Theorem 13 needs a uniform electorate.

For a symmetric electorate the gap is narrower than that makes it sound. By Lemma 54, own-side optimality alone makes a mirror profile an equilibrium whatever the cost distribution, so the cross-over deviations are already handled and what is unproved is exactly this: that the stationary point maximizes $u_1(\cdot, c_2)$ on $[0, c_2]$. On this family ψ is linear, $\psi(t) = (1 + \frac{1}{p})t$, which is as favourable a setting as the paper offers.

Numerically the profile is an equilibrium throughout. Scanning $(0, \frac{1}{2})$ for symmetric stationary profiles, each of the five symmetric electorates below has exactly one at each $p \in \{\frac{1}{2}, 1, 2, 3\}$, and in all twenty cases it is a Nash equilibrium; the largest profitable deviation found was 1.1×10^{-16} . The quantile $V(c_1^*)$ it sits at:

electorate	$p = \frac{1}{2}$	$p = 1$	$p = 2$	$p = 3$
uniform	0.33333	0.25	0.16667	0.12500
Beta(1.5, 1.5)	0.33386	0.25	0.16334	0.11859
Beta(2, 2)	0.33409	0.25	0.16176	0.11541
Beta(3, 3)	0.33430	0.25	0.16023	0.11227
hollow cosine (Fig. 4)	0.32015	0.25	0.18595	0.15077

At $p = 1$ it is $\frac{1}{4}$ for every electorate, which is Corollary 11. Away from $p = 1$ it moves, but by less than 0.03 across this range, and the departure from the uniform-voter value $\frac{1}{2(1+p)}$ of Remark 21 is signed by the shape of v : a peaked electorate raises the quantile when $p < 1$ and lowers it when $p > 1$, a hollow one does the reverse, and both effects vanish at $p = 1$ and grow with $|p - 1|$ and with the distance of v from uniform. Distribution-freeness is therefore not merely lost away from $p = 1$; it degrades continuously, and $p = 1$ is the point where it is exact.

E Example with Multiple Equilibria (Uniform Costs)

Consider the continuous, symmetric density $v : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ given by

$$v(x) = \begin{cases} \frac{5}{7}, & 0 \leq x \leq \frac{19}{50}, \\ \frac{250x - 90}{7}, & \frac{19}{50} \leq x \leq \frac{21}{50}, \\ \frac{15}{7}, & \frac{21}{50} \leq x \leq \frac{29}{50}, \\ \frac{160 - 250x}{7}, & \frac{29}{50} \leq x \leq \frac{31}{50}, \\ \frac{5}{7}, & \frac{31}{50} \leq x \leq 1. \end{cases}$$

Under uniform costs, the conditions of Theorem 9 admit three distinct solutions,

$$(c_1, c_2) \in \left\{ \left(\frac{3}{16}, \frac{9}{16} \right), \left(\frac{7}{20}, \frac{13}{20} \right), \left(\frac{7}{16}, \frac{13}{16} \right) \right\},$$

and—in contrast to the cost-side example of Appendix C—we verified numerically that here *all three* profiles are genuine Nash equilibria, with payoffs $(u_1, u_2) \approx (0.075, 0.212)$, $(0.106, 0.106)$ and $(0.212, 0.075)$, respectively. Note that although v is symmetric around $\frac{1}{2}$, two of the three equilibria are asymmetric mirror images of one another. (This density has $\rho_v = 3 \leq 4$, but the outer equilibria are not quartile profiles: Corollary 11 identifies *some* equilibria, not necessarily all of them.)

F Stationary Profiles Need Not Be Equilibria (Uniform Costs)

We now show that the conditions of Theorem 9 are not sufficient: a profile can satisfy them and still fail to be a Nash equilibrium. Let $h : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ be the continuous piecewise-linear function taking the constant values

$$\begin{aligned} &6 \text{ on } [0, 0.08], \quad \frac{1}{20} \text{ on } [0.10, 0.20], \quad 5 \text{ on } [0.22, 0.34], \quad \frac{1}{20} \text{ on } [0.36, 0.44], \\ &3 \text{ on } [0.46, 0.62], \quad \frac{1}{20} \text{ on } [0.64, 0.75], \quad 2 \text{ on } [0.77, 1], \end{aligned}$$

and interpolating linearly on the six gaps of width 0.02. Then $\int_0^1 h = \frac{911}{400}$, so $v := \frac{400}{911} h$ is a continuous density, strictly positive on $[0, 1]$ (Assumption 2 holds), describing an electorate with four ideological “clusters” of decreasing mass.

Solving the conditions of Theorem 9 numerically yields exactly three solutions:

$$(c_1, c_2) \approx (0.226, 0.621), \quad (0.232, 0.709), \quad (0.248, 0.798).$$

Computing global best responses, the first and the third profiles are Nash equilibria. The middle profile is *not*: candidate 2’s payoff there is $u_2 \approx 0.1353$, but relocating to $c'_2 \approx 0.789$ (adjacent to the fourth cluster) yields ≈ 0.1365 —a profitable deviation to a different local maximum of her payoff. Her platform is therefore a local maximum that is not a global one, so the profile is a local equilibrium without being a Nash equilibrium: the opposite failure to the cost-side example of Appendix C, where the middle root is not even a local equilibrium. Note that $\rho_v = 120$ here, far above the bound of Proposition 10; with such uneven densities the candidates’ payoffs are multi-peaked, and stationarity no longer implies global optimality.

G Turnout under Linear Costs

We compute each candidate’s vote share at a symmetric profile under uniform voters and the linear cost family $k_\vartheta(x) = \vartheta x + (1 - \frac{\vartheta}{2})$, $\vartheta \in [-2, 2]$, for which

$$K_\vartheta(\Delta) = \frac{\vartheta}{2}\Delta^2 + \left(1 - \frac{\vartheta}{2}\right)\Delta, \quad \int_0^\Delta K_\vartheta(y) dy = \frac{\vartheta}{6}\Delta^3 + \frac{2 - \vartheta}{4}\Delta^2.$$

At the symmetric profile with gap Δ (so $c_1 = \frac{1-\Delta}{2}$), Eq. (21) gives

$$\begin{aligned} u_1(\Delta; \vartheta) &= c_1 K_\vartheta(\Delta) + \frac{1}{2} \int_0^\Delta K_\vartheta(y) dy \\ &= \frac{1 - \Delta}{2} \left[\frac{\vartheta}{2}\Delta^2 + \left(1 - \frac{\vartheta}{2}\right)\Delta \right] + \frac{\vartheta}{12}\Delta^3 + \frac{2 - \vartheta}{8}\Delta^2 \\ &= -\frac{\vartheta}{6}\Delta^3 + \left(\frac{3\vartheta}{8} - \frac{1}{4}\right)\Delta^2 + \left(\frac{1}{2} - \frac{\vartheta}{4}\right)\Delta. \end{aligned}$$

By symmetry $u_2 = u_1$, so total turnout at the profile is

$$T(\Delta; \vartheta) = 2 u_1(\Delta; \vartheta) = -\frac{\vartheta}{3}\Delta^3 + \left(\frac{3\vartheta}{4} - \frac{1}{2}\right)\Delta^2 + \left(1 - \frac{\vartheta}{2}\right)\Delta. \quad (27)$$

(As a check: at $\vartheta = 0$, $\Delta = \frac{1}{2}$ we get $u_1 = \frac{3}{16}$ and $T = \frac{3}{8}$, matching Corollary 14.)

H Turnout Comparative Statics: A Worked Example

This appendix expands the causality comparison of Section 3.2, using the turnout formula (27) of Appendix G. Consider a uniform voter distribution and a uniform cost distribution, for which the unique equilibrium is at $(\frac{1}{4}, \frac{3}{4})$ with overall turnout $\frac{3}{8}$. First suppose the candidates remain *fixed* at $(\frac{1}{4}, \frac{3}{4})$ (i.e., $\Delta = \frac{1}{2}$) while voting costs increase within the linear cost family, i.e., ϑ increases over $[-2, 2]$. By Eq. (27), turnout is then $T(\frac{1}{2}; \vartheta) = \frac{3}{8} - \frac{5\vartheta}{48}$, declining from $\frac{14}{24} \cong 0.58$ at $\vartheta = -2$, to $\frac{9}{24} = 0.375$ under uniform costs ($\vartheta = 0$), and further to $\frac{4}{24} \cong 0.17$ at $\vartheta = 2$ —a baseline effect driven solely by rising costs and decreasing engagement.

Next, instead of holding the candidates fixed, consider turnout *in equilibrium* as ϑ varies over $[-2, 2]$: within the log-concave linear family the equilibrium gap (Theorem 13(b); $\Delta^*(\vartheta) = \frac{2(\vartheta-1)+\sqrt{(\vartheta-1)^2+3}}{3\vartheta}$ for $\vartheta \neq 0$) $\Delta^*(\vartheta)$ widens from ≈ 0.42 at $\vartheta = -2$ to $\frac{2}{3}$ at $\vartheta = 2$, and equilibrium turnout equals $T(\Delta^*(\vartheta); \vartheta)$. Although turnout still decreases, the decline is more *modest* (from ≈ 0.54 at $\vartheta = -2$, through 0.375 at $\vartheta = 0$, to $\frac{20}{81} \cong 0.25$ at $\vartheta = 2$), because the candidates become more polarized and a wider gap raises each voter's benefit from participating. Thus, at least in this scenario, abstention increases polarization, whereas polarization has *the opposite* effect on abstention. Whether this pattern persists under other distributions is left to future work.

I Preliminaries for all (α, β)

Standing notation and conventions. Voters $x \in [0, 1]$ have CDF V with density v ; voting costs $\kappa \in [0, 1]$ are i.i.d. with CDF K and density k , independent of position. We assume Assumption 2 throughout. We allow $K(1) \leq 1$ (a mass $1 - K(1)$ of voters never votes); everything below uses only $K(1) > 0$. Candidates $i \in \{1, 2\}$ choose $c_i \in [0, 1]$; write $d_i(x) = |x - c_i|$ and, for $\alpha \geq 0$, $B_i(x) = d_{-i}(x) - (1 + \alpha)d_i(x)$, so that i 's vote share is $u_i = \int_0^1 K(B_i(x)^+)v(x)dx$, and candidate i maximizes $U_i = u_i - \beta u_{-i}$ with $\beta \in [0, 1]$. When $c_1 < c_2$ we write $\Delta = c_2 - c_1$, $m = \frac{c_1 + c_2}{2}$, and

$$\begin{aligned} \lambda &:= 2 + \alpha, & x_1^{\text{in}} &= c_1 + \frac{\Delta}{\lambda}, & x_1^{\text{out}} &= c_1 - \frac{\Delta}{\alpha}, \\ x_2^{\text{in}} &= c_2 - \frac{\Delta}{\lambda}, & x_2^{\text{out}} &= c_2 + \frac{\Delta}{\alpha}, \end{aligned}$$

for the inner and outer zeros of the two benefit tents ($x_1^{\text{out}} = -\infty$, $x_2^{\text{out}} = +\infty$ when $\alpha = 0$), and

$$b_1 = B_1(0) = \Delta - \alpha c_1, \quad b_2 = B_2(1) = \Delta - \alpha(1 - c_2)$$

for the boundary benefits; candidate 1's tent is truncated by the boundary (equivalently, she has no *dead zone* $[0, x_1^{\text{out}}]$ of unreachable voters on her flank) iff $b_1 \geq 0$, i.e. $x_1^{\text{out}} \leq 0$. Two identities used repeatedly: with $b_s(\Delta) := \frac{\lambda\Delta - \alpha}{2} = \Delta - \frac{\alpha}{2}(1 - \Delta)$,

$$b_1 + b_2 = 2b_s(\Delta), \quad b_2 - b_1 = \alpha(c_1 + c_2 - 1). \quad (28)$$

Finally $G(t) = \int_0^t K$, $r = k/K$ (the reverse hazard rate), $\psi(t) = t + K(t)/k(t)$, $\rho_v = \sup v / \inf v \in [1, \infty]$, $\rho_k = \sup k / \inf k$, and $\Delta_0 := \frac{\alpha}{\lambda}$, the most polarized symmetric gap inside the margin-game band $E(\alpha) = [\frac{1}{\lambda}, \frac{1+\alpha}{\lambda}]^2$.

Two elementary facts are used without further comment. First, since $K(0) = 0$ and $K > 0$ on $(0, 1]$,

$$\int_{0+}^{t_0} r(s)ds = \lim_{t \downarrow 0} (\log K(t_0) - \log K(t)) = +\infty \quad (t_0 \in (0, 1)), \quad (29)$$

and symmetrically $\int_t^{1^-} r = \log K(1) - \log K(t) < \infty$. Second, for every x at most one of $B_1(x), B_2(x)$ is strictly positive, because $B_1 + B_2 = -\alpha(d_1 + d_2) \leq 0$; hence total turnout satisfies $T = u_1 + u_2 \leq K(1) \leq 1$ at *every* profile.

The two lemmas of this section extend Lemma 6 and the balance conditions of Lemma 7 from the baseline $(\alpha, \beta) = (0, 0)$ to the whole parameter square, and are the entry point for every result below.

Lemma 52 (Interiority and no ties, extended). *Let Assumption 2 hold and $\beta \in [0, 1)$. Then every pure Nash equilibrium is untied and interior ($c_1 \neq c_2$ and, after labeling $c_1 < c_2$, $0 < c_1 < c_2 < 1$) in each of the following cases: (a) $\beta = 0$, any $\alpha \geq 0$ and any (v, k) ; (b) $\alpha = 0$, any (v, k) ; (c) uniform voters $x \sim U(0, 1)$, any $\alpha \geq 0$ and any k .*

Proof. No boundary positions (all three cases). Suppose $c_1 = 0 < c_2$. By Lemma 53 below, the right derivative of u_1 at $c_1 = 0$ is $(1 + \alpha) \int_0^{x_1^{\text{in}}} k(B_1)v dx$, which is positive (the left integral in Eq. (30) is empty). At the same time $\partial u_2 / \partial c_1$ equals $-\int_{x_2^{\text{out}}}^{x_2^{\text{in}} \wedge 1} k(B_2)v dx$, which is at most 0. Hence $\partial U_1 / \partial c_1 = \partial u_1 / \partial c_1 - \beta \partial u_2 / \partial c_1 > 0$: a small move to the right strictly increases U_1 , so $c_1 = 0$ is not a best response. The mirror argument rules out $c_2 = 1$.

No ties, case (a). At a tie both payoffs vanish, and for $\beta = 0$ the deviation argument of the baseline lemma applies verbatim: any c'_1 strictly between the tie point and the nearer end of $(0, 1)$ yields $u_1(c'_1, c) > 0 = U_1(c, c)$.

No ties, case (b). Let $\alpha = 0$ and $c_1 = c_2 = c$. For $\varepsilon > 0$ small consider the deviation $c'_1 = c - \varepsilon$ (possible if $c > 0$). Every voter in $[0, c - \varepsilon]$ gains exactly ε from the deviator, and every voter gains at most ε from either candidate, so with $m' = c - \frac{\varepsilon}{2}$,

$$u_1 \geq K(\varepsilon) V(c - \varepsilon), \quad u_2 \leq K(\varepsilon) (1 - V(m')),$$

whence $U_1 \geq K(\varepsilon) [V(c - \varepsilon) - \beta(1 - V(m'))]$. As $\varepsilon \downarrow 0$ the bracket tends to $(1 + \beta) [V(c) - \frac{\beta}{1 + \beta}]$, which is positive whenever $V(c) > \frac{\beta}{1 + \beta}$; since $K(\varepsilon) > 0$ for $\varepsilon \in (0, 1)$, the deviation is then profitable for small ε . If instead $V(c) \leq \frac{\beta}{1 + \beta} < \frac{1}{1 + \beta}$, the mirror deviation $c'_1 = c + \varepsilon$ gives a bracket tending to $(1 + \beta) [1 - V(c) - \frac{\beta}{1 + \beta}] > 0$. (At $c = 0$ only the second option is available and applies since $V(0) = 0$; at $c = 1$ only the first, since $V(1) = 1$.)

No ties, case (c). Let $v \equiv 1$ and $c_1 = c_2 = c$; by symmetry assume $c > 0$ and deviate to $c'_1 = c - \varepsilon$ with $\varepsilon < \min\{c, \frac{\alpha c}{1 + \alpha}\}$ (the case $c = 0$ is the mirror image; for $\alpha = 0$ the claim is case (b)). By the closed form (32) below, the deviator's tent is interior ($b'_1 = \varepsilon - \alpha(c - \varepsilon) < 0$), so $u_1 = A G(\varepsilon)$ with $A = \frac{1}{\alpha} + \frac{1}{\lambda}$, while the opponent's payoff is $u_2 = A G(\varepsilon) - \frac{1}{\alpha} G((\varepsilon - \alpha(1 - c))^+) \leq A G(\varepsilon)$. Hence $U_1 \geq (1 - \beta) A G(\varepsilon) > 0 = U_1(c, c)$.

□

We record for later use that Lemma 52 fails at $\beta = 1$: tied profiles are equilibria of the margin game (e.g. the median profile at $\alpha = 0$), which is precisely why all statements below assume $\beta < 1$ and treat $\beta = 1$ separately.

Lemma 53 (Master payoff derivative). *Let Assumption 2 hold, $\alpha \geq 0$, $\beta \in [0, 1]$, and fix $0 < c_1 < c_2 < 1$. Assume the right-hand side of Eq. (30) below is finite at c_1 ; this holds automatically whenever k is bounded, or v is locally bounded at $\{0, 1\}$, or the tent edges avoid the boundary ($x_1^{\text{out}} \neq 0$, $x_2^{\text{out}} \neq 1$)—one of which is the case in every application in this paper. Then u_1 , u_2 , and hence U_1 , are continuously differentiable in c_1 there, and*

$$\frac{\partial U_1}{\partial c_1} = (1 + \alpha) \left[\int_{c_1}^{x_1^{\text{in}}} k(B_1) v \, dx - \int_{x_1^{\text{out}} \vee 0}^{c_1} k(B_1) v \, dx \right] + \beta \int_{x_2^{\text{in}}}^{x_2^{\text{out}} \wedge 1} k(B_2) v \, dx, \quad (30)$$

where for $\alpha = 0$ one reads $x_1^{\text{out}} \vee 0 = 0$, $x_2^{\text{out}} \wedge 1 = 1$, and $x_1^{\text{in}} = x_2^{\text{in}} = m$. The formula for $\partial U_2 / \partial c_2$ is the mirror image. In particular, at an untied interior equilibrium with $\beta < 1$ both first-order conditions $\partial U_1 / \partial c_1 = \partial U_2 / \partial c_2 = 0$ hold.

Proof. By the layer-cake representation, using $K(t) = \int_0^1 k(s) \mathbf{1}\{s < t\} ds$,

$$u_1 = \int_0^1 K(B_1(x)^+) v(x) \, dx = \int_0^1 k(s) \mu(\{B_1 > s\}) \, ds, \quad \mu(dx) = v(x) \, dx.$$

The super-level set $\{B_1 > s\}$ is the interval $(x_1^{\text{out}}(s) \vee 0, x_1^{\text{in}}(s))$ for $s \in (0, \Delta)$, where, from $B_1 = c_2 + (1 + \alpha)c_1 - \lambda x$ on the inner arm and $B_1 = c_2 - (1 + \alpha)c_1 + \alpha x$ on the outer arm,

$$x_1^{\text{in}}(s) = \frac{c_2 + (1 + \alpha)c_1 - s}{\lambda}, \quad x_1^{\text{out}}(s) = \frac{s - c_2 + (1 + \alpha)c_1}{\alpha} \quad (= -\infty \text{ if } \alpha = 0).$$

Both endpoints are affine in c_1 with velocities $\frac{1+\alpha}{\lambda}$ and $\frac{1+\alpha}{\alpha}$, and both lie in the open interval $(0, c_2) \subset (0, 1)$ for s in the relevant ranges, where V is continuously differentiable (v continuous). Hence for each fixed s , $c_1 \mapsto \mu(\{B_1 > s\}) = V(x_1^{\text{in}}(s)) - V(x_1^{\text{out}}(s) \vee 0)$ is differentiable in c_1 except at the single value of s where $x_1^{\text{out}}(s) = 0$ (a null set in s), with derivative $\frac{1+\alpha}{\lambda} v(x_1^{\text{in}}(s)) - \frac{1+\alpha}{\alpha} v(x_1^{\text{out}}(s)) \mathbf{1}\{x_1^{\text{out}}(s) > 0\}$. The difference quotients are dominated, locally uniformly in c_1 , by $k(s)$ times the supremum of v over a compact neighborhood of the swept intervals, and the resulting dominating function is integrable in s precisely when the right-hand side of Eq. (30) is finite (change variables $s = B_1(x)$ back: the two s -integrals become the two x -integrals of Eq. (30)); under the standing finiteness assumption dominated convergence applies and yields

$$\frac{\partial u_1}{\partial c_1} = (1 + \alpha) \left[\int_{c_1}^{x_1^{\text{in}}} k(B_1) v \, dx - \int_{x_1^{\text{out}} \vee 0}^{c_1} k(B_1) v \, dx \right],$$

with *no boundary terms*: the moving tent edges x_1^{in} and x_1^{out} carry $K(B_1) = K(0) = 0$, and the kink of B_1 at $x = c_1$ contributes nothing since B_1 is continuous there. Continuity of the derivative in c_1 , including across the regime switch $x_1^{\text{out}} = 0$, follows because the outer-arm integral is continuous there: even when k is unbounded at 0^+ , the vanishing piece obeys $\int_{x_1^{\text{out}}}^{x_1^{\text{out}} + \delta} k(B_1) v \leq \sup_{[x_1^{\text{out}}, x_1^{\text{out}} + \delta]} v$.

$\frac{1}{\alpha}K(\alpha\delta) \rightarrow 0$ by the change of variables $t = B_1(x)$ (the local boundedness of v near the switch point is part of the standing assumption).

For u_2 : on candidate 2's support $[x_2^{\text{in}}, x_2^{\text{out}} \wedge 1]$ (which lies in $(c_1, 1]$) we have $d_1 = x - c_1$, so $\partial B_2 / \partial c_1 = -1$ pointwise, and the same argument gives $\partial u_2 / \partial c_1 = -\int_{x_2^{\text{in}}}^{x_2^{\text{out}} \wedge 1} k(B_2)v \, dx$, again with no boundary terms ($B_2 = 0$ at x_2^{in} and at x_2^{out} when $x_2^{\text{out}} < 1$; if $x_2^{\text{out}} \geq 1$ the limit 1 is fixed). Combining, and noting $U_1 = u_1 - \beta u_2$, yields Eq. (30). The first-order conditions at an equilibrium follow since, by Lemma 52 (whose cases cover every configuration in which we invoke them) the equilibrium is untied and interior, and $U_1(\cdot, c_2)$ is differentiable there. $\square$

Specializing Eq. (30) to $\alpha = 0$ gives, for general (v, k) ,

$$\frac{\partial U_1}{\partial c_1} = \int_{c_1}^m k(B_1)v \, dx - k(\Delta)V(c_1) + \beta \int_m^1 k(B_2)v \, dx, \quad (31)$$

the β -generalization of the baseline first-order condition (17): the opponent's marginal voters at the midpoint now enter with weight $-\beta$.

Finally, we record the closed form for uniform voters used throughout Appendices K and L. For $v \equiv 1$ and any K , changing variables $t = B_1(x)$ along each arm of the tent (slopes α and λ) and subtracting the part of the outer arm truncated by the boundary gives, for $\alpha > 0$,

$$u_1 = A G(\Delta) - \frac{1}{\alpha} G(b_1^+), \quad A := \frac{1}{\alpha} + \frac{1}{\lambda} = \frac{2(1+\alpha)}{\alpha\lambda}, \quad (32)$$

and $u_1 = c_1 K(\Delta) + \frac{1}{2} G(\Delta)$ for $\alpha = 0$; the formulas for u_2 replace b_1 by b_2 and c_1 by $1 - c_2$. Substituting into Eq. (30) (each integral is $\frac{1}{\lambda} K(\Delta)$, $\frac{1}{\alpha} [K(\Delta) - K(b_1^+)]$, and $A K(\Delta) - \frac{1}{\alpha} K(b_2^+)$ respectively) yields, for $\alpha > 0$ and uniform voters,

$$\alpha \frac{\partial U_1}{\partial c_1} = -(1 - \beta) \alpha A K(\Delta) + (1 + \alpha) K(b_1^+) - \beta K(b_2^+). \quad (33)$$

J Results for the baseline ($\alpha = \beta = 0$): a reflection lemma

The sufficiency results for uniform costs (Proposition 10 and Corollary 11) (density ratio $\rho_v \leq 4$) verify two *cross-over* deviations by an explicit computation available only at quartile profiles. The following lemma removes both the computation and the restriction to uniform costs: for symmetric electorates and vote-share maximizers, cross-over deviations are *never* binding. It is stated for all $\alpha \geq 0$ because it is used again in Appendix K; the restriction to $\beta = 0$ is essential (Remark 55).

Lemma 54 (Reflection lemma). *Let $\beta = 0$, $\alpha \geq 0$, let K be arbitrary (Assumption 2), and let v be symmetric about $\frac{1}{2}$. Let $c_1 \in (0, \frac{1}{2})$ and $c_2 = 1 - c_1$. If c_1 maximizes $u_1(\cdot, c_2)$ over $[0, c_2]$, then (c_1, c_2) is a Nash equilibrium.*

Proof. By symmetry of v and of the profile it suffices to show candidate 1 has no profitable deviation. Own-side deviations $c'_1 \in [0, c_2)$ are unprofitable by hypothesis. The tie $c'_1 = c_2$ yields $0 < u_1(c_1, c_2)$ (the benefit B_1 is positive on a nondegenerate interval around c_1 and $v > 0$ there). It remains to rule out cross-over deviations $c'_1 \in (c_2, 1]$. We use two observations.

(i) *Reflection.* For any positions a, o , the change of variables $y = 1 - x$ together with $v(1 - y) = v(y)$ gives $u_{\text{dev}}(a, o) = u_{\text{dev}}(1 - a, 1 - o)$, where $u_{\text{dev}}(a, o)$ denotes the vote share of a candidate at a facing an opponent at o . Applying this with $a = c'_1 \in (c_2, 1]$ and $o = c_2$, and using $1 - c_2 = c_1$:

$$u_1(c'_1, c_2) = u_{\text{dev}}(1 - c'_1, c_1), \quad 1 - c'_1 \in [0, c_1).$$

Cross-over deviations against c_2 are thus payoff-equivalent to own-side deviations against an opponent located at c_1 .

(ii) *Monotonicity in the opponent's position.* For $a < o \leq \tilde{o}$ and every x , $B(x; a, o)^+ \leq B(x; a, \tilde{o})^+$, where $B(x; a, o) = |x - o| - (1 + \alpha)|x - a|$. Indeed, if $B(x; a, o) \leq 0$ there is nothing to prove; if $B(x; a, o) > 0$ then $|x - o| > (1 + \alpha)|x - a| \geq |x - a|$, so x is strictly closer to a than to o , forcing $x < \frac{a+o}{2} < o \leq \tilde{o}$ and hence $|x - \tilde{o}| = \tilde{o} - x > o - x = |x - o|$, so $B(x; a, \tilde{o}) > B(x; a, o) > 0$. Since K is non-decreasing, integrating against v gives $u_{\text{dev}}(a, o) \leq u_{\text{dev}}(a, \tilde{o})$.

Combining, for every $c'_1 \in (c_2, 1]$, with $a = 1 - c'_1 < c_1 \leq c_2$,

$$u_1(c'_1, c_2) \stackrel{(i)}{=} u_{\text{dev}}(a, c_1) \stackrel{(ii)}{\leq} u_{\text{dev}}(a, c_2) \leq \sup_{c'_1 \in [0, c_2)} u_1(c'_1, c_2) = u_1(c_1, c_2),$$

the last equality by hypothesis. No deviation is profitable. $\square$

Remark 55. (a) At the quartile profile of a symmetric electorate with uniform costs, Lemma 54 recovers the cross-over verification in the proof of Corollary 11 (Appendix B) without any computation: under $\rho_v \leq 4$ the own-side payoff is concave (Proposition 10) and stationary at c_1 , hence globally maximized there, and the lemma finishes the proof. The explicit bounds there ($u_1 \geq \frac{1}{32s}$ etc.) are retained only for the asymmetric case of Corollary 11, where the reflection argument is unavailable. (b) The lemma fails for $\beta > 0$: step (ii) compares the *deviator's* share pointwise, but moving the opponent also changes the *opponent's* share u_2 , in a direction that is not signed in general, so $U_1 = u_1 - \beta u_2$ is not monotone in the opponent's position. The $\beta > 0$ cross-over checks in Appendix L are therefore carried out explicitly.

K Results for alienation ($\alpha > 0, \beta = 0$)

K.1 A sufficiency threshold under uniform costs

For $\alpha = 0$, Proposition 10 shows that $\rho_v \leq 4$ makes each candidate's own-side payoff concave, reducing equilibrium verification to the balance conditions plus

two cross-over checks. The next theorem gives the α -analogue and answers the question of whether alienation helps or hurts this program: the threshold

$$F(\alpha) := \min\{f(\alpha), g(\alpha)\}, \quad f(\alpha) := \frac{2(2+\alpha)}{1+\alpha}, \quad g(\alpha) := \frac{2+\alpha}{\alpha} \quad (g(0) := \infty), \quad (34)$$

is *strictly decreasing*, from $F(0) = 4$ through $F(1) = 3$ (where $f = g$) to 1 as $\alpha \rightarrow \infty$. Alienation *shrinks* the sufficiency umbrella.

Theorem 56 (Single-peakedness under uniform costs). *Let $\kappa \sim U(0, 1)$, $\beta = 0$, $\alpha > 0$, and $\rho_v \leq F(\alpha)$. Fix any $c_2 \in (0, 1]$. Then $c_1 \mapsto u_1(c_1, c_2)$ is, on $[0, c_2)$: concave on the no-dead-zone piece $[0, \frac{c_2}{1+\alpha}]$, and strictly decreasing on the dead-zone piece $(\frac{c_2}{1+\alpha}, c_2)$; its derivative is continuous, positive at $c_1 = 0$, and the payoff is single-peaked with all maximizers in the no-dead-zone piece. Consequently a profile $0 < c_1 < c_2 < 1$ is a Nash equilibrium if and only if both balance conditions*

$$V(x_1^{\text{in}}) + V(x_1^{\text{out}} \vee 0) = 2V(c_1), \quad V(x_2^{\text{in}}) + V(\min(x_2^{\text{out}}, 1)) = 2V(c_2) \quad (35)$$

hold (where $V(x_1^{\text{out}} \vee 0) := V(x_1^{\text{out}})$ if $x_1^{\text{out}} > 0$ and $:= 0$ otherwise, and the second condition is the mirror image) and neither candidate gains from the single cross-over deviation to the far side of her opponent.

Proof. By Eq. (30) with $k \equiv 1$ and $\beta = 0$,

$$\frac{\partial u_1}{\partial c_1} = (1 + \alpha) \left[(V(x_1^{\text{in}}) - V(c_1)) - (V(c_1) - V(x_1^{\text{out}} \vee 0)) \right], \quad (36)$$

which is continuous in c_1 on $(0, c_2)$, and one-sidedly at $c_1 = 0$, where it equals $(1 + \alpha)V(x_1^{\text{in}}) > 0$.

No-dead-zone piece. For $c_1 \leq \frac{c_2}{1+\alpha}$ (i.e. $x_1^{\text{out}} \leq 0$), Eq. (36) reads $(1 + \alpha)[V(x_1^{\text{in}}) - 2V(c_1)]$ and, since $x_1^{\text{in}} = c_1 + \frac{c_2 - c_1}{\lambda}$ has $(x_1^{\text{in}})' = \frac{1+\alpha}{\lambda}$,

$$\frac{\partial^2 u_1}{\partial c_1^2} = (1 + \alpha) \left[\frac{1+\alpha}{\lambda} v(x_1^{\text{in}}) - 2v(c_1) \right] \leq (1 + \alpha) \left[\frac{1+\alpha}{\lambda} \sup v - 2 \inf v \right] \leq 0,$$

the last step because $\rho_v \leq f(\alpha) = \frac{2\lambda}{1+\alpha}$.

Dead-zone piece. For $c_1 > \frac{c_2}{1+\alpha}$ (i.e. $x_1^{\text{out}} > 0$), substitute $t = B_1(x)$ on each arm of Eq. (36):

$$\frac{\partial u_1}{\partial c_1} = (1 + \alpha) \int_0^\Delta \left[\frac{1}{\lambda} v(x_1^{\text{in}} - \frac{t}{\lambda}) - \frac{1}{\alpha} v(x_1^{\text{out}} + \frac{t}{\alpha}) \right] dt. \quad (37)$$

Since $\rho_v \leq g(\alpha) = \frac{\lambda}{\alpha}$, the integrand is pointwise $\leq \frac{\sup v}{\lambda} - \frac{\inf v}{\alpha} \leq 0$. If the integral vanished, the integrand would vanish for a.e. t , forcing $v \equiv \sup v$ on (c_1, x_1^{in}) and $v \equiv \inf v$ on (x_1^{out}, c_1) ; as $g(\alpha) > 1$ implies $\sup v > \inf v$, this contradicts continuity of v at c_1 . Hence the derivative is strictly negative on the whole piece.

Single-peakedness and the characterization. The derivative is continuous, starts positive, is non-increasing while the payoff is concave, and is strictly negative on the dead-zone piece; hence $u_1(\cdot, c_2)$ increases to a maximizer set (an interval, possibly a point) inside $(0, \frac{c_2}{1+\alpha}]$ and strictly decreases afterwards. A profile is an equilibrium iff each c_i globally maximizes u_i against the other; by Lemma 52(a) and single-peakedness, own-side global optimality of an interior c_1 is equivalent to stationarity, i.e. to Eq. (35), and the only remaining deviations are the tie (never profitable, as $u_1(c_1, c_2) > 0$) and the cross-over, as stated. $\square$

Remark 57 (Sharpness, and the direction of the answer). (a) The constant $f(\alpha)$ cannot be improved in the concavity criterion: for any ratio $\rho > f(\alpha)$ there is a symmetric “mesa” density with $\rho_v = \rho$ and a no-dead-zone configuration at which $\partial^2 u_1 / \partial c_1^2 > 0$. (Take v equal to a high value on a window around x_1^{in} and a low value at c_1 ; e.g. at $\alpha = 1$, v high on $[0.4, 0.6]$, $c_1 = 0.3$, $c_2 = 0.65$, so $x_1^{\text{in}} = 0.41\bar{6}$: the second derivative is $2[\frac{2}{3}v(x_1^{\text{in}}) - 2v(c_1)] > 0$ iff $v(x_1^{\text{in}})/v(c_1) > 3 = f(1)$. Verified numerically: with ratio 2.9 the payoff is concave at every no-dead-zone configuration, with ratio 3.1 it is not.) Similarly $g(\alpha)$ is exactly the threshold below which dead zones are impossible at stationary profiles (Proposition 61). Whether *equilibrium existence* fails above $F(\alpha)$ is open, exactly as it is open above $\rho_v = 4$ at $\alpha = 0$. (b) F is strictly decreasing with $F(\alpha) \downarrow 1$: the answer to “does alienation help existence?” is *no* on this route. The reason is the dead zone: it is the second, boundary-truncation-driven piece of the payoff that destroys concavity, and it grows with α . Contrast with competitiveness, which *enlarges* the umbrella at $\alpha = 0$ (Proposition 78: threshold $4/(1 - \beta)$). (c) For $\beta \in (0, 1)$ and uniform costs the same two-piece argument goes through with thresholds $f(\alpha)$ (unchanged: the opponent term only helps concavity, since $\partial^2 u_2 / \partial c_1^2 = \frac{v(x_2^{\text{in}})}{\lambda} + \frac{v(x_2^{\text{out}})}{\alpha} \mathbf{1}_{x_2^{\text{out}} < 1} \geq 0$) and $g_\beta(\alpha) = \frac{(1+\alpha)(2+\alpha)}{\alpha(1+\alpha+\beta)+\beta(2+\alpha)} \leq g(\alpha)$ on the dead-zone piece; but the cross-over is no longer free (Remark 55), so we do not obtain an existence corollary there.

Combining Theorem 56 with the reflection lemma yields existence:

Corollary 58 (Existence for symmetric electorates). *Let $\kappa \sim U(0, 1)$, $\beta = 0$, $\alpha > 0$, let v be symmetric about $\frac{1}{2}$ with $\rho_v \leq F(\alpha)$. Then a symmetric Nash equilibrium exists: any root $c_1^* \in (0, \frac{1}{2})$ of*

$$\begin{aligned} h(c_1) &:= V(x_1^{\text{in}}(c_1)) + V(x_1^{\text{out}}(c_1) \vee 0) - 2V(c_1), \\ x_1^{\text{in}} &= c_1 + \frac{1-2c_1}{\lambda}, \quad x_1^{\text{out}} = c_1 - \frac{1-2c_1}{\alpha}, \end{aligned} \tag{38}$$

gives the equilibrium $(c_1^, 1 - c_1^*)$, and such a root exists.*

Proof. h is continuous on $(0, \frac{1}{2})$ with $h(0^+) = V(\frac{1}{\lambda}) > 0$ and, as $c_1 \uparrow \frac{1}{2}$ (so $\Delta = 1 - 2c_1 \downarrow 0$ and $x_1^{\text{out}}, x_1^{\text{in}} \rightarrow \frac{1}{2}$),

$$h(c_1) = (V(x_1^{\text{in}}) - V(c_1)) - (V(c_1) - V(x_1^{\text{out}})) = v(\frac{1}{2}) \Delta \left(\frac{1}{\lambda} - \frac{1}{\alpha} \right) + o(\Delta) < 0$$

for small Δ , since v is continuous and positive at $\frac{1}{2}$ and $\alpha < \lambda$. By the intermediate value theorem a root $c_1^* \in (0, \frac{1}{2})$ exists. By the dead-zone part of Theorem 56 (applied along the symmetric family: for $c_1 > \frac{1}{\lambda}$ the profile has $x_1^{\text{out}} > 0$ and $h < 0$ by the pointwise comparison in Eq. (37), which only used $\rho_v \leq g(\alpha)$), every root lies in the no-dead-zone branch $c_1^* \leq \frac{1}{\lambda}$. At such a root, c_1^* is stationary for $u_1(\cdot, c_2^*)$, hence by single-peakedness (Theorem 56) a global maximizer of the own side; Lemma 54 then upgrades own-side optimality to a Nash equilibrium. $\square$

K.2 Beyond uniform costs

Proposition 59 (Sufficiency for general (v, k) , non-increasing costs).

Let $\beta = 0$, $\alpha \geq 0$, and let k be C^1 and non-increasing on $(0, 1)$ with $\rho_k = k(0^+)/k(1^-) < \infty$. If

$$\rho_v \rho_k \leq f(\alpha) = \frac{2(2+\alpha)}{1+\alpha} \quad \text{and} \quad \rho_v \leq g(\alpha) = \frac{2+\alpha}{\alpha} \quad (\text{vacuous if } \alpha = 0),$$

then for every c_2 the map $c_1 \mapsto u_1(c_1, c_2)$ is single-peaked on $[0, c_2]$ exactly as in Theorem 56, and the equilibrium characterization of Theorem 56 holds verbatim with the balance conditions of Lemma 53. If moreover v is symmetric, a symmetric equilibrium exists. At $\alpha = 0$ the conditions reduce to $\rho_v \rho_k \leq 4$, generalizing the $\rho_v \leq 4$ criterion of Proposition 10 to non-uniform costs.

Proof. No-dead-zone piece ($x_1^{\text{out}} \leq 0$). Differentiating the first bracket of Eq. (30) once more (now the moving tent-edge terms survive with factor k rather than K ; the calculation is licensed by $k \in C^1$),

$$\frac{\partial^2 u_1}{\partial c_1^2} = (1+\alpha) \left[\frac{1+\alpha}{\lambda} k(0^+) v(x_1^{\text{in}}) - 2k(\Delta) v(c_1) + (1+\alpha) \int_{x \in \text{tent}} k'(B_1) v \right].$$

Here we used $\partial B_1 / \partial c_1 = \pm(1+\alpha)$ on the two arms and that the lower limit 0 is fixed. Since $k' \leq 0$, the integral term is ≤ 0 ; since k is non-increasing, $k(0^+) = \sup k$ and $k(\Delta) \geq \inf k$, so the bracket is at most $\frac{1+\alpha}{\lambda} \sup k \sup v - 2 \inf k \inf v \leq 0$ by $\rho_v \rho_k \leq \frac{2\lambda}{1+\alpha}$.

Dead-zone piece ($x_1^{\text{out}} > 0$, only if $\alpha > 0$). As in Eq. (37), but keeping k :

$$\frac{\partial u_1}{\partial c_1} = (1+\alpha) \int_0^\Delta k(t) \left[\frac{1}{\lambda} v(x_1^{\text{in}} - \frac{t}{\lambda}) - \frac{1}{\alpha} v(x_1^{\text{out}} + \frac{t}{\alpha}) \right] dt < 0,$$

by the same pointwise comparison and strictness argument as in Theorem 56; note this piece needs *no* assumption on k at all. The junction continuity, single-peakedness, characterization, and (for symmetric v) the existence argument via Lemma 54 are verbatim as before; for existence, the sign analysis of Eq. (38) carries over with the integrands weighted by $k > 0$. $\square$

Remark 60. The C^1 assumption is for the displayed second-derivative formula only; the conclusion extends to continuous non-increasing k by approximating

k uniformly with C^1 non-increasing densities (concavity and single-peakedness survive pointwise limits of payoffs). For $\beta > 0$ and general (v, k) the opponent term $\partial^2 u_2 / \partial c_1^2$ contains $k'(B_2)$ -integrals of indefinite sign, and we make no claim; the uniform-cost case is Remark 57(c), and the uniform-voter case is settled completely in Appendix L.

K.3 Dead zones

Proposition 61 (When equilibria have dead zones). *Let $\alpha > 0$.*

- (a) (Necessity; any K , $\beta = 0$.) *If a profile $0 < c_1 < c_2 < 1$ with $x_1^{\text{out}} > 0$ satisfies candidate 1's first-order condition, then $\rho_v \geq \frac{2+\alpha}{\alpha}$. Equivalently, if $\alpha < \frac{2}{\rho_v-1}$, no stationary profile—in particular no equilibrium—has a dead zone.*
- (b) (Uniform voters never; any K , any $\beta \in [0, 1)$.) *If $x \sim U(0, 1)$, then no Nash equilibrium of the β -game has a dead zone: at every equilibrium $b_1 > 0$ and $b_2 > 0$.*
- (c) (Sufficiency for symmetric electorates; uniform costs, $\beta = 0$.) *If $\kappa \sim U(0, 1)$ and v is symmetric about $\frac{1}{2}$ with*

$$V\left(\frac{2(1+\alpha)}{(2+\alpha)^2}\right) > 2V\left(\frac{1}{2+\alpha}\right), \quad (39)$$

then a symmetric stationary profile with a dead zone exists.

Proof. (a) With $x_1^{\text{out}} > 0$, the first-order condition from Eq. (30) is the vanishing of Eq. (37) (with the weight $k(t)$ for general K). If $\rho_v < \frac{\lambda}{\alpha}$ the integrand is pointwise negative as in Theorem 56, and if $\rho_v = \frac{\lambda}{\alpha}$ the strictness argument there applies; either way the integral cannot vanish. Hence stationarity forces $\rho_v \geq \frac{\lambda}{\alpha}$, i.e. $\alpha \geq \frac{2}{\rho_v-1}$.

(b) By Lemma 52(c) and Lemma 53, an equilibrium is untied, interior, and satisfies both first-order conditions. By Eq. (33) and its mirror image,

$$(1+\alpha)K(b_1^+) - \beta K(b_2^+) = (1-\beta)\alpha A K(\Delta) = (1+\alpha)K(b_2^+) - \beta K(b_1^+),$$

so $(1+\alpha+\beta)[K(b_1^+) - K(b_2^+)] = 0$, i.e. $K(b_1^+) = K(b_2^+)$. If this common value is 0, the first-order condition gives $(1-\beta)\alpha A K(\Delta) = 0$, impossible since $\beta < 1$ and $K(\Delta) > 0$ for $\Delta \in (0, 1)$. Hence $K(b_1^+) = K(b_2^+) > 0$, which forces $b_1 > 0$ and $b_2 > 0$.

(c) Consider the symmetric family $c_2 = 1 - c_1$ and h from Eq. (38). The dead-zone branch is $c_1 \in (\frac{1}{\lambda}, \frac{1}{2})$ (indeed $x_1^{\text{out}} > 0 \iff \alpha c_1 > \Delta = 1 - 2c_1 \iff c_1 > \frac{1}{\lambda}$). At the junction $c_1 = \frac{1}{\lambda}$ we have $\Delta = \frac{\alpha}{\lambda}$, $x_1^{\text{in}} = \frac{1}{\lambda} + \frac{\alpha}{\lambda^2} = \frac{2(1+\alpha)}{\lambda^2}$ and $x_1^{\text{out}} = 0$, so $h(\frac{1}{\lambda}) = V(\frac{2(1+\alpha)}{\lambda^2}) - 2V(\frac{1}{\lambda}) > 0$ by Eq. (39), while $h(\frac{1}{2}) < 0$ as in the proof of Corollary 58. By the intermediate value theorem, h has a root in the open dead-zone branch. $\square$

Remark 62. (a) Part (b) removes the last gap in the uniform-voter theory: the characterizations of Appendix L never need a dead-zone case. At $\beta = 1$ the conclusion fails—inside the band $E(\alpha)$ both tents are interior, i.e. $b_1, b_2 \leq 0$, which is exactly what produces the band—so the dichotomy at $\beta = 1$ is sharp. (b) Criterion (39) compares the mass of the widest possible inner arm with twice the mass below the junction position; for uniform voters its left side minus right side is $-\frac{2}{\lambda^2} < 0$, consistent with (b). Numerically the criterion is exact for symmetric Beta electorates: for $v = \text{Beta}(n, n)$, $n \in \{2, 4, 8\}$ and $\alpha \in \{1, 2, 4\}$, dead-zone stationary profiles exist in precisely the cases where Eq. (39) holds (it fails only for $n = 2$, $\alpha = 1$). For $v = \text{Beta}(8, 8)$ and $\alpha = 2$ the (unique) symmetric stationary profile $(0.381, 0.619)$ has $x_1^{\text{out}} = 0.26 > 0$ and is a genuine equilibrium (mutual global best responses, verified numerically): dead-zone equilibria are non-empty. (c) Parts (a) and (c) bracket the phenomenon: dead zones require $\alpha \geq \frac{2}{\rho_v - 1}$ and concentrated electorates, and are guaranteed (at the stationary level) once the center is heavy enough that Eq. (39) holds.

L Results for competitive candidates ($\beta > 0$)

Throughout this section $\beta \in [0, 1)$ unless stated otherwise; all results specialize at $\beta = 0$ to (and strengthen) the vote-share results of Section 4. Write

$$\gamma_\beta := \frac{1 + \beta}{1 - \beta} \in [1, \infty), \quad \psi_\beta(t) := t + \gamma_\beta \frac{K(t)}{k(t)}, \quad \chi_\beta(t) := t + \frac{1 + \beta}{2} \frac{K(t)}{k(t)}.$$

The two deformations of $\psi(t) = t + K(t)/k(t)$ play distinct roles below: ψ_β governs *stationarity*, χ_β governs *single-peakedness*; at $\beta = 0$ they coincide only in their monotonicity class, not in value, and the wedge between the coefficients γ_β and $\frac{1 + \beta}{2}$ is what Example 70 exploits.

L.1 Uniform voters, $\alpha = 0$: complete characterization

Theorem 63 (Uniform voters, $\alpha = 0$: characterization and uniqueness).

Let $x \sim U(0, 1)$, $\alpha = 0$, $\beta \in [0, 1)$, and let K satisfy Assumption 2.

- (a) (Necessity.) Every Nash equilibrium is symmetric, $c_1 + c_2 = 1$, with gap $\Delta \in (0, 1)$ satisfying $\psi_\beta(\Delta) = 1$.
- (b) (Characterization.) A symmetric profile with gap Δ and $c_2 = \frac{1 + \Delta}{2}$ is a Nash equilibrium if and only if Δ maximizes

$$F(t) := (c_2 - t - \beta(1 - c_2))K(t) + \frac{1 - \beta}{2} G(t) \quad \text{over } t \in (0, c_2].$$

- (c) (Uniqueness under monotone K/k .) If K/k is non-decreasing on $(0, 1)$, then ψ_β is strictly increasing, $\psi_\beta(t) = 1$ has a unique root $\Delta_\beta^* \in (0, 1)$, and the game has the unique Nash equilibrium $(\frac{1 - \Delta_\beta^*}{2}, \frac{1 + \Delta_\beta^*}{2})$.

Proof. Throughout, fix the opponent at c_2 and parameterize candidate 1's own-side positions by the distance $t = c_2 - c_1 \in (0, c_2]$. By the $\alpha = 0$ closed forms below Eq. (32), $u_1 = (c_2 - t)K(t) + \frac{1}{2}G(t)$ and $u_2 = (1 - c_2)K(t) + \frac{1}{2}G(t)$, so $U_1 = u_1 - \beta u_2 = F(t)$ as displayed. By Lemma 53, $F \in C^1$ with

$$F'(t) = (Q - t)k(t) - \frac{1+\beta}{2}K(t) = k(t) [Q - \chi_\beta(t)], \quad Q := c_2 - \beta(1 - c_2). \quad (40)$$

Cross-over deviations are dominated whenever $c_2 \geq \frac{1}{2}$. For $s \in (0, 1 - c_2]$ compare the mirror pair $c'_1 = c_2 - s$ (own side) and $c''_1 = c_2 + s$ (far side). The closed forms give

$$\begin{aligned} U_1(c'_1, c_2) - U_1(c''_1, c_2) &= [(c_2 - s) - (1 - c_2 - s)]K(s) + \beta[c_2 - (1 - c_2)]K(s) \\ &= (1 + \beta)(2c_2 - 1)K(s) \geq 0. \end{aligned}$$

Since the far-side range $s \leq 1 - c_2$ is contained in the own-side range $s \leq c_2$, we get $\sup_{c'_1 > c_2} U_1 \leq \sup_{c'_1 < c_2} U_1 = \sup_t F(t)$. Deviating to the tie $c'_1 = c_2$ yields 0, which is dominated whenever $\sup F > 0$.

(b). ($\Leftarrow$) Let Δ maximize F over $(0, c_2]$ at a symmetric profile. Then own-side deviations are unprofitable by definition; $c_2 = \frac{1+\Delta}{2} > \frac{1}{2}$, so cross-overs are dominated; and $F(\Delta) = (1 - \beta)u_1 > 0$ dominates the tie. Candidate 2's problem is the mirror image (uniform voters are symmetric), so the profile is an equilibrium. ($\Rightarrow$) If the symmetric profile is an equilibrium, c_1 is a global maximizer of $U_1(\cdot, c_2)$ over $[0, c_2]$, i.e. Δ maximizes F over $(0, c_2]$.

(a). By Lemma 52(b)(c), an equilibrium is untied and interior; Lemma 53 gives both first-order conditions. In the t -parameterization these read $Q_1 = \chi_\beta(\Delta)$ and $Q_2 = \chi_\beta(\Delta)$ with $Q_1 = c_2 - \beta(1 - c_2)$ and, mirroring, $Q_2 = (1 - c_1) - \beta c_1$. Subtracting, $(1 + \beta)(c_1 + c_2 - 1) = 0$, so the equilibrium is symmetric. Substituting $c_2 = \frac{1+\Delta}{2}$ into $Q = \chi_\beta(\Delta)$ and simplifying yields the identity

$$Q - \chi_\beta(\Delta) = \frac{1 - \beta}{2} [1 - \psi_\beta(\Delta)], \quad (41)$$

so stationarity of the symmetric profile is exactly $\psi_\beta(\Delta) = 1$.

(c). If K is super-regular then ψ_β and χ_β are strictly increasing. *Root existence.* First, $\lim_{t \downarrow 0} K/k = 0$: the limit $L \geq 0$ exists by monotonicity, and if $L > 0$ then $r = k/K \leq 1/L$ near 0, contradicting Eq. (29); hence $\psi_\beta(0^+) = 0 < 1$. Second, $\psi_\beta(t) > 1$ for some $t < 1$: otherwise $\gamma_\beta K/k \leq 1 - t$ near 1, i.e. $r \geq \gamma_\beta/(1 - t)$, making $\int_{t_0}^{1^-} r = \infty$ for some $t_0 < 1$ —impossible since $\int_{t_0}^{1^-} r = \log K(1) - \log K(t_0) < \infty$. By continuity and strict monotonicity there is a unique root $\Delta_\beta^* \in (0, 1)$. *The root is an equilibrium.* At the symmetric profile with gap Δ_β^* : χ_β is strictly increasing and crosses Q exactly once; by Eq. (41) the crossing is at Δ_β^* , so by Eq. (40), $F' > 0$ before and $F' < 0$ after: F is single-peaked with peak Δ_β^* , which is therefore its maximizer (the peak is interior: $\chi_\beta(c_2) - Q = \frac{1+\beta}{2} \frac{K}{k}(c_2) + \beta(1 - c_2) > 0$ shows $F'(c_2^-) < 0$, ruling out $c_1 = 0$). By (b) the profile is an equilibrium. *Uniqueness.* By (a) any equilibrium is symmetric with $\psi_\beta(\Delta) = 1$; the root is unique. $\square$

L.2 Uniform voters, $\alpha > 0$: complete characterization

Define, for $\Delta \in (\Delta_0, 1)$,

$$J(\Delta) := \log \frac{K(\Delta)}{K(b_s(\Delta))} = \int_{b_s(\Delta)}^{\Delta} r(s) ds, \quad L_\beta := \log \frac{(1 + \alpha - \beta)\lambda}{2(1 - \beta)(1 + \alpha)}. \quad (42)$$

Note $L_\beta > 0$ always: the ratio minus one has the sign of $(1 + \alpha - \beta)\lambda - 2(1 - \beta)(1 + \alpha) = \alpha(1 + \alpha + \beta) > 0$. Note also that L_β does not depend on K ; this is what makes the comparative statics in Propositions 66 and 67 clean.

Theorem 64 (Uniform voters, $\alpha > 0$: unique equilibrium and its gap).

Let $x \sim U(0, 1)$, $\alpha > 0$, $\beta \in [0, 1)$, and let K satisfy Assumption 2 and be super-regular. Then the game has a unique Nash equilibrium. It is symmetric, has no dead zones, and its gap is the unique root $\Delta^* = \Delta^*(\alpha, \beta) \in (\Delta_0, 1)$ of $J(\Delta) = L_\beta$; equivalently, of the first-order condition

$$(1 + \alpha - \beta) K(b_s(\Delta)) = (1 - \beta) \frac{2(1 + \alpha)}{2 + \alpha} K(\Delta). \quad (43)$$

In particular the equilibrium is strictly more polarized than every symmetric margin-game equilibrium: $\Delta^* > \Delta_0$.

Proof. Existence and uniqueness of the root. On $(\Delta_0, 1)$, J is differentiable with $J'(\Delta) = r(\Delta) - \frac{\lambda}{2} r(b_s(\Delta)) \leq r(\Delta)(1 - \frac{\lambda}{2}) = -\frac{\alpha}{2} r(\Delta) < 0$, using $b_s(\Delta) < \Delta$ and r non-increasing. Moreover $J(\Delta_0^+) = +\infty$ by Eq. (29) and $J(1^-) = \log \frac{K(1)}{K(1)} = 0 < L_\beta$. Hence a unique root $\Delta^* \in (\Delta_0, 1)$ exists, with $J > L_\beta$ on (Δ_0, Δ^*) and $J < L_\beta$ on $(\Delta^*, 1)$; exponentiating $J = L_\beta$ gives Eq. (43).

Necessity. Exactly as in Proposition 61(b): an equilibrium is untied and interior with both first-order conditions (Lemma 52(c), Lemma 53); subtracting them yields $K(b_1^+) = K(b_2^+) > 0$, hence $b_1 = b_2 > 0$ (K is strictly increasing on $(0, 1)$), which by Eq. (28) forces $c_1 + c_2 = 1$ and $b_1 = b_2 = b_s(\Delta) > 0$, i.e. $\Delta > \Delta_0$. The first-order condition (33) at the symmetric profile becomes exactly Eq. (43), i.e. $\Delta = \Delta^*$.

Sufficiency. Fix $c_2 = c_2^* = \frac{1 + \Delta^*}{2}$ and let candidate 1 vary $c_1 \in (0, c_2^*)$; write $\Delta := c_2^* - c_1$, $\delta := \frac{b_2 - b_1}{2} = \frac{\alpha}{2}(c_1 - c_1^*)$ by Eq. (28) (using $1 - c_2^* = c_1^*$), so $b_1 = b_s(\Delta) - \delta$ and $b_2 = b_s(\Delta) + \delta$. Decompose Eq. (33) as

$$\alpha \frac{\partial U_1}{\partial c_1} = \Phi(\Delta) + (1 + \alpha)[K(b_1^+) - K(b_s(\Delta)^+)] - \beta[K(b_2^+) - K(b_s(\Delta)^+)], \quad (44)$$

where $\Phi(\Delta) := (1 + \alpha - \beta)K(b_s(\Delta)^+) - (1 - \beta)\alpha A K(\Delta)$ is the value at the “centered” benefits $\delta = 0$. For $\Delta \leq \Delta_0$, $\Phi(\Delta) = -(1 - \beta)\alpha A K(\Delta) < 0$; for $\Delta \in (\Delta_0, 1)$, $\Phi(\Delta) < 0 \iff J(\Delta) > L_\beta \iff \Delta < \Delta^*$ (and $\Phi(\Delta^*) = 0$, $\Phi > 0$ for $\Delta > \Delta^*$). Now: if $c_1 > c_1^*$ then $\delta > 0$ and $\Delta < \Delta^*$, so both bracketed corrections in Eq. (44) are ≤ 0 (monotonicity of $t \mapsto K(t^+)$) and $\Phi(\Delta) < 0$: the derivative is strictly negative. If $c_1 < c_1^*$ then $\delta < 0$ and $\Delta \in (\Delta^*, c_2^*] \subset (\Delta^*, 1)$,

so both corrections are ≥ 0 and $\Phi(\Delta) > 0$: the derivative is strictly positive. Hence $U_1(\cdot, c_2^*)$ is strictly single-peaked on $(0, c_2^*)$ with peak c_1^* , and the one-sided derivative at $c_1 = 0$ is positive, so c_1^* is the own-side global maximizer.

Cross-overs and the tie. For $s \in (0, 1 - c_2^*]$ compare $c'_1 = c_2^* - s$ with $c''_1 = c_2^* + s$ using Eq. (32). The deviator's boundary benefits are $b_L = s - \alpha(c_2^* - s)$ on the near side and $b_R = s - \alpha(1 - c_2^* - s)$ on the far side; $c_2^* \geq \frac{1}{2}$ gives $b_L \leq b_R$, hence $u_1(c'_1, c_2^*) = AG(s) - \frac{1}{\alpha}G(b_L^+) \geq AG(s) - \frac{1}{\alpha}G(b_R^+) = u_1(c''_1, c_2^*)$. The opponent's boundary benefit is $s - \alpha(1 - c_2^*)$ when the deviator is on the near side and $s - \alpha c_2^*$ on the far side; $c_2^* \geq \frac{1}{2}$ gives $u_2(c_2^*, c'_1) \leq u_2(c_2^*, c''_1)$. Hence $U_1(c'_1, c_2^*) \geq U_1(c''_1, c_2^*)$, and since the far-side range is contained in the own-side range, cross-overs are dominated. The tie yields $0 < (1 - \beta)u_1(c_1^*, c_2^*)$ (the equilibrium turnout is positive because $\Delta^* > b_s(\Delta^*)$ makes Eq. (32) strictly positive). By symmetry candidate 2 has no profitable deviation either, so the profile is the unique Nash equilibrium. $\square$

Remark 65 (Consistency as $\alpha \rightarrow 0$). Applying the mean value theorem to $K(\Delta) - K(b_s(\Delta))$ in Eq. (43) produces the exact identity

$$\frac{1 + \alpha + \beta}{2 + \alpha} K(\Delta^*) = \frac{1 + \alpha - \beta}{2} (1 - \Delta^*) k(\xi_\alpha), \quad \xi_\alpha \in (b_s(\Delta^*), \Delta^*).$$

As $\alpha \downarrow 0$, $b_s(\Delta^*) \rightarrow \Delta^*$, so $\xi_\alpha \rightarrow \Delta^*$ and, by continuity of k , the identity converges to $(1 + \beta)K(\Delta) = (1 - \beta)(1 - \Delta)k(\Delta)$, i.e. $\psi_\beta(\Delta) = 1$: the $\alpha > 0$ theory deforms continuously onto Theorem 63. (Verified numerically: the closed forms for doubly-uniform and polynomial K match the roots of Eq. (43) and of $\psi_\beta = 1$ to 10^{-6} across $\alpha \in \{0, \frac{1}{2}, 1, 2\}$, $\beta \in \{0, \dots, 0.9\}$.)

L.3 Comparative statics and their limits

Proposition 66 (Monotone comparative statics). *In the setting of Theorem 63(c) (resp. Theorem 64), the equilibrium gap $\Delta^*(\alpha, \beta)$ is strictly decreasing in β on $[0, 1)$, and (for $\alpha > 0$) strictly increasing in α .*

Proof. Decreasing in β , $\alpha = 0$. γ_β is strictly increasing in β and $K/k(\Delta^*) > 0$, so $\psi_\beta(\Delta^*)$ is strictly increasing in β ; since ψ_β is strictly increasing in t , the root of $\psi_\beta = 1$ strictly decreases. (No differentiability in β is needed.)

Decreasing in β , $\alpha > 0$. Write $\Psi(\Delta, \alpha, \beta) = J(\Delta) - L_\beta$, so $\Psi(\Delta^*, \cdot) = 0$, $\partial\Psi/\partial\Delta = J' < 0$, and

$$\frac{\partial\Psi}{\partial\beta} = -\frac{dL_\beta}{d\beta} = -\left(\frac{1}{1-\beta} - \frac{1}{1+\alpha-\beta}\right) = -\frac{\alpha}{(1-\beta)(1+\alpha-\beta)} < 0,$$

whence $d\Delta^*/d\beta = -\Psi_\beta/\Psi_\Delta < 0$.

Increasing in α . Here $\partial\Psi/\partial\alpha = \partial J/\partial\alpha - dL_\beta/d\alpha$ with $\partial J/\partial\alpha = -r(b_s)\frac{\partial b_s}{\partial\alpha} = r(b_s(\Delta))\frac{1-\Delta}{2} > 0$ and $dL_\beta/d\alpha = \frac{1}{\lambda} + \frac{1}{1+\alpha-\beta} - \frac{1}{1+\alpha} =: R$. At $\Delta = \Delta^*$, since r is non-increasing,

$$L_\beta = J(\Delta^*) = \int_{b_s}^{\Delta^*} r \leq r(b_s)(\Delta^* - b_s) = r(b_s)\frac{\alpha}{2}(1 - \Delta^*),$$

so $r(b_s) \frac{1-\Delta^*}{2} \geq \frac{L_\beta}{\alpha}$ and it suffices to show $g(\beta) := L_\beta - \alpha R > 0$. Compute

$$g'(\beta) = \frac{\alpha}{(1-\beta)(1+\alpha-\beta)} - \frac{\alpha}{(1+\alpha-\beta)^2} = \frac{\alpha^2}{(1-\beta)(1+\alpha-\beta)^2} > 0,$$

and, with $s = \frac{\alpha}{2}$, $g(0) = \log \frac{\lambda}{2} - \frac{\alpha}{\lambda} = \log(1+s) - \frac{s}{1+s} > 0$. Hence $g(\beta) > 0$ for all $\beta \in [0, 1]$, $\partial \Psi / \partial \alpha \geq g(\beta) / \alpha > 0$, and $d\Delta^* / d\alpha = -\Psi_\alpha / \Psi_\Delta > 0$. $\square$

Proposition 67 (Reverse-hazard-rate comparison at every β). *Let $K, \tilde{K}$ satisfy the hypotheses of Theorem 63(c) (resp. Theorem 64) and suppose $\tilde{r}(t) \geq r(t)$ for all $t \in (0, 1)$. Then for every $\beta \in [0, 1]$ (and every α), $\tilde{\Delta}^*(\alpha, \beta) \geq \Delta^*(\alpha, \beta)$: more eager electorates sustain weakly more polarization, uniformly in competitiveness.*

Proof. $\alpha = 0$: $\tilde{r} \geq r$ means $\tilde{K}/\tilde{k} \leq K/k$ pointwise, so $\tilde{\psi}_\beta \leq \psi_\beta$ pointwise; hence $\tilde{\psi}_\beta(\Delta^*) \leq \psi_\beta(\Delta^*) = 1 = \tilde{\psi}_\beta(\tilde{\Delta}^*)$ and, since $\tilde{\psi}_\beta$ is increasing, $\tilde{\Delta}^* \geq \Delta^*$. $\alpha > 0$: $\tilde{J}(\Delta) = \int_{b_s}^\Delta \tilde{r} \geq J(\Delta)$ pointwise, while L_β is the same for both games (K -free). Thus $\tilde{J}(\tilde{\Delta}^*) = L_\beta = J(\Delta^*) \leq \tilde{J}(\Delta^*)$, and $\tilde{J}$ decreasing gives $\tilde{\Delta}^* \geq \Delta^*$. $\square$

Proposition 68 (Collapse as $\beta \uparrow 1$, with exact rate). *Let $x \sim U(0, 1)$ and $\alpha > 0$.*

- (a) (Universal collapse; arbitrary K .) *Let $\beta_n \uparrow 1$ and let (c_1^n, c_2^n) be symmetric stationary profiles of the β_n -games; by the proof of Proposition 61(b) (which uses no super-regularity) this includes every Nash equilibrium. Then $\Delta_n \rightarrow \Delta_0 = \frac{\alpha}{2+\alpha}$: polarization collapses onto the most polarized symmetric margin-game profile.*
- (b) (Exact finite- β identity.) *Under the hypotheses of Theorem 64, for every β ,*

$$\Delta^* - \Delta_0 = \frac{2}{\lambda} K^{-1}\left((1-\beta) c_\beta K(\Delta^*)\right), \quad c_\beta := \frac{2(1+\alpha)}{\lambda(1+\alpha-\beta)}.$$

- (c) (Rates.) *If $k(0^+) \in (0, \infty)$ then $\Delta^* - \Delta_0 = (1-\beta) \frac{4(1+\alpha)K(\Delta_0)}{\lambda^2 \alpha k(0^+)} (1 + o(1))$: linear in $1-\beta$, with K entering only through $K(\Delta_0)/k(0^+)$. If instead $K(t) \sim Ct^p$ as $t \downarrow 0$ ($p > 1$), the rate slows to order $(1-\beta)^{1/p}$. At $\alpha = 0$ (under Theorem 63(c)): if $K(t)/k(t) \sim ct$ as $t \downarrow 0$ then $\Delta_\beta^* = \frac{1-\beta}{2c} (1 + o(1))$; in particular $c = 1$, and the rate is the universal $\frac{1-\beta}{2}$, whenever $k(0^+) \in (0, \infty)$.*

Proof. (a) A symmetric stationary profile satisfies Eq. (43) with $\beta = \beta_n$ (the derivation of Eq. (43) from Eq. (33) used no super-regularity), so

$$K(b_s(\Delta_n)^+) = (1-\beta_n) c_{\beta_n} K(\Delta_n) \leq (1-\beta_n) \frac{2(1+\alpha)}{\lambda \alpha} K(1) \rightarrow 0.$$

Since $K \geq K(t_0) > 0$ on $[t_0, 1]$ for each $t_0 > 0$, this forces $b_s(\Delta_n)^+ \rightarrow 0$, i.e. $\Delta_n \rightarrow \Delta_0$ (using $b_s(\Delta) = \frac{\lambda}{2}(\Delta - \Delta_0)$ and, for the lower side, that $\Delta_n \leq \Delta_0$ would make the right-hand side of Eq. (43) positive and the left zero).

(b) Immediate from Eq. (43): $K(b_s(\Delta^*)) = (1 - \beta)c_\beta K(\Delta^*)$ and $b_s(\Delta^*) = \frac{\lambda}{2}(\Delta^* - \Delta_0)$.

(c) For $k(0^+) \in (0, \infty)$, $K^{-1}(\varepsilon) = \varepsilon/k(0^+)(1 + o(1))$; insert $\varepsilon = (1 - \beta)c_\beta K(\Delta^*)$, $c_\beta \rightarrow \frac{2(1+\alpha)}{\lambda\alpha}$ and $K(\Delta^*) \rightarrow K(\Delta_0)$ (by (a) and continuity). For $K \sim Ct^p$, $K^{-1}(\varepsilon) \asymp \varepsilon^{1/p}$. At $\alpha = 0$: $\psi_\beta(\Delta^*) = 1$ reads $\gamma_\beta(K/k)(\Delta^*) = 1 - \Delta^*$; the right side tends to 1 (as $\Delta^* \rightarrow 0$ by the same argument as (a)), so $(K/k)(\Delta^*) \sim \gamma_\beta^{-1}$ and $K/k \sim ct$ gives $\Delta^* \sim \frac{1}{c\gamma_\beta} \sim \frac{1-\beta}{2c}$. $\square$

Remark 69 (Numerical confirmation). For $K = t^2$ ($p = 2$, $k(0^+) = 0$) at $\alpha = 0$ the root is exactly $\Delta_\beta^* = \frac{2(1-\beta)}{3-\beta}$: rate $(1 - \beta)$, constant $1/c = 2$, not the universal $\frac{1-\beta}{2}$. At $\alpha = 1$, $\beta = 0.99$, $K = t^2$: the computed gap is $\Delta^* = 0.360984$ and $\frac{2}{\lambda}K^{-1}((1-\beta)c_\beta K(\Delta^*)) = 0.027651 = \Delta^* - \frac{1}{3}$ to six digits, matching (b) exactly. The reverse-hazard-rate comparison of Proposition 67 was confirmed at every $\beta \in \{0, 0.5, 0.9, 0.99\}$ for $K = t$ versus $K = t^2$ at $\alpha \in \{0, 1\}$.

The monotone-reverse-hazard-rate hypothesis in Proposition 66 is not removable: without it, *more competition can increase polarization*.

Example 70 (An explicit cost distribution with three equilibria and an increasing branch). Let $\alpha = 0$, $x \sim U(0, 1)$, and define K through $\varphi := K/k$ by $K(t) = \exp(-\int_t^1 \varphi(s)^{-1} ds)$, where φ is piecewise linear:

$$\varphi(t) = \begin{cases} 2t, & t \in [0, \frac{17}{50}], \\ \frac{17}{25} - \frac{3}{2}(t - \frac{17}{50}), & t \in [\frac{17}{50}, \frac{1}{2}], \\ \frac{11}{25} + \frac{1}{5}(t - \frac{1}{2}), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Then $\varphi > 0$ on $(0, 1]$, K is a valid cost CDF satisfying Assumption 2 with $K(1) = 1$ ($k(t) = K(t)/\varphi(t) \sim \text{const} \cdot t^{-1/2}$ as $t \downarrow 0$, continuous and positive on $(0, 1)$), and:

- (i) For every $\beta \in [0, \frac{1}{3})$, all symmetric stationary profiles are Nash equilibria; the equilibrium gaps are exactly the roots of $\psi_\beta(t) = t + \gamma_\beta \varphi(t) = 1$.
- (ii) For $\beta \in [0, \frac{3}{47})$ there are exactly three equilibria, with gaps (writing $\gamma = \gamma_\beta$)

$$\Delta_1 = \frac{1}{1 + 2\gamma}, \quad \Delta_2 = \frac{100 - 119\gamma}{100 - 150\gamma}, \quad \Delta_3 = \frac{50 - 17\gamma}{50 + 10\gamma},$$

and $\frac{d\Delta_2}{d\gamma} = \frac{3100}{(100-150\gamma)^2} > 0$: along the middle branch, *polarization strictly increases in β* , from $\Delta_2 = \frac{19}{50}$ at $\beta = 0$ to $\frac{1}{2}$ at the fold. The outer branches are strictly decreasing.

- (iii) At $\beta^* = \frac{3}{47}$ (where $\gamma_{\beta^*} = \frac{25}{22}$) the middle and upper branches collide at $\Delta = \frac{1}{2}$ and annihilate: for $\beta \in (\beta^*, \frac{1}{3})$ the branch Δ_1 is the unique equilibrium. The equilibrium count drops from three to one through a fold—a non-smooth one, since the two gaps meet at the kink of ψ_β at $t = \frac{1}{2}$ and so approach $\frac{1}{2}$ at finite opposite rates rather than at the square-root rate of a smooth saddle-node: increasing competitiveness *discontinuously* de-polarizes the equilibrium set.

Proof. K is a CDF with $K(0) = 0$ by divergence of $\int_{0+} \varphi^{-1} = \int_{0+} \frac{ds}{2s}$, and Assumption 2 is immediate. (i) By Theorem 63(b) and Eqs. (40) to (41), a symmetric stationary profile is an equilibrium as soon as χ_β is strictly increasing (then F is single-peaked with peak at the stationary gap). Here $\chi_\beta = t + \frac{1+\beta}{2}\varphi(t)$ is piecewise linear with slopes

$$1 + \frac{1+\beta}{2} \cdot 2 = 2 + \beta, \quad 1 - \frac{3}{4}(1 + \beta), \quad 1 + \frac{1+\beta}{10},$$

all strictly positive iff $\beta < \frac{1}{3}$. (ii) $\psi_\beta = t + \gamma\varphi(t)$ is piecewise linear with slopes $1 + 2\gamma > 0$, $1 - \frac{3}{2}\gamma < 0$, $1 + \frac{\gamma}{5} > 0$; solving $\psi_\beta = 1$ on each piece gives the displayed formulas, and the membership conditions $\Delta_1 \leq \frac{17}{50}$, $\Delta_2 \in (\frac{17}{50}, \frac{1}{2}]$, $\Delta_3 \in (\frac{1}{2}, 1)$ hold precisely for $\gamma < \frac{25}{22}$, i.e. $\beta < \frac{3}{47}$ (at $\gamma = 1$: $\Delta_1 = \frac{1}{3} \leq 0.34$, $\Delta_2 = \frac{19}{50}$, $\Delta_3 = \frac{11}{20}$). The derivative of Δ_2 in γ is as displayed, and γ_β is strictly increasing in β ; monotonicity of the outer branches is a one-line computation. (iii) At $\gamma = \frac{25}{22}$, $\psi_\beta(\frac{1}{2}) = \frac{1}{2} + \gamma \cdot \frac{11}{25} = 1$, and for $\gamma > \frac{25}{22}$ the minimum of ψ_β over $[\frac{17}{50}, 1]$ exceeds 1 while the first piece still crosses: exactly one root remains; for $\beta \in (\frac{3}{47}, \frac{1}{3})$ it is, by (i), the unique equilibrium. $\square$

Remark 71 (A local criterion, and the three-root example of Appendix C). The proof of Example 70 isolates a criterion of independent use: at a symmetric stationary profile with gap Δ at which $\varphi = K/k$ is differentiable,

$$\chi'_\beta(\Delta) = 1 + \frac{1+\beta}{2} \varphi'(\Delta) < 0 \implies \Delta \text{ is a strict local minimum of } F,$$

hence not an equilibrium, since $F' = k[Q - \chi_\beta]$ then crosses 0 from below. This explains, conceptually rather than numerically, why the three-root cost distribution has non-equilibrium middle root: there $\chi'_0 = 1 + \varphi'/2$ evaluates (numerically) to +4.5, −0.5, +0.94 at the three roots $\Delta \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ —the middle root is a local minimum of the deviator’s payoff, while in Example 70 the middle root has $\chi'_\beta > 0$ and is a genuine equilibrium. Which of the two behaviors a middle root exhibits is thus read off the slope of K/k at the root.

L.4 The $\beta \uparrow 1$ limit for general electorates

Theorem 72 (Upper hemicontinuity onto the margin game). *Let Assumption 2 hold and $\alpha \geq 0$. The payoffs (u_1, u_2) are jointly continuous on $[0, 1]^2$, and $\sup_c |U_i^\beta(c) - U_i^1(c)| \leq (1 - \beta)$ for all i and β . Consequently, if $\beta_n \uparrow 1$ and (c^n) are Nash equilibria of the β_n -games with $c^n \rightarrow c^\infty$, then c^∞ is a Nash equilibrium of the margin game ($\beta = 1$). In particular:*

- (a) *If $\alpha = 0$, then $c^n \rightarrow (q, q)$, the median profile: equilibria of the nearly-zero-sum games select the margin game’s unique equilibrium. (No monotonicity of $r = k/K$ and no bound on ρ_v is assumed.)*
- (b) *If $\alpha > 0$ and $x \sim U(0, 1)$, every limit point lies in the margin-game equilibrium set $\{(c_1, c_2) : \frac{1}{\lambda} \leq c_1 \leq c_2 \leq \frac{1+\alpha}{\lambda}\}$; under the hypotheses of Theorem 64 the full limit is the maximally polarized symmetric profile, $\Delta^*(\alpha, \beta) \downarrow \Delta_0$.*

Proof. Continuity. For positions (a, o) and (a', o') and every voter x , $|B(x; a, o) - B(x; a', o')| \leq (1 + \alpha)|a - a'| + |o - o'|$, and $t \mapsto K(t^+)$ is uniformly continuous on $[-1, 1]$ (continuous on a compact); since $\int v = 1$,

$$|u_1(a, o) - u_1(a', o')| \leq \omega_K((1 + \alpha)|a - a'| + |o - o'|),$$

with ω_K the modulus of continuity of $K(\cdot^+)$ —valid for *arbitrary* v , including unbounded densities. The same bound holds for u_2 . Continuity holds across ties: as $a \rightarrow o$ all benefits tend to 0 uniformly, so $u_i \rightarrow 0$, the tie payoff. Next, $|U_i^\beta - U_i^1| = (1 - \beta)u_{-i} \leq (1 - \beta)K(1) \leq 1 - \beta$.

Limit. Fix any c'_1 . For each n , $U_1^{\beta_n}(c_1^n, c_2^n) \geq U_1^{\beta_n}(c'_1, c_2^n)$. Passing to the limit using joint continuity of (u_1, u_2) and the uniform bound, $U_1^1(c^\infty) \geq U_1^1(c'_1, c_2^\infty)$; similarly for candidate 2. Hence c^∞ is a margin-game equilibrium. (a) then follows from Theorem 45 (at $\alpha = 0$ the margin game has the unique equilibrium (q, q) , q the median), plus compactness of $[0, 1]^2$: every subsequence has a subsubsequence converging to (q, q) , so the whole sequence does. (b) follows from the uniform-voter margin-game characterization and Propositions 66 and 68. $\square$

Upper hemicontinuity presumes the β_n -equilibria exist. The next results show that for concentrated-at-the-extremes electorates they may *fail to exist* for all β close to 1—so the median selection of Theorem 72(a) is genuinely a statement about limits, not a guarantee. The engine is a quantitative margin-game certificate.

Lemma 73 (Lipschitz transfer). *Let V be arbitrary, $\alpha \geq 0$, and let K be L_K -Lipschitz on $[0, 1]$ (e.g. $L_K = 1$ for $\kappa \sim U(0, 1)$). Then the margin $w(c, c') := u_{\text{dev}}(c, c') - u_{\text{dev}}(c', c)$ satisfies, for all $c, \tilde{c}, c'$,*

$$|w(c, c') - w(\tilde{c}, c')| \leq (2 + \alpha)L_K |c - \tilde{c}|, \quad |w(c, c') - w(c, \tilde{c}')| \leq (2 + \alpha)L_K |c' - \tilde{c}'|.$$

Proof. Moving one candidate by h changes every voter's benefit toward her by at most $(1 + \alpha)|h|$ and toward her opponent by at most $|h|$; since $t \mapsto K(t^+)$ is L_K -Lipschitz and $\int v = 1$, the two vote shares move by at most $(1 + \alpha)L_K|h|$ and $L_K|h|$. Sum. (No boundedness of v is used.) $\square$

Proposition 74 (Grid certificate). *In the setting of Lemma 73, let $g_1, \dots, g_N$ be the uniform grid on $[0, 1]$ with step h , and suppose that*

$$\max_i \min_j w(g_i, g_j) \leq -\hat{\varepsilon} < 0.$$

Set $\varepsilon' := \hat{\varepsilon} - (2 + \alpha)L_K h/2$ and assume $\varepsilon' > 0$. Then:

- (a) $\min_{c' \in [0, 1]} w(c, c') \leq -\varepsilon'$ for every $c \in [0, 1]$; in particular the margin game ($\beta = 1$) has no pure Nash equilibrium.
- (b) For every $c' \in [0, 1]$ there exists c'' with $w(c'', c') \geq \varepsilon'$.

Proof. (a) Given c , take the nearest grid point g_i ($|c - g_i| \leq h/2$) and the certified g_j with $w(g_i, g_j) \leq -\hat{\varepsilon}$; then $w(c, g_j) \leq w(g_i, g_j) + (2 + \alpha)L_K h/2 \leq -\varepsilon'$.

A pure equilibrium (c_1, c_2) of the symmetric zero-sum margin game would need $w(c_1, \cdot) \geq 0$ everywhere (its value is 0 by antisymmetry), which (a) excludes. (b) Apply (a) at $c = c'$ to get g_j with $w(c', g_j) \leq -\varepsilon'$, hence by antisymmetry $w(g_j, c') \geq \varepsilon'$. $\square$

Theorem 75 (Non-existence for hollow electorates). *[computer-assisted]*
 Let $\alpha = 1$, $\kappa \sim U(0, 1)$.

- (a) For $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$ (the arcsine electorate): on the uniform grid with $N = 1201$ ($h = \frac{1}{1200}$), every $w(g_i, g_j)$ evaluates exactly (in terms of the regularized incomplete beta function) and $\max_i \min_j w(g_i, g_j) = -8.2244 \times 10^{-3}$; the Lipschitz slack is $(2 + \alpha)h/2 = 1.25 \times 10^{-3}$, so Proposition 74 applies with $\varepsilon' = 6.97 \times 10^{-3}$. The margin game has no pure equilibrium.
- (b) The same holds for $v = \text{Beta}(0.8, 0.8)$ with $N = 2401$, $\hat{\varepsilon} = 3.0238 \times 10^{-3}$, slack 6.25×10^{-4} , $\varepsilon' = 2.40 \times 10^{-3}$: non-existence is not an artifact of the unbounded arcsine density.

Proof. Lemma 73 and Proposition 74 reduce the claim to the finite computation, which is exact modulo floating-point evaluation of the incomplete beta function: each $w(g_i, g_j)$ is a finite signed combination of terms $V(\cdot)$ and $\int x v(x) dx$ at algebraic break points (the closed-form uniform-cost payoff), evaluated in double precision; we deduct a numerical guard of 10^{-10} —more than seven orders of magnitude below the certified margins—inside ε' . The computation is in the accompanying verification suite (`check_competition.py`, claim `C1.arcsine`). A sanity check on $v = \text{Beta}(2, 2)$ (centrally peaked), at the same $\alpha = 1$, returns grid security exactly 0, attained at the median and only there on the grid, consistent with Theorem 48 ($v''(\frac{1}{2}) = -12 < 0$, so the median is a strict local equilibrium of the margin game). $\square$

Corollary 76 (Non-existence for an interval of β). *In the setting of Theorem 75, the β -game has no pure Nash equilibrium for any*

$$\beta > \frac{1 - \varepsilon'}{1 + \varepsilon'} = \begin{cases} 0.98615 \dots & v = \text{Beta}(\frac{1}{2}, \frac{1}{2}), \\ 0.99521 \dots & v = \text{Beta}(0.8, 0.8). \end{cases}$$

Proof. Suppose (c_1, c_2) is a pure equilibrium of the β -game. Decompose $U_1 = \frac{1-\beta}{2}T + \frac{1+\beta}{2}w$ with $T = u_1 + u_2$. By Proposition 74(b) there is c'' with $w(c'', c_2) \geq \varepsilon'$, so candidate 1's equilibrium payoff satisfies $U_1 \geq U_1(c'', c_2) \geq \frac{1+\beta}{2}\varepsilon'$ (both terms of the decomposition are then nonnegative). Likewise $U_2 \geq \frac{1+\beta}{2}\varepsilon'$. Summing, $(1-\beta)T(c_1, c_2) = U_1 + U_2 \geq (1+\beta)\varepsilon'$, while $T \leq 1$ always; hence $\beta \leq \frac{1-\varepsilon'}{1+\varepsilon'}$. $\square$

Remark 77 (How conservative is the bound?). *[numerical observation]* The corollary certifies non-existence on a left neighborhood of 1; it is not tight. Numerically (best-response iteration with global verification), for $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$, $\alpha = 1$: pure equilibria exist for $\beta \leq 0.90$ (at $\beta = 0.90$: $(0.234, 0.766)$) and none are found for $\beta \geq 0.91$. We label the location of the true threshold a numerical

observation; the theorem-grade statements are Corollary 76 and, sharper, Theorem 97 ($\beta \geq 0.9168$). Together with Theorems 63 and 64 (existence for all $\beta < 1$ under uniform voters and super-regularity) and Theorem 94 and Corollary 95 (existence for all $\beta < 1$ at $\alpha = 0$ under uniform costs, the arcsine electorate included), this places the certified failure of pure existence at the joint corner $\alpha = 1$, β near 1, hollow v : non-existence is a phenomenon of alienation and competitiveness together, not of competitiveness alone. The two existence conjectures stay well clear of that corner: Remark 8 is the baseline, with $\alpha = \beta = 0$, and Conjecture 100 switches off alienation alone.

L.5 Quantile structure at $\alpha = 0$ under uniform costs

Proposition 78 (Mass-gap law and concavity under uniform costs).

Let $\alpha = 0$, $\kappa \sim U(0, 1)$, $\beta \in [0, 1)$, and let v satisfy Assumption 2.

(a) Every Nash equilibrium satisfies

$$V(c_2) - V(c_1) = \frac{1 - \beta}{2}, \quad 2V(c_1) = V(m) + \beta(1 - V(m)) : \quad (45)$$

the platforms trap voter mass exactly $\frac{1-\beta}{2}$, decreasing in competitiveness, independently of v .

(b) If $\rho_v \leq \frac{4}{1-\beta}$ then $U_1(\cdot, c_2)$ is concave on $[0, c_2]$ for every c_2 (and mirror), so equilibria are exactly the profiles satisfying Eq. (45) at which neither candidate gains from her cross-over deviation.

Proof. (a) By Lemma 52(b) and Eq. (31) with $k \equiv 1$: $\partial U_1 / \partial c_1 = V(m) - 2V(c_1) + \beta(1 - V(m)) = 0$ and, mirroring, $\partial U_2 / \partial c_2 = 1 - V(m) - 2(1 - V(c_2)) + \beta V(m) = 0$. The first display is the second condition of Eq. (45); the second rearranges to $2(1 - V(c_2)) = (1 - V(m)) + \beta V(m)$. Adding this to $2V(c_1) = V(m) + \beta(1 - V(m))$ gives $2 + 2V(c_1) - 2V(c_2) = 1 + \beta$, i.e. the mass-gap law. (b) Differentiating once more, $\partial^2 U_1 / \partial c_1^2 = \frac{1-\beta}{2} v(m) - 2v(c_1) \leq \frac{1-\beta}{2} \sup v - 2 \inf v \leq 0$. Concavity makes stationarity equivalent to own-side global optimality; ties are dominated as before, and the only remaining deviations are cross-overs. $\square$

Corollary 79 (Quantile equilibrium for symmetric electorates). *In the setting of Proposition 78, let v be symmetric about $\frac{1}{2}$ with $\rho_v \leq \frac{4}{1-\beta}$. Then the profile placing the candidates at the $\frac{1+\beta}{4}$ and $\frac{3-\beta}{4}$ quantiles,*

$$V(c_1) = \frac{1 + \beta}{4}, \quad V(c_2) = \frac{3 - \beta}{4},$$

is a Nash equilibrium. At $\beta = 0$ this is the quartile rule of Corollary 11; as $\beta \uparrow 1$ the platforms converge to the median from the $\frac{1}{4}/\frac{3}{4}$ quantiles, sweeping the entire inner quartile range.

Proof. By symmetry of v , $c_2 = 1 - c_1$, $m = \frac{1}{2}$, $V(m) = \frac{1}{2}$, and Eq. (45) holds: $2 \cdot \frac{1+\beta}{4} = \frac{1}{2} + \beta \cdot \frac{1}{2}$. By Proposition 78(b), c_1 is a global maximizer of $U_1(\cdot, c_2)$

on $[0, c_2]$. It remains to compare the equilibrium payoff with the tie and with cross-over deviations. Write $s := \inf v > 0$ and $\rho := \rho_v$; the mass-gap law gives $\frac{1-\beta}{2} = \int_{c_1}^{c_2} v \leq (\sup v) \Delta \leq \rho s \Delta$, so

$$\Delta \geq \frac{1-\beta}{2\rho s}. \quad (46)$$

Equilibrium payoff. By symmetry $u_2 = u_1$ at the profile, so $U_1 = (1-\beta)u_1$. With $K(t) = t$,

$$\begin{aligned} u_1 &= \Delta V(c_1) + \int_{c_1}^m (c_1 + c_2 - 2x) v \, dx \\ &\geq \frac{1+\beta}{4} \Delta + s \int_{c_1}^m (c_1 + c_2 - 2x) \, dx = \frac{1+\beta}{4} \Delta + \frac{s\Delta^2}{4}, \end{aligned}$$

hence

$$U_1(c_1, c_2) \geq \frac{(1-\beta^2) \Delta}{4} > 0, \quad (47)$$

which in particular dominates the tie.

Cross-over. Let $c'_1 = c_2 + t$, $t \in (0, 1 - c_2]$, and $m' = c_2 + \frac{t}{2}$. The deviator's supporters lie in $(m', 1]$ with benefit at most t , so, using $V(m') \geq V(c_2) + s\frac{t}{2}$,

$$u'_1 \leq t(1 - V(m')) \leq t \frac{1+\beta}{4} - \frac{s t^2}{2}.$$

The opponent's supporters include all of $[0, c_2]$, each with benefit exactly t , so her share satisfies $u'_2 \geq t V(c_2) = t \frac{3-\beta}{4}$. Hence

$$U_1(c'_1, c_2) \leq t \frac{1+\beta}{4} - \beta t \frac{3-\beta}{4} - \frac{s t^2}{2} = \frac{(1-\beta)^2}{4} t - \frac{s}{2} t^2 \leq \frac{(1-\beta)^4}{32s},$$

the last step by maximizing the concave quadratic over $t \geq 0$. Finally, by Eqs. (46) to (47),

$$U_1(c_1, c_2) \geq \frac{(1-\beta^2)}{4} \cdot \frac{1-\beta}{2\rho s} = \frac{(1+\beta)(1-\beta)^2}{8\rho s} \geq \frac{(1-\beta)^4}{32s} \iff \rho \leq \frac{4(1+\beta)}{(1-\beta)^2},$$

and $\rho \leq \frac{4}{1-\beta} \leq \frac{4(1+\beta)}{(1-\beta)^2}$ always. Candidate 2 is the mirror image. $\square$

Remark 80. (a) The mass-gap law in Eq. (45) is distribution-free and gives a clean empirical signature of competitiveness: the fraction of the electorate strictly between the platforms is $\frac{1-\beta}{2}$, whatever v . It was verified numerically to 10^{-11} at asymmetric densities, and the quantile equilibrium of Corollary 79 to 10^{-4} against best-response search for $v = 1 + \frac{1}{2} \cos 2\pi x$ at $\beta \in \{0.2, 0.5, 0.8\}$ (unique equilibrium found in each case). (b) The concavity threshold $\frac{4}{1-\beta}$ increases in β : competitiveness enlarges the sufficiency umbrella at $\alpha = 0$, in exact antithesis to alienation, which shrinks it (Theorem 56: $F(\alpha) \downarrow$). The two extensions are not symmetric levers on existence. (c) Multiplicity is not excluded and is real: non-uniqueness examples at $\beta = 0$ deform continuously to small $\beta > 0$ (Example 70 is the cost-side analogue).

M The joint corner: $\alpha > 0$, $\beta > 0$, non-uniform voters

Standing notation. As in Appendix I: voters $x \sim V$ on $[0, 1]$ with density v , costs $\kappa \sim K$ with density k , both under Assumption 2 (with $K(1) \in (0, 1]$ permitted). Candidate positions $c_1 \leq c_2$, $\Delta = c_2 - c_1$, $\lambda = 2 + \alpha$, benefit $B_i(x) = d_{-i}(x) - (1 + \alpha)d_i(x)$, payoffs $u_i = \int K(B_i^+) dV$ and $U_i = u_i - \beta u_{-i}$ with $\beta \in [0, 1]$. Tent edges for $c_1 < c_2$:

$$\begin{aligned} x_1^{\text{in}} &= c_1 + \frac{\Delta}{\lambda}, & x_1^{\text{out}} &= c_1 - \frac{\Delta}{\alpha}, \\ x_2^{\text{in}} &= c_2 - \frac{\Delta}{\lambda}, & x_2^{\text{out}} &= c_2 + \frac{\Delta}{\alpha} \quad (\alpha > 0), \end{aligned}$$

with the conventions $x_1^{\text{out}} = -\infty$, $x_2^{\text{out}} = +\infty$ at $\alpha = 0$. We write $\rho_v = \sup v / \inf v$ (possibly ∞), $S = \sup v$, $s = \inf v$. The master formula (Lemma 53) reads, for $c_1 < c_2$ interior,

$$\frac{\partial U_1}{\partial c_1} = (1 + \alpha)(I_1 - O_1) + \beta(I_2 + O_2), \quad (48)$$

where

$$\begin{aligned} I_1 &= \int_{c_1}^{x_1^{\text{in}}} k(B_1)v, & O_1 &= \int_{x_1^{\text{out}} \vee 0}^{c_1} k(B_1)v, \\ I_2 &= \int_{x_2^{\text{in}}}^{c_2} k(B_2)v, & O_2 &= \int_{c_2}^{x_2^{\text{out}} \wedge 1} k(B_2)v, \end{aligned}$$

the k -weighted voter masses on the *inner* and *outer* arms of the two benefit tents (at $\alpha = 0$ the outer arms carry the constant benefit Δ , so $O_1 = k(\Delta)V(c_1)$ and $O_2 = k(\Delta)(1 - V(c_2))$).

M.1 Two distribution-free laws of stationarity

The first result generalizes the mass-gap law of Proposition 78: that law is the $\alpha = 0$, uniform-cost shadow of a ratio law that is free of *both* distributions and holds at every interior stationary profile of the joint game.

Lemma 81 (Mirror first-order condition). *For $c_1 < c_2$ interior,*

$$\frac{\partial U_2}{\partial c_2} = (1 + \alpha)(O_2 - I_2) - \beta(I_1 + O_1). \quad (49)$$

Proof. Apply the reflection $\sigma(x) = 1 - x$. In the reflected game the voter density is $\hat{v}(x) = v(1 - x)$, the cost distribution is unchanged, and the profile (c_1, c_2) becomes $(\hat{c}_1, \hat{c}_2) = (1 - c_2, 1 - c_1)$ with $\hat{c}_1 < \hat{c}_2$; the change of variables $x \mapsto 1 - x$ in the payoff integrals gives $\hat{u}_1(\hat{c}_1, \hat{c}_2) = u_2(c_1, c_2)$ and $\hat{u}_2 = u_1$, hence $\hat{U}_1 = U_2$. Since $c_2 = 1 - \hat{c}_1$, $\partial U_2 / \partial c_2 = -\partial \hat{U}_1 / \partial \hat{c}_1$, and Eq. (48) applied in the reflected game gives $\partial \hat{U}_1 / \partial \hat{c}_1 = (1 + \alpha)(\hat{I}_1 - \hat{O}_1) + \beta(\hat{I}_2 + \hat{O}_2)$. Changing variables back, $\hat{I}_1 = I_2$, $\hat{O}_1 = O_2$, $\hat{I}_2 = I_1$, $\hat{O}_2 = O_1$, which is Eq. (49). $\square$

Proposition 82 (Inner–outer ratio law and asymmetry law). *Let (c_1, c_2) with $0 < c_1 < c_2 < 1$ be a stationary profile of the joint game, i.e. a profile at which both first-order conditions hold — in particular, any interior untied Nash equilibrium. Write $I := I_1 + I_2$ and $O := O_1 + O_2$ for the total k -weighted inner- and outer-arm masses. Then, for every (v, k, α, β) under Assumption 2:*

$$(1 + \alpha + \beta) I = (1 + \alpha - \beta) O, \quad (50)$$

$$(1 + \alpha - \beta) (I_1 - I_2) = (1 + \alpha + \beta) (O_1 - O_2). \quad (51)$$

Proof. Subtracting Eq. (49) from Eq. (48), both being zero,

$$0 = (1 + \alpha) [(I_1 - O_1) - (O_2 - I_2)] + \beta [(I_2 + O_2) + (I_1 + O_1)] = (1 + \alpha)(I - O) + \beta(I + O),$$

which rearranges to Eq. (50). Adding the two conditions instead,

$$0 = (1 + \alpha) [(I_1 - I_2) - (O_1 - O_2)] - \beta [(I_1 - I_2) + (O_1 - O_2)],$$

which rearranges to Eq. (51). $\square$

Corollary 83 (Special cases). *At every interior stationary profile:*

- (a) $\beta = 0$: $I = O$ — each candidate pair balances inner against outer k -mass in aggregate (the balance conditions of Lemma 53, summed).
- (b) $\alpha = 0$, $k \equiv 1$: $I = V(c_2) - V(c_1)$ and $O = 1 - I$, so Eq. (50) reads $V(c_2) - V(c_1) = \frac{1-\beta}{2}$: the mass-gap law of Proposition 78 is the $\alpha = 0$ shadow of Eq. (50).
- (c) For any $\alpha > 0$, the inner share of the total marginal k -mass is the universal constant $I/(I + O) = \frac{1+\alpha-\beta}{2(1+\alpha)}$; it decreases from $\frac{1}{2}$ at $\beta = 0$ to $\frac{\alpha}{2(1+\alpha)} > 0$ as $\beta \uparrow 1$. Competition drains the contested middle, but under alienation it can never empty it.
- (d) By Eq. (51), since $\frac{1+\alpha+\beta}{1+\alpha-\beta} > 1$, any asymmetry between the candidates' outer masses is amplified in their inner masses, with the universal gain $\frac{1+\alpha+\beta}{1+\alpha-\beta}$; at symmetric profiles both sides vanish.

Remark 84 (Verification). [numerical observation] The suite verifies Eq. (50) and the inner-share constant of Corollary 83(c) at polished stationary profiles of thirteen games. The games span $\alpha \in \{0, 0.5, 1, 2\}$ and $\beta \in \{0, \dots, 0.5\}$, with uniform, cosine, Beta(2, 2), Beta(8, 8), and asymmetric Beta(2, 5) electorates, and with uniform and quadratic ($K(t) = t^2$) costs. The cases include a dead-zone equilibrium — Beta(8, 8), $\alpha = 2$, $\beta = 0.2$, where $I/O = 7/8$ exactly — and asymmetric equilibria. Relative errors are 10^{-8} – 10^{-12} .

M.2 The joint single-peakedness threshold

For uniform costs, Theorem 56 gave the $\beta = 0$ umbrella $\rho_v \leq F(\alpha) = \min\{f(\alpha), g(\alpha)\}$ with $f(\alpha) = \frac{2\lambda}{1+\alpha}$ (concavity off the dead zone) and $g(\alpha) = \frac{\lambda}{\alpha}$ (monotone decrease inside it). The joint threshold deforms the two branches in opposite directions.

Theorem 85 (Joint single-peakedness, uniform costs). *Let $\kappa \sim U(0, 1)$, $\alpha > 0$, $\beta \in [0, 1)$, and let v satisfy Assumption 2 with*

$$\begin{aligned} \rho_v &\leq F_\beta(\alpha) := \min\{f_\beta(\alpha), g_\beta(\alpha)\}, \\ f_\beta(\alpha) &= \frac{2\lambda(1+\alpha) + \beta}{(1+\alpha)^2}, \quad g_\beta(\alpha) = \frac{(1+\alpha)\lambda}{\alpha(1+\alpha+\beta) + \beta\lambda}. \end{aligned} \quad (52)$$

Then for every opponent position $c_2 \in (0, 1]$ the own-side map $c_1 \mapsto U_1(c_1, c_2)$ on $[0, c_2)$ is concave on the no-dead-zone piece $\{c_1 : x_1^{\text{out}} \leq 0\} = [0, \frac{c_2}{1+\alpha}]$ and strictly decreasing on the dead-zone piece $(\frac{c_2}{1+\alpha}, c_2)$; hence it is single-peaked, with its peak in the no-dead-zone piece. The mirror statement holds for candidate 2 on $(c_1, 1]$.

Moreover f_β is strictly increasing and g_β strictly decreasing in β , with $f_0 = f$ and $g_0 = g$.

Proof. Fix c_2 and write derivatives in c_1 . With uniform costs Eq. (48) reads

$$U'_1 = (1+\alpha) \left[(V(x_1^{\text{in}}) - V(c_1)) - (V(c_1) - V(x_1^{\text{out}} \vee 0)) \right] + \beta \left[V(x_2^{\text{out}} \wedge 1) - V(x_2^{\text{in}}) \right]. \quad (53)$$

No-dead-zone piece ($x_1^{\text{out}} \leq 0$). Here $V(x_1^{\text{out}} \vee 0) = 0$ is constant. Differentiating Eq. (53), using $(x_1^{\text{in}})' = \frac{1+\alpha}{\lambda}$, $(x_2^{\text{in}})' = \frac{1}{\lambda}$, $(x_2^{\text{out}})' = -\frac{1}{\alpha}$, and treating the kink of $V(x_2^{\text{out}} \wedge 1)$ at $x_2^{\text{out}} = 1$ one-sidedly,

$$\begin{aligned} U''_1 &= (1+\alpha) \left[\frac{1+\alpha}{\lambda} v(x_1^{\text{in}}) - 2v(c_1) \right] \\ &\quad - \beta \left[\frac{1}{\lambda} v(x_2^{\text{in}}) + \frac{1}{\alpha} v(x_2^{\text{out}}) \mathbf{1}_{\{x_2^{\text{out}} < 1\}} \right] \quad \text{a.e.} \end{aligned}$$

Dropping the nonnegative x_2^{out} -term and pairing the two $\frac{1}{\lambda}$ -terms,

$$\begin{aligned} U''_1 &\leq \frac{(1+\alpha)^2 v(x_1^{\text{in}}) - \beta v(x_2^{\text{in}})}{\lambda} - 2(1+\alpha)v(c_1) \\ &\leq \frac{(1+\alpha)^2 S - \beta s}{\lambda} - 2(1+\alpha)s \leq 0 \end{aligned}$$

whenever $(1+\alpha)^2 \rho_v \leq 2(1+\alpha)\lambda + \beta$, i.e. $\rho_v \leq f_\beta(\alpha)$. Since U'_1 is continuous in c_1 (the integrands and limits in Eq. (53) are continuous), a continuous function with a.e. nonpositive derivative of its derivative is concave.

Dead-zone piece ($x_1^{\text{out}} > 0$). From Eq. (53), using $V(x_1^{\text{in}}) - V(c_1) \leq S\frac{\Delta}{\lambda}$, $V(c_1) - V(x_1^{\text{out}}) \geq s\frac{\Delta}{\alpha}$, and $V(x_2^{\text{out}} \wedge 1) - V(x_2^{\text{in}}) \leq S\Delta(\frac{1}{\lambda} + \frac{1}{\alpha})$,

$$U'_1 \leq (1+\alpha)\Delta\left(\frac{S}{\lambda} - \frac{s}{\alpha}\right) + \beta S\Delta\left(\frac{1}{\lambda} + \frac{1}{\alpha}\right), \quad (54)$$

which is negative iff $(1+\alpha)\lambda > \rho_v[\alpha(1+\alpha+\beta) + \beta\lambda]$, i.e. $\rho_v < g_\beta(\alpha)$. At $\rho_v = g_\beta(\alpha)$ equality in Eq. (54) would force $v \equiv S$ on (c_1, x_1^{in}) and $v \equiv s$ on (x_1^{out}, c_1) simultaneously, contradicting continuity of v at c_1 unless $S = s$, in

which case $\rho_v = 1 < g_\beta$. Hence $U'_1 < 0$ strictly on the dead-zone piece whenever $\rho_v \leq g_\beta(\alpha)$.

A function that is concave on $[0, \frac{c_2}{1+\alpha}]$, strictly decreasing on $(\frac{c_2}{1+\alpha}, c_2)$, and continuous, is single-peaked with its maximum in the first piece. The mirror statement follows from Lemma 81 (equivalently, reflection). Monotonicity of f_β in β is immediate; for g_β , the denominator is strictly increasing in β since $\alpha + \lambda > 0$. Finally $f_0 = \frac{2\lambda}{1+\alpha}$ and $g_0 = \frac{\lambda}{\alpha}$. $\square$

Remark 86 (Competition rescues concavity but taxes the dead zone). The joint threshold $F_\beta(\alpha)$ is neither a β -dependent rescaling of the $\beta = 0$ threshold $F(\alpha)$ nor moved by β in a single direction. On the no-dead-zone piece the competitiveness term $-\beta u_2$ contributes the *stabilizing* second derivative $-\beta[\frac{v(x_2^{\text{in}})}{\lambda} + \frac{v(x_2^{\text{out}})}{\alpha} \mathbf{1}]$ (hurting the opponent is easiest exactly where the deviator’s own tent is thinnest), so f_β *grows* with β : competition partly rescues the umbrella that alienation shrinks. Inside the dead zone the same term is destabilizing — moving toward the opponent hurts her even when it no longer helps oneself — so g_β *shrinks*. Consequently for small α (where f binds) the umbrella widens with β , while for large α (where g binds) it narrows further; the crossover alienation level solves $f_\beta = g_\beta$ and moves left as β grows.

At $\alpha = 0$ the exact computation of Proposition 78 gives the threshold $\frac{4}{1-\beta}$, strictly better than $\lim_{\alpha \downarrow 0} f_\beta = 4 + \beta$: there the inner tent edges x_1^{in} and x_2^{in} coincide at the midpoint, so the stabilizing term cancels against the destabilizing one *pointwise* rather than through the worst-case pair (S, s) . For $\alpha > 0$ the two edges are distinct and the worst case is achievable; see the sharpness evidence below.

Remark 87 (Sharpness). [numerical observation] A numerical search over continuous piecewise-linear “mesa” densities of ratio ρ with the high plateau covering x_1^{in} and verifies, at $(\alpha, \beta) \in \{(1, .5), (.5, .8), (2, .3)\}$, that U''_1 changes sign as ρ crosses $f_\beta(\alpha)$ ($\pm 7\%$). Appendix M.4 shows g_β is essentially attained as well (Remark 93). The constant f_β is therefore the exact worst-case concavity threshold within this proof scheme, though densities realizing the worst case are non-generic.

Remark 88 (General k : the obstruction). [open] For non-increasing $k \in C^1$ the threshold $\rho_v \rho_k \leq 2\lambda/(1 + \alpha)$ of Proposition 59 does not deform as cleanly. Differentiating the β -term of Eq. (48) in c_1 produces, besides stabilizing boundary terms proportional to $-\beta k(0^+)$, the bulk term $-\beta \int_{x_2^{\text{in}}}^{x_2^{\text{out}} \wedge 1} k'(B_2) v$, which is *nonnegative* (destabilizing) and is controlled pointwise by $|k'|$ on the opponent’s tent, not by the ratio ρ_k : no bound in terms of the pair (ρ_v, ρ_k) alone can absorb it, because $|k'|$ can be made arbitrarily large on a set where K barely moves. A clean joint threshold for general non-increasing k therefore requires an additional hypothesis (e.g. a bound on $\sup |k'|/\inf k$); we record the failure rather than a lemma we cannot make tight. For uniform costs $k' \equiv 0$ and Theorem 85 is complete.

M.3 Stationary profiles and a continuation theorem

Proposition 89 (Symmetric stationary profiles exist for every $\beta < 1$).

Let $\kappa \sim U(0, 1)$, $\alpha > 0$, $\beta \in [0, 1)$, and let v be symmetric about $\frac{1}{2}$ and satisfy Assumption 2. Define, for $c_1 \in (0, \frac{1}{2})$ and $c_2 = 1 - c_1$,

$$h_\beta(c_1) := \left. \frac{\partial U_1}{\partial c_1} \right|_{(c_1, 1-c_1)} = (1+\alpha+\beta)V(x_1^{\text{in}}) + (1+\alpha-\beta)V(x_1^{\text{out}} \vee 0) - 2(1+\alpha)V(c_1). \quad (55)$$

Then h_β is continuous, $h_\beta(0^+) = (1+\alpha+\beta)V(\frac{1}{\lambda}) > 0$ and $h_\beta < 0$ on a left neighborhood of $\frac{1}{2}$; hence a symmetric stationary profile exists. If moreover $\rho_v \leq g_\beta(\alpha)$, every symmetric stationary profile lies on the no-dead-zone branch $c_1 \in (0, \frac{1}{\lambda})$.

Proof. For symmetric v and symmetric profiles, $x_2^{\text{in}} = 1 - x_1^{\text{in}}$ and $x_2^{\text{out}} = 1 - x_1^{\text{out}}$, so $V(x_2^{\text{out}} \wedge 1) - V(x_2^{\text{in}}) = V(x_1^{\text{in}}) - V(x_1^{\text{out}} \vee 0)$ and Eq. (53) becomes Eq. (55). As $c_1 \downarrow 0$: $\Delta \rightarrow 1$, $x_1^{\text{in}} \rightarrow \frac{1}{\lambda}$, $x_1^{\text{out}} \rightarrow -\frac{1}{\alpha} < 0$, $V(c_1) \rightarrow 0$, giving the stated limit. As $c_1 \uparrow \frac{1}{2}$: $\Delta = 1 - 2c_1 \downarrow 0$, $x_1^{\text{out}} > 0$, and expanding V around $\frac{1}{2}$ (where v is continuous and positive),

$$\begin{aligned} h_\beta(c_1) &= v(\tfrac{1}{2}) \Delta \left[(1+\alpha) \left(\tfrac{1}{\lambda} - \tfrac{1}{\alpha} \right) + \beta \left(\tfrac{1}{\lambda} + \tfrac{1}{\alpha} \right) \right] + o(\Delta) \\ &= v(\tfrac{1}{2}) \Delta \frac{2(1+\alpha)}{\alpha\lambda} (\beta - 1) + o(\Delta) < 0 \end{aligned}$$

for $\beta < 1$ and Δ small. The intermediate value theorem gives a root. On $c_1 \in [\frac{1}{\lambda}, \frac{1}{2})$ we have $x_1^{\text{out}} \geq 0$ and $x_1^{\text{out}} \vee 0 = x_1^{\text{out}}$, and the dead-zone estimate (54) (with its strictness argument) shows $h_\beta < 0$ there whenever $\rho_v \leq g_\beta(\alpha)$; roots are therefore confined to $c_1 \in (0, \frac{1}{\lambda})$. $\square$

Theorem 90 (Continuation of the symmetric equilibrium into $\beta > 0$).

Let $\kappa \sim U(0, 1)$, $\alpha > 0$, and let v be symmetric about $\frac{1}{2}$, satisfy Assumption 2, and obey the strict $\beta = 0$ umbrella $\rho_v < \min\{f(\alpha), g(\alpha)\}$. Assume the $\beta = 0$ symmetric stationarity equation $h_0(c_1) = 0$ has a unique root c_1^0 . Then there is $\bar{\beta} > 0$ such that for every $\beta \in [0, \bar{\beta})$ the game has a symmetric Nash equilibrium $(c_1^\beta, 1 - c_1^\beta)$, and $c_1^\beta \rightarrow c_1^0$ as $\beta \downarrow 0$.

Proof. Step 1 (roots, single-peakedness, for small β). Since g_β is continuous and strictly decreasing with $g_0 = g(\alpha)$, the condition $\rho_v < g(\alpha)$ gives $\beta_g := \frac{(1+\alpha)(\lambda - \rho_v \alpha)}{\rho_v(\alpha + \lambda)} > 0$ with $\rho_v < g_\beta(\alpha)$ for all $\beta < \beta_g$; and $f_\beta \geq f_0 > \rho_v$ for all β . Hence for $\beta < \beta_g$ Theorem 85 applies: the own side is single-peaked, and by Proposition 89 a symmetric stationary profile $(c_1^\beta, 1 - c_1^\beta)$ exists, with $c_1^\beta \in (0, \frac{1}{\lambda})$. By single-peakedness the stationary point is the *global own-side maximum* of $U_1(\cdot, 1 - c_1^\beta)$ on $[0, 1 - c_1^\beta)$, and by symmetry likewise for candidate 2 on her own side. It remains to control far-side deviations.

Step 2 (the $\beta = 0$ far-side gap). Let $c_2^0 = 1 - c_1^0$ and $P(0) = u_1(c_1^0, c_2^0) > 0$. For $t \in [0, 1 - c_2^0]$, reflection and the strict monotonicity of $u_1(a, \cdot)$ (its derivative $I_1 + O_1$ is strictly positive for $a < o$, since $k, v > 0$ and $\Delta > 0$) give

$$u_1(c_2^0 + t, c_2^0) = u_1(1 - c_2^0 - t, 1 - c_2^0) < u_1(1 - c_2^0 - t, c_2^0) \leq P(0) \quad (t > 0),$$

using $1 - c_2^0 < c_2^0$ (as $c_1^0 < \frac{1}{2}$) for the strict middle inequality, while the deviation $t = 0$ (the tie) yields payoff $0 < P(0)$. The far-side supremum $\text{far}(0) := \sup_{t \in [0, 1 - c_2^0]} u_1(c_2^0 + t, c_2^0)$ is attained by continuity on a compact interval; at any maximizer the displayed strict inequality holds, so $\delta_0 := P(0) - \text{far}(0) > 0$.

Step 3 (continuity in β). On the compact family of symmetric profiles, $h_\beta \rightarrow h_0$ uniformly as $\beta \rightarrow 0$ (indeed $|h_\beta - h_0| \leq \beta$ pointwise, since $0 \leq V(x_1^{\text{in}}) - V(x_1^{\text{out}} \vee 0) \leq 1$). Moreover all roots of h_β , $\beta \leq \hat{\beta} < \min(\beta_g, 1)$, lie in a fixed compact $[\varepsilon_0, \frac{1}{2} - \varepsilon_0]$: near 0, $h_\beta \geq h_0 > 0$ (the β -term in Eq. (55) is nonnegative); near $\frac{1}{2}$, the expansion in Proposition 89 is negative uniformly for $\beta \leq \hat{\beta}$. By uniqueness of c_1^0 , every sequence of roots $c_1^{\beta_n}$ with $\beta_n \rightarrow 0$ converges to c_1^0 ; hence $c_1^\beta \rightarrow c_1^0$ (for any measurable selection of roots). Finally, uniformly over positions, $|U_1^\beta - U_1^0| = \beta u_2 \leq \beta$, and u_1 is jointly continuous; therefore $P(\beta) = (1 - \beta) u_1(c_1^\beta, 1 - c_1^\beta) \rightarrow P(0)$ and

$$\text{far}(\beta) := \sup_{c' \in [1 - c_1^\beta, 1]} U_1^\beta(c', 1 - c_1^\beta) \leq \sup_{c' \in [1 - c_1^\beta, 1]} u_1(c', 1 - c_1^\beta) \rightarrow \text{far}(0),$$

where the convergence of the right-hand side uses joint continuity of u_1 and $c_1^\beta \rightarrow c_1^0$ (a parametric maximum over a continuously moving compact interval). Choose $\bar{\beta} \leq \min(\beta_g, \hat{\beta})$ so small that $P(\beta) - \text{far}(\beta) \geq \delta_0/2 > 0$ for all $\beta < \bar{\beta}$. Then no far-side deviation (including the tie) is profitable, no own-side deviation is profitable by Step 1, and by symmetry the same holds for candidate 2: $(c_1^\beta, 1 - c_1^\beta)$ is a Nash equilibrium. $\square$

Remark 91 (The moderate- β regime, numerically). [numerical observation] The continuation theorem is qualitative; a numerical continuation shows the phenomenon is not confined to small β . Verified global equilibria (maximal deviation gain 0 at resolution 10^{-7}) include: cosine voters $\rho_v = 3$ at $(\alpha, \beta) = (0.2, 0.2)$ inside the umbrella $F_\beta = 3.67$; $\rho_v = 1.86$ at $(0.5, 0.4)$ inside $F_\beta = 1.92$; the same electorate at $(0.5, 0.8)$, outside the umbrella ($g_\beta = 1.19$); and Beta(2, 2) (unbounded ρ_v) at $(0.5, 0.5)$. The umbrella is sufficient, visibly not necessary.

M.4 Dead zones and competitiveness

Proposition 92 (The dead-zone threshold moves with β). *Let $\kappa \sim U(0, 1)$ and $\alpha > 0$.*

- (a) (Necessity.) *If an interior profile $c_1 < c_2$ satisfies candidate 1's first-order condition $U_1' = 0$ and candidate 1 has a dead zone ($x_1^{\text{out}} > 0$), then $\rho_v \geq g_\beta(\alpha)$. No symmetry of v or of the profile is assumed.*
- (b) (Sufficiency, symmetric v .) *If v is symmetric and*

$$(1 + \alpha + \beta) V\left(\frac{2(1 + \alpha)}{\lambda^2}\right) > 2(1 + \alpha) V\left(\frac{1}{\lambda}\right), \quad (56)$$

then a symmetric stationary profile with dead zones exists.

- (c) (Uniform voters are immune at every β .) If $v \equiv 1$ then no profile with $x_1^{\text{out}} > 0$ satisfies candidate 1's first-order condition, for any $\beta \in [0, 1]$ (recovering Proposition 61(b)).

Since g_β strictly decreases in β and the left side of Eq. (56) strictly increases in β , competition makes alienation dead zones strictly easier to sustain at stationarity: the $\beta = 0$ threshold $\rho_v \geq \lambda/\alpha$ of Proposition 61(a) drops to $\rho_v \geq g_\beta(\alpha)$.

Proof. (a) is the contrapositive of the dead-zone estimate in the proof of Theorem 85: if $x_1^{\text{out}} > 0$ and $\rho_v < g_\beta$ then Eq. (54) gives $U'_1 < 0$, and at $\rho_v = g_\beta$ the strictness argument applies; so stationarity forces $\rho_v \geq g_\beta$.

(b) On the symmetric family, the dead-zone branch is $c_1 \in (\frac{1}{\lambda}, \frac{1}{2})$, on which h_β of Eq. (55) is continuous. At the junction $c_1 = \frac{1}{\lambda}$ we have $x_1^{\text{out}} = 0$, $\Delta = \frac{\alpha}{\lambda}$ and $x_1^{\text{in}} = \frac{1}{\lambda} + \frac{\alpha}{\lambda^2} = \frac{2(1+\alpha)}{\lambda^2}$, so $h_\beta(\frac{1}{\lambda}) = (1+\alpha+\beta)V(\frac{2(1+\alpha)}{\lambda^2}) - 2(1+\alpha)V(\frac{1}{\lambda}) > 0$ by Eq. (56), while $h_\beta < 0$ near $\frac{1}{2}$ by Proposition 89. The intermediate value theorem gives a root with $c_1 > \frac{1}{\lambda}$, i.e. $x_1^{\text{out}} > 0$; by symmetry both candidates then have dead zones.

(c) With $v \equiv 1$ and $x_1^{\text{out}} > 0$, Eq. (54) with $S = s = 1$ gives $U'_1 \leq (1 + \alpha)\Delta(\frac{1}{\lambda} - \frac{1}{\alpha}) + \beta\Delta(\frac{1}{\lambda} + \frac{1}{\alpha}) = \Delta \frac{2(1+\alpha)}{\alpha\lambda}(\beta - 1) < 0$ for $\beta < 1$. $\square$

Remark 93 (The bound g_β is essentially attained). [numerical observation] A numerical check takes $\alpha = 2$ and continuous piecewise-linear densities with a low plateau on the deviator's outer arm (x_1^{out}, c_1) (ramp width $3 \cdot 10^{-3}$), and bisects for the minimal density ratio at which candidate 1's first-order condition admits a dead-zone root. The minima are 2.0152, 1.5447, 1.2524, 1.0531 at $\beta = 0, 0.3, 0.6, 0.9$, against $g_\beta(2) = \frac{2}{1+\beta} = 2.0000, 1.5385, 1.2500, 1.0526$: the proved necessity bound is sharp up to the ramp width. (These witnesses are asymmetric densities, as part (a) permits; symmetric witnesses are constrained by the mirror arm and stay slightly further from the bound.)

N Existence at $\alpha = 0$ for every $\beta < 1$

Theorem 75 shows that hollow electorates destroy pure equilibria at $\alpha = 1$ for β near 1. The next theorem shows that at $\alpha = 0$ they do not: for uniform costs and a symmetric electorate satisfying a local ratio condition — which every convex (U-shaped) symmetric density satisfies once mild, and the arcsine density satisfies for all β — the quantile equilibrium of Proposition 78 exists for every $\beta \in [0, 1]$. This makes the median-selection statement Theorem 72(a) non-vacuous precisely for the electorates that make it interesting.

Theorem 94 (Quantile equilibria at $\alpha = 0$, all β). *Let $\alpha = 0$, $\kappa \sim U(0, 1)$, and let v be symmetric about $\frac{1}{2}$ and satisfy Assumption 2, with $s_0 := \inf_{(0,1)} v > 0$. Define the quantile profile $c_2^* = V^{-1}(\frac{3-\beta}{4})$, $c_1^* = 1 - c_2^*$, and $\bar{v} := \sup_{[c_1^*, c_2^*]} v$.*

If

$$\sup_{c_1 \in (0, c_2^*)} \frac{v\left(\frac{c_1 + c_2^*}{2}\right)}{v(c_1)} \leq \frac{4}{1 - \beta}, \quad (\text{E1-a})$$

$$\frac{\bar{v}}{s_0} \leq \frac{4(1 + \beta)}{(1 - \beta)^2}, \quad (\text{E1-b})$$

then (c_1^*, c_2^*) is a Nash equilibrium of the β -game.

Proof. By symmetry of v and of the profile it suffices to rule out deviations of candidate 1; candidate 2's problem is the mirror image.

Stationarity. At $\alpha = 0$, Eq. (53) becomes, with $m = \frac{c_1 + c_2}{2}$,

$$h(c_1) := U_1'(c_1, c_2^*) = V(m) + \beta(1 - V(m)) - 2V(c_1), \quad (57)$$

and $V(c_1^*) = \frac{1+\beta}{4}$, $V(m^*) = V(\frac{1}{2}) = \frac{1}{2}$ give $h(c_1^*) = \frac{1}{2} + \frac{\beta}{2} - \frac{1+\beta}{2} = 0$.

Own side. $h'(c_1) = \frac{1-\beta}{2}v(m) - 2v(c_1) \leq 0$ for all $c_1 \in (0, c_2^*)$ by Eq. (E1-a), so h is non-increasing; with $h(c_1^*) = 0$ this gives $h \geq 0$ on $(0, c_1^*]$ and $h \leq 0$ on $[c_1^*, c_2^*)$: $U_1(\cdot, c_2^*)$ is single-peaked with global own-side maximum at c_1^* . Its value is $P = (1 - \beta)u^*$, where $u^* = u_1(c_1^*, c_2^*) = u_2(c_1^*, c_2^*)$ by symmetry, and

$$u^* \geq \Delta V(c_1^*) + \int_{c_1^*}^{1/2} (1-2x) v(x) dx \geq \frac{(1+\beta)\Delta}{4}, \quad \text{so} \quad P \geq \frac{(1-\beta^2)\Delta}{4}, \quad (58)$$

using $B_1 = \Delta$ on $[0, c_1^*]$ and $B_1 = 1 - 2x$ on $[c_1^*, \frac{1}{2}]$ (here $c_1^* + c_2^* = 1$), and $K(t) = t$.

Far side. Let $c' = c_2^* + t$, $t \in (0, 1 - c_2^*]$, and write $m' = c_2^* + \frac{t}{2}$. The deviator's supporters lie in $(m', 1]$ with benefit at most t , so, using $v \geq s_0$ on $[c_2^*, m']$,

$$u_1(c', c_2^*) \leq t(1 - V(m')) \leq t\left(\frac{1+\beta}{4} - \frac{s_0 t}{2}\right).$$

Every voter in $[0, c_2^*]$ obtains benefit exactly t against the deviator (at $\alpha = 0$, $B_2 = d_1 - d_2 = t$ there), so $u_2(c', c_2^*) \geq tV(c_2^*) = t\frac{3-\beta}{4}$. Hence

$$U_1(c', c_2^*) \leq t\frac{1+\beta}{4} - \frac{s_0 t^2}{2} - \beta t\frac{3-\beta}{4} = \frac{(1-\beta)^2}{4}t - \frac{s_0}{2}t^2 \leq \frac{(1-\beta)^4}{32s_0},$$

maximizing the concave quadratic over $t \geq 0$. The tie $c' = c_2^*$ yields payoff $0 < P$.

Closing the gap. The mass-gap law $V(c_2^*) - V(c_1^*) = \frac{1-\beta}{2}$ and $v \leq \bar{v}$ on $[c_1^*, c_2^*]$ give $\Delta \geq \frac{1-\beta}{2\bar{v}}$, so by Eq. (58)

$$P \geq \frac{(1-\beta^2)(1-\beta)}{8\bar{v}} \geq \frac{(1-\beta)^4}{32s_0} \iff \frac{\bar{v}}{s_0} \leq \frac{4(1+\beta)}{(1-\beta)^2},$$

which is Eq. (E1-b). Hence no far-side deviation is profitable either, and (c_1^*, c_2^*) is an equilibrium. $\square$

Corollary 95 (Convex symmetric electorates; the arcsine for all β). *Let $\alpha = 0$, $\kappa \sim U(0, 1)$, and let v be symmetric, convex and satisfy Assumption 2. Set $\eta(\beta) := v(c_2^*)/v(\frac{1}{2})$. If $\eta(\beta) \leq \frac{4}{1-\beta}$, then the quantile profile is a Nash equilibrium. In particular, for the arcsine electorate $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$,*

$$c_2^* = \sin^2\left(\frac{\pi(3-\beta)}{8}\right), \quad \eta(\beta) = \frac{1}{\sin\left(\frac{\pi(3-\beta)}{4}\right)} \in (1, \sqrt{2}],$$

so the condition holds and a pure equilibrium exists for every $\beta \in [0, 1)$.

Proof. For convex symmetric v : v is non-increasing on $[0, \frac{1}{2}]$ and non-decreasing on $[\frac{1}{2}, 1]$, so $s_0 = v(\frac{1}{2})$ and, on $[c_1^*, c_2^*]$, v attains its maximum at the endpoints: $\bar{v} = v(c_1^*) = v(c_2^*)$. For Eq. (E1-a): with $m = \frac{c_1^* + c_2^*}{2}$ the midpoint of $[c_1, c_2^*]$, convexity gives $v(m) \leq \max\{v(c_1), v(c_2^*)\}$, so the ratio is at most $\max\{1, v(c_2^*)/v(c_1)\} \leq \max\{1, \eta(\beta)\} = \eta(\beta)$ (as $v(c_1) \geq s_0$ and $\eta \geq 1$), and $\eta \leq \frac{4}{1-\beta}$ by hypothesis. For Eq. (E1-b): $\bar{v}/s_0 = \eta \leq \frac{4}{1-\beta} \leq \frac{4(1+\beta)}{(1-\beta)^2}$. For the arcsine, $V(x) = \frac{2}{\pi} \arcsin \sqrt{x}$, so $V(c_2^*) = \frac{3-\beta}{4}$ gives the stated c_2^* , and $v(c_2^*)/v(\frac{1}{2}) = \frac{1/2}{\sqrt{c_2^*(1-c_2^*)}} = \frac{1}{\sin(\frac{\pi(3-\beta)}{4})}$, using $c_2^*(1-c_2^*) = \frac{1}{4} \sin^2(\frac{\pi(3-\beta)}{4})$. As β runs over $[0, 1)$ the argument runs over $(\frac{\pi}{2}, \frac{3\pi}{4}]$, so $\eta \in (1, \sqrt{2}]$, which is below $\frac{4}{1-\beta} \geq 4$ for every β . $\square$

Remark 96 (Contrast with $\alpha > 0$; scope; verification). (i) The contrast with Theorem 75 is now a theorem: the same hollow electorate $\text{Beta}(\frac{1}{2}, \frac{1}{2})$ that has *no* pure equilibrium at $\alpha = 1$ for $\beta \geq 0.9168$ (Theorem 97) has a pure equilibrium at $\alpha = 0$ for *every* $\beta < 1$. Hollowness is lethal only in combination with alienation. Consequently Theorem 72(a) is a genuine selection statement on this class: the quantile equilibria converge to the median profile as $\beta \uparrow 1$ (here explicitly: $c_2^* \downarrow \frac{1}{2}$). (ii) For general (v, k) , the question of existence at $\alpha = 0$ for every $\beta < 1$ contains, at $\beta = 0$, the open problem of pure-equilibrium existence for arbitrary (v, k) , and is not resolved here; Conjecture 100 records the conjecture. A numerical check verifies the quantile equilibrium as a global best response for $\text{Beta}(\frac{1}{2}, \frac{1}{2})$ and $\text{Beta}(0.8, 0.8)$ at $\beta \in \{0, 0.3, 0.6, 0.9, 0.97\}$ (deviation gains ≤ 0 , first-order residuals $\sim 10^{-11}$), and finds verified equilibria for the *non-uniform* costs $K = t(2-t)/2$ and $K = t^2$ with the arcsine electorate at β up to 0.99 (e.g. $K = t^2$: (0.075, 0.925) at $\beta = 0$ collapsing to (0.492, 0.508) at $\beta = 0.99$).

O The certified non-existence interval

Corollary 76 gave non-existence for $\beta > 0.98615$ ($\alpha = 1$, $\kappa \sim U(0, 1)$, $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$), while numerically equilibria persist to $\beta = 0.90$ and are gone by $\beta = 0.91$ (Remark 77). The following closes most of that gap.

Theorem 97 (Certified interval). *[computer-assisted] Let $\alpha = 1$, $\kappa \sim U(0, 1)$, $v = \text{Beta}(\frac{1}{2}, \frac{1}{2})$. The β -game has no pure Nash equilibrium for any*

$$\beta \geq 0.9168.$$

A pure equilibrium is found numerically at $\beta = 0.90$ (Remark 77); that lower bound is a numerical observation, not part of the certificate, and places any existence threshold β_c in $(0.90, 0.9168]$.

Proof. The proof is a finite, exactly evaluated certificate, reproduced by the verification suite as the three E2 claims. Fix β and define the regret $R(c_1, c_2) = \max_i [\sup_{c'} U_i(c', c_{-i}) - U_i(c_1, c_2)]$; a pure equilibrium exists iff $\min_{[0,1]^2} R = 0$, so $\min R > 0$ certifies non-existence. All payoffs $u(a, o) = \int K(B^+) dV$ are evaluated exactly through the regularized incomplete beta function. From the proof of Lemma 73, u is 2-Lipschitz in the own and 1-Lipschitz in the opponent position (for $\alpha = 1$, $L_K = 1$); hence $U_1(a, o) = u(a, o) - \beta u(o, a)$ is (3, 3)-Lipschitz, the best-response value $s(o) = \sup_a U_1(a, o)$ is 3-Lipschitz, and the two one-sided regrets are (3, 6)- and (6, 3)-Lipschitz, so the regret varies by at most 4.5 times the side of a square cell, measured from its center.

On the $N \times N$ grid, $N = 2401$, $h = \frac{1}{2400}$, the matrix $U_1 = U - \beta U^\top$ and the column maxima give a *conservative* grid regret (the grid best-response value underestimates the true supremum, so grid regret underestimates true regret at grid profiles). Cells whose grid regret exceeds $\tau + 4.5h$ certify $R \geq \tau$ on the whole cell. The remaining (flagged) cells are covered by a refined lattice of step $h/16$ (step $h/64$ for the residual worst cells), on which the regret is bounded below by the exact *fixed-attacker* margins $\max\{U_1(a_j^{\text{col}}, c_2) - U_1(c_1, c_2), U_1(a_i^{\text{row}}, c_1) - U_2(c_1, c_2)\}$ with a_j^{col} , a_i^{row} the grid best responses in the cell’s column and row; these bounds have the same Lipschitz structure, so the refined slack is $4.5h/16$ resp. $4.5h/64$. This yields a verified margin $\varepsilon'(\beta) > 0$ with $R \geq \varepsilon'(\beta)$ on all of $[0, 1]^2$.

Finally $|R_\beta - R_{\beta'}| \leq 2u_{\text{sup}}|\beta - \beta'|$ with $u_{\text{sup}} = 0.2839$ an upper bound on $\sup u$ (grid maximum plus Lipschitz correction $1.5h$), so each certified β covers the interval $(\beta - \varepsilon'(\beta)/(2u_{\text{sup}}), \beta]$. A 65-step walk descending from $\beta = 0.9862$ with step $0.85\varepsilon'/(2u_{\text{sup}})$ produced overlapping certified intervals covering $[0.91671, 0.9862]$; margins ranged from $6.1 \cdot 10^{-3}$ at $\beta = 0.9862$ down to $4.6 \cdot 10^{-5}$ at $\beta = 0.9170$. Together with Corollary 76 ($\beta > 0.98615$) and Theorem 75 ($\beta = 1$), non-existence holds for all $\beta \geq 0.9168 > 0.91671$. $\square$

Remark 98 (Where the certificate stops). [numerical observation] The exact grid regret at $N = 2401$ falls from $6.2 \cdot 10^{-3}$ at $\beta = 0.9862$ through $1.9 \cdot 10^{-4}$ at $\beta = 0.92$ to $6.9 \cdot 10^{-6}$ at $\beta = 0.91$ and (numerically) 0 at $\beta = 0.90$: the true threshold sits near 0.908 (a pure equilibrium is still found at $\beta = 0.908$ and none at $\beta = 0.909$), and the certificate stops where the dying regret meets the finest refinement slack $4.5h/64 \approx 2.9 \cdot 10^{-5}$ — less than 0.01 above it. The same machinery applies verbatim to Beta(0.8, 0.8) (where Corollary 76 gives 0.99521); we have not run that walk.

P Independent verification of the Sections 3–5 results

The numerical content of the seven results listed in Table 5—drawn from Sections 3 to 5, the scale-invariance proposition among them—is recomputed from the model’s definitions by an independent implementation: 14 checks, all passing.

Table 5. Re-verification of results from Sections 3 to 5 by an independent implementation of the model. “exact” means closed forms (uniform-voter payoffs, incomplete-beta integrals) reproduced to the stated precision.

Result	Verified statement	Outcome
Proposition 32	zK : equilibria fixed, turnout $\times z$ ($\alpha \in \{0, 1\}$, $z \in \{1, \frac{1}{2}, \frac{1}{10}\}$)	exact
Theorems 24 and 30	$\Delta^*(\alpha)$ solves $\int_b^\Delta r = \log \frac{\lambda}{2}$; unique; increasing in α ; doubly-uniform value $\frac{2+\alpha}{4+\alpha}$	$\leq 2 \cdot 10^{-5}$
Theorem 27	$K = t^2 \succeq_{\text{thr}} K = t$ polarizes at $\alpha \in \{0, 1, 2\}$	exact
Proposition 29	T' formula (10^{-10}); peak $\bar{\Delta} = \frac{\lambda^2}{\alpha^2 + 6\alpha + 4}$, $\bar{\Delta}(1) = \frac{9}{11}$; $\Delta^* < \bar{\Delta}$; $T^* = \frac{2(3+\alpha)}{(4+\alpha)^2}$	exact
Proposition 28	isomorphism $u_{(V_\sigma, K)} = u_{(V_0, K_\sigma)}$ pointwise ($5 \cdot 10^{-9}$); power-law costs exactly proportional; exponential cost super-proportional	exact
Theorem 48	$w_1/\delta^4 \rightarrow C(\alpha)v''(\frac{1}{2})$ with $C(1) = \frac{13}{27}$; Beta(2, 2) and arcsine match the prediction at $\delta = 0.02$ (values below); uniform: 0 to machine precision	confirmed
Proposition 49	certified in stronger form by Theorems 75 and 97	certified

Quartic check values. At $\delta = 0.02$, $\alpha = 1$: Beta(2, 2) gives $w_1 = -9.2444 \cdot 10^{-7}$ against the prediction $C(1)v''(\frac{1}{2})\delta^4 = -9.2444 \cdot 10^{-7}$; the arcsine gives $+1.9666 \cdot 10^{-7}$ against $+1.9617 \cdot 10^{-7}$ (the residual is the $o(\delta^4)$ term and shrinks at $\delta = 0.01, 0.005$).

Security values. Under the exact quadrature, $\max_c \min_{c'} w(c, c') = -8.2244 \cdot 10^{-3}$ (attained at $c = 0.8292$; grid $n = 1201$) for the arcsine electorate and, for Beta(0.8, 0.8), $-3.0238 \cdot 10^{-3}$ ($n = 2401$, from Theorem 75); the certified Lipschitz bounds of Theorem 75 ($\leq -6.97 \cdot 10^{-3}$, $\leq -2.40 \cdot 10^{-3}$) are unchanged. For Beta(2, 2) the security value attains 0 at the median and only there; for uniform voters it is 0 (to 10^{-16}) on the equilibrium band, as it must be.

The security-value curves. The three curves $c \mapsto \min_{c'} w(c, c')$ of Fig. 11 ($\alpha = 1$, $\kappa \sim U(0, 1)$; Beta(2, 2), uniform and arcsine voters) are computed from the exact 1201×1201 payoff matrices and plotted at every fifth grid point (`figdata.py` in the verification suite).

Remark 99 (Why the security value must be computed exactly). [numerical observation] Naive quadrature of the arcsine density’s endpoint singularity produces security values that are wrong by orders of magnitude, in two distinct regimes. A midpoint rule with 10^3 cells misweights the mass near the boundary by $\sim 10\%$ (e.g. $w(0.999, 0.5)$: exact -0.0284 , midpoint -0.0314) — visible but modest. Any rule that samples v at $x \in \{0, 1\}$ (trapezoid, Simpson, or a grid containing the endpoints), where the density is unbounded, is arbitrarily wrong: the same $w(0.999, 0.5)$ evaluates to $+79.2$ under a 10^3 -cell trapezoid rule. Security values

of order -0.387 arise this way: some near-boundary attacker c' acquires spurious mass and appears to beat every position by $O(0.1-1)$. The curves of Fig. 11 use the exact incomplete-beta integrals and are free of both effects.

Q Conjecture and open items

Conjecture 100 (Existence at $\alpha = 0$). Under Assumption 2 with $\alpha = 0$, the β -game has a pure Nash equilibrium for every $\beta \in [0, 1)$ and every (v, k) .

Evidence for. Theorem 94 and Corollary 95 (uniform costs, symmetric v under Eqs. (E1-a) to (E1-b), including all hollow convex electorates for all β); Proposition 78's corollary ($\rho_v \leq \frac{4}{1-\beta}$); the margin game $\beta = 1$ has the median equilibrium for arbitrary (v, k) (Theorem 45), so the conjectured family has the right limit; verified equilibria for non-uniform costs and hollow v at β up to 0.99 (Remark 96); and the ratio law fixes the candidate profile's quantiles in a way that degenerates gracefully as $\beta \uparrow 1$. *Evidence against, and why it stays a conjecture.* At $\beta = 0$ the statement contains the still-open existence problem for arbitrary (v, k) ; the concavity route fails for $\rho_v = \infty$ with general k (the local condition (E1-a) has no analogue once k varies, cf. Remark 88); and the $\alpha > 0$ counterexamples (Theorems 75 and 97) show the effect that must fail at $\alpha = 0$ is genuinely present nearby in parameter space.

Two technical provisos. The finiteness proviso of Lemma 53 and the C^1 approximation step in Proposition 59 remain open exactly as stated; neither affects any result in the main text.

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