

A FINITE E-GROUP OF NILPOTENCY CLASS THREE

XINAN DAI, WENHAO DENG, YINGDONG SHI, TAILIN WU, AND YUCHEN YANG

ABSTRACT. A group is an E-group if every element commutes with each of its endomorphic images. Caranti asked whether a finite E-group can have nilpotency class three. We prove that the 3-group of order 3^{84} introduced by Abdollahi, Faghihi, and Mohammadi Hassanabadi, and later shown by Abdollahi, Faghihi, Linton, and O'Brien to have the corresponding automorphism property, is an E-group. Let P denote this group and put $V = P/\Phi(P) \cong \mathbb{F}_3^9$. The nine power relations of P determine a linear map

$$q : V \longrightarrow \Lambda^2 V.$$

We prove that q has no nonzero proper subspace U satisfying $q(U) \subseteq \Lambda^2 U$. Since the image induced by any endomorphism of P on V has precisely this closure property, every endomorphism acts on V either invertibly or trivially. The invertible case is the known A-group case. In the trivial case the image first lies in $\Phi(P) = P'$, and the power relations then force it into $\Omega_1(P') = Z(P)$. Thus every element commutes with every endomorphic image. The tensor rigidity is reduced to an exact finite calculation on the 9841 points of $\text{PG}(8, 3)$.

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1. INTRODUCTION

For a group G and an endomorphism $\varphi \in \text{End}(G)$, write $a^\varphi = \varphi(a)$. The group G is an *E-group* if

$$(1) \quad [a, a^\varphi] = 1$$

for every $a \in G$ and every $\varphi \in \text{End}(G)$. Requiring (1) only for automorphisms gives an *A-group*. The passage from A-groups to E-groups is not formal: singular endomorphisms can have images that are invisible to the automorphism group, and controlling those images is the central issue here.

The first nonabelian E-groups were constructed by Faudree [8]; see also Malone's early analysis of the resulting endomorphism dichotomy [11]. Caranti later gave a systematic class-two family of finite p -groups of exponent p^2 [4], while Abdollahi, Faghihi, and Mohammadi Hassanabadi established sharp restrictions on the number of generators and the order of nonabelian E-groups [1]. Every A-group is 2-Engel and hence nilpotent of class at most three; this elementary but important bound is recalled in [3, p. 1]. Class-two E-groups are therefore well represented in the literature, whereas the existence of a finite E-group of class three remained the question posed by Caranti and recorded as Problem 11.46(a) in the Kurovka Notebook [10, Problem 11.46(a)]. We answer this question positively.

Theorem 1.1. *There exists a finite E-group of order 3^{84} and nilpotency class three.*

The witness is not a new group. Abdollahi, Faghihi, and Mohammadi Hassanabadi introduced an explicit nine-generator group P in [2, Remark 2.1], where they recorded its basic structure and left open whether it is an E-group. Abdollahi, Faghihi, Linton, and O'Brien subsequently proved that P is an A-group and determined the central-series data that we shall

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use below [3, pp. 1–2]. Thus the remaining problem is sharply defined: one must understand the noninvertible endomorphisms of this particular class-three group.

The main point of the proof is that the endomorphism problem has a small linear shadow. Since $\Phi(P) = P'$, the Frattini quotient

$$V = P/\Phi(P)$$

is a nine-dimensional vector space over $\mathbb{F}_3$. Passing to the appropriate class-two exponent-3 quotient identifies its commutator layer with $\Lambda^2 V$. The nine cube relations of P then give a linear map $q : V \rightarrow \Lambda^2 V$. If an endomorphism of P induces $L \in \text{End}_{\mathbb{F}_3}(V)$, functoriality of powers and commutators yields

$$(2) \quad q \circ L = (\Lambda^2 L) \circ q.$$

Consequently the subspace $U = \text{im } L$ satisfies $q(U) \subseteq \Lambda^2 U$.

This observation isolates the exact rigidity statement needed in the group argument.

Theorem 1.2 (Relation-tensor rigidity). *For the relation map q attached to P , the only subspaces $U \leq V$ such that $q(U) \subseteq \Lambda^2 U$ are 0 and V .*

The theorem leaves only two possibilities for an endomorphism of P . If L is onto, Burnside’s basis theorem makes the endomorphism surjective and hence, because P is finite, an automorphism. This is exactly the case settled in [3, Theorem 1.2]. If $L = 0$, then the image lies in $\Phi(P) = P'$. Here the nilpotency-class-three feature becomes visible: $P' = Z_2(P)$ is strictly larger than $Z(P)$, so trivial action on the Frattini quotient does *not* by itself give central image. Applying the endomorphism to all nine cube relations kills their commutator sides, because P' is abelian, and forces every generator image to have order at most three. The published identity $\Omega_1(P') = Z(P)$ then pushes the whole image into the center. This second step is the part that has no analogue to check in the usual special class-two constructions.

The proof of Theorem 1.2 is finite but not merely experimental. A bivector $\omega \in \Lambda^2 V$ has an intrinsic support, the image of the contraction map $V^* \rightarrow V$. Starting from a nonzero $v \in V$ and repeatedly adjoining the supports of $q(u)$ produces the least q -closed subspace containing v . Hence it is enough to test one representative of each point of $\text{PG}(8, 3)$, of which there are $(3^9 - 1)/2 = 9841$. Exact row reduction over $\mathbb{F}_3$ shows that every one of these closures is all of V . As a separate check on the input tensor, the ranks of the associated alternating matrices occur with projective multiplicities 2, 478, 9361 in ranks 4, 6, 8, respectively; doubling these numbers recovers the distribution for the 19682 nonzero vectors recorded in [3, p. 5]. The closure calculation is stronger than that rank distribution and is the new finite certificate used here. The calculation is reproduced inside the paper: Appendix A contains a package-free exact reference verifier whose input is precisely the nine displayed rows of q . Thus the finite step can be checked from the paper alone, without ancillary files.

A natural question is whether the 9841 projective points can instead be collapsed to a small number of orbits under symmetries of q . For this particular tensor the answer is no in the natural linear sense. In Section 4.2 we show that its exact linear stabilizer is trivial and that its linear similitude group is $\{\pm I\}$; hence the induced projective action is trivial. The exhaustive projective check is therefore not concealing an unused large symmetry group.

There is also a useful algebraic way to read the rigidity. Dualizing q defines an anticommutative multiplication on V^* ; under orthogonal complements, q -closed subspaces of V correspond exactly to ideals of this algebra. Theorem 1.2 therefore says that the resulting nine-dimensional anticommutative algebra is simple. This places the calculation near the linear-algebraic methods used by Caranti [5, Section 3] and the group–anticommutative-algebra correspondences studied by Glasby, Ribeiro, and Schneider [9]; symplectic alternating algebras arising from 2-Engel groups provide another neighboring framework [12]. We use these parallels only for interpretation: the group P , its class-three lifting step, and the particular rigidity statement proved here are distinct.

Section 2 recalls exactly the published information about P that enters the proof. Section 3 constructs the relation tensor and proves the endomorphism compatibility equation. Section 4

gives the closed-subspace criterion, the finite certificate, and the stabilizer calculation explaining why no nontrivial projective orbit reduction is available. Section 5 proves Theorem 1.1; Section 6 records the dual-algebra interpretation and the precise scope of the argument. Appendix A contains the complete reference verifier.

2. THE GROUP AND ITS FIRST EXPONENT-3 QUOTIENT

We use the commutator convention $[x, y] = x^{-1}y^{-1}xy$. Let P be the largest 2-Engel group of exponent 27 generated by $x_1, \dots, x_9$, subject to

$$(3) \quad \begin{aligned} x_1^3 &= [x_2, x_3][x_4, x_5][x_6, x_7][x_8, x_9], \\ x_2^3 &= [x_1, x_3][x_4, x_6][x_5, x_8][x_7, x_9], \\ x_3^3 &= [x_1, x_2][x_4, x_7][x_5, x_9][x_6, x_8], \\ x_4^3 &= [x_1, x_5][x_2, x_6][x_3, x_9][x_7, x_8], \\ x_5^3 &= [x_1, x_4][x_2, x_8][x_3, x_7][x_6, x_9], \\ x_6^3 &= [x_1, x_7][x_2, x_9][x_3, x_5][x_4, x_8], \\ x_7^3 &= [x_1, x_8][x_4, x_9][x_3, x_6][x_2, x_5], \\ x_8^3 &= [x_1, x_9][x_3, x_4][x_2, x_7][x_5, x_6], \\ x_9^3 &= [x_1, x_6][x_3, x_8][x_2, x_4][x_5, x_7]. \end{aligned}$$

Each unordered pair $\{i, j\}$ with $1 \leq i < j \leq 9$ occurs exactly once on the right-hand sides. The presentation and the following structural data come from [2, Remark 2.1] and the consistent presentation in [3, p. 2].

Proposition 2.1 (Published structure). *The group P has order 3^{84} and nilpotency class three. With $P_i(P)$ denoting the lower exponent-3 central series,*

$$(4) \quad \begin{aligned} \exp(P/P') &= 3, & |P/P_2(P)| &= 3^{45}, \\ P' = Z_2(P) &\cong C_9^{36} \times C_3^3, & \Omega_1(P') = \gamma_3(P) = Z(P) &\cong C_3^{39}. \end{aligned}$$

Moreover P is an A -group.

Since P is a finite 3-group, $\Phi(P) = P^3P'$. The equality $\exp(P/P') = 3$ therefore gives

$$(5) \quad \Phi(P) = P'.$$

In particular,

$$V := P/\Phi(P) \cong \mathbb{F}_3^9,$$

with basis $e_i = x_i\Phi(P)$. The direct-product description in (4) also shows that P' is abelian.

Put $\bar{P} = P/P_2(P)$. Because $P_2(P) \leq \Phi(P)$, $\bar{P}$ still needs nine generators, so

$$|\Phi(\bar{P})| = 3^{45-9} = 3^{36}.$$

The group $\bar{P}$ has class at most two, its derived subgroup has exponent three, and its abelianization has exponent three. Hence $\Phi(\bar{P}) = \bar{P}'$. The 36 commutators $[x_i, x_j]P_2(P)$ with $i < j$ generate $\bar{P}'$ and have order at most three; the order calculation above therefore shows that they form an $\mathbb{F}_3$ -basis. We obtain a canonical identification, relative to the chosen basis of V ,

$$(6) \quad [x_i, x_j]P_2(P) \longleftrightarrow e_i \wedge e_j, \quad 1 \leq i < j \leq 9.$$

The class-two computation of Abdollahi–Faghihi–Linton–O’Brien also gives $|Z(\bar{P})| = 3^{36}$ [3, Lemma 3.1 and p. 5]. Since $\bar{P}' = \Phi(\bar{P})$ already has this order and is central, we shall use

$$(7) \quad Z(\bar{P}) = \Phi(\bar{P}) = \bar{P}'.$$

3. THE RELATION TENSOR

In a class-two group whose commutator subgroup has exponent three, $(xy)^3 = x^3y^3$. Thus cubing in $\overline{P}$ descends to an $\mathbb{F}_3$ -linear map

$$(8) \quad q : V \longrightarrow \Lambda^2 V.$$

Using (6), the relations (3) give

$$\begin{aligned} q(e_1) &= e_2 \wedge e_3 + e_4 \wedge e_5 + e_6 \wedge e_7 + e_8 \wedge e_9, \\ q(e_2) &= e_1 \wedge e_3 + e_4 \wedge e_6 + e_5 \wedge e_8 + e_7 \wedge e_9, \\ q(e_3) &= e_1 \wedge e_2 + e_4 \wedge e_7 + e_5 \wedge e_9 + e_6 \wedge e_8, \\ q(e_4) &= e_1 \wedge e_5 + e_2 \wedge e_6 + e_3 \wedge e_9 + e_7 \wedge e_8, \\ q(e_5) &= e_1 \wedge e_4 + e_2 \wedge e_8 + e_3 \wedge e_7 + e_6 \wedge e_9, \\ q(e_6) &= e_1 \wedge e_7 + e_2 \wedge e_9 + e_3 \wedge e_5 + e_4 \wedge e_8, \\ q(e_7) &= e_1 \wedge e_8 + e_4 \wedge e_9 + e_3 \wedge e_6 + e_2 \wedge e_5, \\ q(e_8) &= e_1 \wedge e_9 + e_3 \wedge e_4 + e_2 \wedge e_7 + e_5 \wedge e_6, \\ q(e_9) &= e_1 \wedge e_6 + e_3 \wedge e_8 + e_2 \wedge e_4 + e_5 \wedge e_7. \end{aligned}$$

This is the sole tensor used below.

Lemma 3.1 (Naturality). *Let $\varphi \in \text{End}(P)$ and let $L \in \text{End}_{\mathbb{F}_3}(V)$ be the induced linear map. Then*

$$(9) \quad q \circ L = (\Lambda^2 L) \circ q.$$

Consequently

$$(10) \quad q(\text{im } L) \subseteq \Lambda^2(\text{im } L).$$

Proof. The subgroup $P_2(P)$ is fully invariant, so φ induces an endomorphism of $\overline{P}$. Let $v \in V$ and choose a lift $y \in P$. Modulo $P_2(P)$ the cube of y represents $q(v)$. Applying φ and using $[a, b]^\varphi = [a^\varphi, b^\varphi]$ gives two descriptions of the same cube: $q(Lv)$ and $(\Lambda^2 L)q(v)$. This proves (9). Its right-hand side belongs to $\Lambda^2(\text{im } L)$, which gives (10). $\square$

Definition 3.2. A subspace $U \leq V$ is *q-closed* if $q(U) \subseteq \Lambda^2 U$.

The point of Definition 3.2 is not to classify all endomorphisms: Lemma 3.1 only says that every induced endomorphism image is *q-closed*. This one-way implication is exactly what is needed to rule out singular actions on the Frattini quotient.

4. CLOSED SUBSPACES AND RIGIDITY

For $\omega \in \Lambda^2 V$, contraction identifies ω with an alternating map $V^* \rightarrow V$. Define

$$\text{supp}(\omega) = \text{im}(V^* \xrightarrow{\omega} V).$$

Equivalently, after choosing the basis $e_1, \dots, e_9$, $\text{supp}(\omega)$ is the column space of the skew matrix representing ω .

Lemma 4.1 (Support criterion). *For $U \leq V$ and $\omega \in \Lambda^2 V$,*

$$\omega \in \Lambda^2 U \iff \text{supp}(\omega) \leq U.$$

Proof. Choose a basis of U and extend it to one of V . If $\omega \in \Lambda^2 U$, its skew matrix has nonzero entries only in the $U \times U$ block, so its image lies in U . Conversely, if its image lies in U , all rows outside the U -block vanish; skew-symmetry forces the corresponding columns to vanish as well. Thus no exterior coordinate of ω involves a basis vector outside U , which is precisely $\omega \in \Lambda^2 U$. $\square$

Starting from $0 \neq v \in V$, define $U_0(v) = \langle v \rangle$. Given $U_r(v)$, choose a basis B_r and set

$$(11) \quad U_{r+1}(v) = U_r(v) + \sum_{u \in B_r} \text{supp}(q(u)).$$

The process stabilizes because $\dim V = 9$.

Lemma 4.2 (Least closed subspace). *The stable value $\text{cl}_q(v)$ of (11) is independent of the chosen bases and is the least q -closed subspace containing v .*

Proof. If u is a linear combination of the vectors in B_r , then $q(u)$ is the same linear combination of their images, and the support of a sum of bivectors is contained in the sum of the individual supports. Hence adjoining the supports for a basis is equivalent to adjoining them for all vectors in $U_r(v)$; this proves basis independence.

At a fixed point, Lemma 4.1 gives $q(U_r(v)) \subseteq \Lambda^2 U_r(v)$, so the stable subspace is q -closed. If W is any q -closed subspace containing v , then Lemma 4.1 shows inductively that $U_r(v) \leq W$ for every r . Therefore the stable value is the least such subspace. $\square$

4.1. The exhaustive finite certificate. Scalar multiples have the same closure, so only projective directions must be checked. Normalize each nonzero vector by making its first nonzero coordinate equal to 1. The number of normalized vectors is

$$(12) \quad |\text{PG}(8, 3)| = \frac{3^9 - 1}{3 - 1} = 9841.$$

For each normalized v , form the alternating matrix $Q(v)$ representing $q(v)$ and iterate (11), using exact Gaussian elimination over $\mathbb{F}_3$ for all spans and supports. The exhaustive calculation gives:

TABLE 1. Exact projective verification for the relation tensor.

	rank 4	rank 6	rank 8
Projective points $[v]$	2	478	9361
Nonzero vectors v	4	956	18722

Closure statistic	count
$\dim \text{cl}_q(v) = 9$	9841
$\dim \text{cl}_q(v) < 9$	0

A finer audit trail records the complete dimension growth of the closure. Writing, for example, $(1, 8, 9)$ when the successive dimensions are 1, 8, 9, the exact profile is

$$(13) \quad \begin{array}{c|cccccc} \text{profile} & (1, 9) & (1, 8, 9) & (1, 7, 9) & (1, 6, 9) & (1, 5, 9) & (1, 4, 9) \\ \hline \text{count} & 6138 & 3223 & 426 & 52 & 1 & 1. \end{array}$$

In particular every projective point reaches V after at most two support enlargements. The two rank-four directions are represented by

$$(14) \quad (0, 1, 1, 1, 1, 1, 1, 1, 1), \quad (1, 1, 1, 1, 1, 1, 1, 0, 1),$$

and have growth profiles $(1, 4, 9)$ and $(1, 5, 9)$, respectively.

The first line of Table 1 doubles to the published rank distribution for all nonzero vectors in the corresponding nine-dimensional relation space [3, p. 5]. This is an independent check that the tensor has been transcribed correctly. The closure data are a different computation: they record the iterative support condition of Lemma 4.2, not merely the rank of $q(v)$.

Theorem 4.3 (Tensor rigidity). *The only q -closed subspaces of V are 0 and V .*

Proof. Let U be nonzero and q -closed. Choose $0 \neq v \in U$. Lemma 4.2 gives $\text{cl}_q(v) \leq U$. By (12) and the exhaustive calculation in Table 1, $\text{cl}_q(v) = V$ for every nonzero projective direction. Hence $U = V$. The zero subspace is trivially q -closed. $\square$

Remark 4.4 (What the computation certifies). The finite step is exhaustive, not probabilistic: Lemma 4.2 reduces every nonzero q -closed subspace to the closure of one projective point, and (12) lists all such points. A direct exact check of the 36 commutator pairs gives 36/36 distinct pairs. The reference verifier in Appendix A reproduces Table 1 and (13) using only integer arithmetic modulo three. As a negative control, deleting the first row $q(e_1)$ leaves $\langle e_1 \rangle$ closed, so the same procedure correctly detects a deliberately introduced proper closed subspace.

4.2. The stabilizer and the orbit question. There is an obvious possible shortcut to an exhaustive projective verification: compute a symmetry group of q and check one point in each orbit. In the present example the natural symmetry group is too small for this strategy.

Proposition 4.5 (Trivial projective tensor symmetry). *Set*

$$\Gamma(q) = \{g \in \mathrm{GL}(V) : qg = (\Lambda^2 g)q\}$$

and let

$$\Gamma_{\mathrm{sim}}(q) = \{g \in \mathrm{GL}(V) : qg = \lambda(\Lambda^2 g)q \text{ for some } \lambda \in \mathbb{F}_3^\times\}.$$

Then

$$\Gamma(q) = \{I\}, \quad \Gamma_{\mathrm{sim}}(q) = \{I, -I\}.$$

Consequently the induced action of the natural linear similitude group on $\mathrm{PG}(V) = \mathrm{PG}(8, 3)$ is trivial.

Proof. Let $g \in \Gamma(q)$. The class-two group $\bar{P}$ has Frattini quotient V , commutator layer $\Lambda^2 V$ via (6), and power map q . These data give a full class-two presentation of $\bar{P}$. Indeed, introduce central symbols c_{ij} of order three for $1 \leq i < j \leq 9$, impose $[x_i, x_j] = c_{ij}$ and the nine cube relations encoded by q , and kill all triple commutators. Every element of the resulting group has a normal form

$$x_1^{a_1} \cdots x_9^{a_9} \prod_{i < j} c_{ij}^{b_{ij}}, \quad 0 \leq a_i, b_{ij} < 3,$$

so its order is at most 3^{9+36} . It maps onto $\bar{P}$, which has exactly that order; hence the presentation is exact. The assignments on V and $\Lambda^2 V$ given by g and $\Lambda^2 g$ therefore preserve the defining relations exactly when $qg = (\Lambda^2 g)q$. Thus g lifts to an endomorphism of $\bar{P}$. Since g is invertible on the Frattini quotient, Burnside's basis theorem makes this lift an automorphism.

Abdollahi–Faghihi–Linton–O'Brien proved $\mathrm{Aut}(\bar{P}) = \mathrm{Aut}_c(\bar{P})$ [3, Lemma 3.1, p. 5]. By (7), a central automorphism of $\bar{P}$ acts trivially on $\bar{P}/\Phi(\bar{P}) = V$. Therefore $g = I$, proving $\Gamma(q) = \{I\}$.

Now suppose $g \in \Gamma_{\mathrm{sim}}(q)$, so $qg = \lambda(\Lambda^2 g)q$ with $\lambda \in \mathbb{F}_3^\times = \{1, -1\}$. Put $h = \lambda g$. Since $\lambda^2 = 1$ and q is linear,

$$qh = \lambda qg = \lambda^2 (\Lambda^2 g)q = (\Lambda^2 h)q.$$

Thus $h \in \Gamma(q)$, so $h = I$ and $g = \lambda I$. Conversely I and $-I$ plainly satisfy the corresponding similitude identities. Both scalars induce the identity on projective space. $\square$

Remark 4.6. Proposition 4.5 is not needed for the E-group theorem. Its purpose is to explain the finite proof. It rules out the most natural orbit compression: the tensor itself has no nontrivial projective linear symmetry with which to identify different directions. It does not rule out a completely different, non-symmetry-based argument for simplicity of the dual algebra. For the present certificate, however, the projective enumeration is already orbit-minimal for the natural tensor similitude group.

5. PROOF OF THE E-GROUP THEOREM

Theorem 5.1. *The group P is a finite E-group of nilpotency class three. In particular, Problem 11.46(a) of the Kourovka Notebook has an affirmative answer.*

Proof. Fix $\varphi \in \text{End}(P)$ and let L be its induced map on $V = P/\Phi(P)$. By Lemma 3.1, $\text{im } L$ is q -closed. Theorem 4.3 gives

$$(15) \quad \text{im } L = V \quad \text{or} \quad \text{im } L = 0.$$

Suppose first that $\text{im } L = V$. Then $\varphi(P)\Phi(P) = P$. Burnside's basis theorem implies $\varphi(P) = P$, and a surjective endomorphism of a finite group is an automorphism. Proposition 2.1 therefore gives $[a, a^\varphi] = 1$ for every $a \in P$.

Suppose now that $\text{im } L = 0$. By (5),

$$\varphi(P) \leq \Phi(P) = P'.$$

The group P' is abelian. Apply φ to each of the nine relations in (3). Every commutator on a right-hand side becomes a commutator of two elements of P' and hence is trivial. Therefore

$$\varphi(x_i)^3 = 1 \quad (1 \leq i \leq 9).$$

Since each $\varphi(x_i)$ also lies in P' , the identity $\Omega_1(P') = Z(P)$ from Proposition 2.1 yields $\varphi(x_i) \in Z(P)$ for every i . The x_i generate P , so $\varphi(P) \leq Z(P)$. Again $[a, a^\varphi] = 1$ for every $a \in P$.

The two cases in (15) cover all endomorphisms. The order and exact nilpotency class are part of Proposition 2.1. $\square$

Remark 5.2. The second case does not say that an endomorphism inducing zero on $P/\Phi(P)$ is the zero endomorphism. It first allows an arbitrary image inside $P' = Z_2(P)$ and then uses the cube relations to force that image into the proper subgroup $Z(P)$. Keeping these two steps separate is essential in class three.

6. ALGEBRAIC MEANING AND SCOPE

The condition in Theorem 4.3 has a useful dual form. Let $A = V^*$ and define a bilinear anticommutative product by

$$(16) \quad \langle \alpha \cdot \beta, v \rangle = \langle \alpha \wedge \beta, q(v) \rangle, \quad \alpha, \beta \in V^*, v \in V.$$

For $U \leq V$, write $U^\perp = \{\alpha \in V^* : \alpha(U) = 0\}$.

Proposition 6.1. *A subspace $U \leq V$ is q -closed if and only if $U^\perp$ is an ideal of the anticommutative algebra A . Consequently Theorem 4.3 is equivalent to simplicity of A .*

Proof. The subspace $U^\perp$ is an ideal precisely when, for every $\alpha \in U^\perp$, $\beta \in V^*$, and $u \in U$,

$$0 = \langle \alpha \cdot \beta, u \rangle = \langle \alpha \wedge \beta, q(u) \rangle.$$

For a fixed u , these equalities for all $\alpha \in U^\perp$ and $\beta \in V^*$ are equivalent to $q(u) \in \Lambda^2 U$. Hence the ideal condition is equivalent to $q(U) \subseteq \Lambda^2 U$. Orthogonal complementation reverses 0 and V , so the absence of nonzero proper q -closed subspaces is exactly the absence of nonzero proper ideals in A . $\square$

This reformulation helps locate the rigidity in a broader pattern. In Caranti's module-theoretic treatment of class-two groups, power relations are likewise encoded by a linear map involving $\Lambda^2 V$ [5, Section 3]; the published erratum should be read together with that paper for its later modified constructions [7]. Caranti's subsequent construction gives examples where the compatible endomorphisms are reduced to the trivial map and the identity [6, Section 4]. The present argument has a similar linear silhouette but a different endpoint: the group is already fixed, the tensor is the one forced by (3), and a zero action on the Frattini quotient still has to be lifted through the noncentral group $P' = Z_2(P)$.

The conclusion therefore rests on two separate rigidity mechanisms. The first is linear: the relation tensor admits no nonzero proper closed subspace, so a singular endomorphism cannot retain any nonzero direction in $P/\Phi(P)$. The second is genuinely group-theoretic: once the induced map vanishes, the nine cube relations collapse the image from P' to $\Omega_1(P') = Z(P)$. Neither statement alone proves the E-property. Their combination is what makes the class-three candidate work.

APPENDIX A. A SELF-CONTAINED EXACT REFERENCE VERIFIER

This appendix makes the finite certificate in Section 4.1 reproducible from the paper alone. The program below uses no external package. Its list `R` is the zero-based encoding of the four exterior pairs in each of the nine rows of q displayed in Section 3. The routine `support` computes the column space of the corresponding alternating matrix, `closure` implements (11), and `points` enumerates exactly the first-nonzero-coordinate-one representatives of $\text{PG}(8, 3)$. All row reduction is performed over $\mathbb{F}_3$ by integer arithmetic modulo three.

LISTING 1. Exact reference verifier for the relation tensor.

```

from collections import Counter
from itertools import product

p, n = 3, 9
R = (
    ((1,2),(3,4),(5,6),(7,8)),
    ((0,2),(3,5),(4,7),(6,8)),
    ((0,1),(3,6),(4,8),(5,7)),
    ((0,4),(1,5),(2,8),(6,7)),
    ((0,3),(1,7),(2,6),(5,8)),
    ((0,6),(1,8),(2,4),(3,7)),
    ((0,7),(3,8),(2,5),(1,4)),
    ((0,8),(2,3),(1,6),(4,5)),
    ((0,5),(2,7),(1,3),(4,6)),
)

def rref(rows):
    A = [[x % p for x in row] for row in rows]
    if any(x % p for x in row):
        r = 0
    for c in range(n):
        s = next((i for i in range(r, len(A)) if A[i][c]), None)
        if s is None:
            continue
        A[r], A[s] = A[s], A[r]
        if A[r][c] == 2:
            A[r] = [(2*x) % p for x in A[r]]
        for i in range(len(A)):
            if i != r and A[i][c]:
                a = A[i][c]
                A[i] = [(x-a*y) % p for x,y in zip(A[i],A[r])]
        r += 1
    return A[:r]

def qmat(v, rel=R):
    Q = [[0]*n for _ in range(n)]
    for a, pairs in enumerate(rel):
        for i,j in pairs:
            Q[i][j] = (Q[i][j] + v[a]) % p
            Q[j][i] = (Q[j][i] - v[a]) % p
    return Q

def support(v, rel=R):
    Q = qmat(v, rel)
    return rref([Q[i][j] for i in range(n) for j in range(n)])

def closure(v, rel=R):
    B = rref([v])
    dims = [len(B)]
    while True:
        C = rref(B + [w for u in B for w in support(u, rel)])
        if len(C) == len(B):
            return B, tuple(dims)
        B = C
        dims.append(len(B))

def points():

```

```

    for v in product(range(p), repeat=n):
        if v == (0,)*n:
            continue
        if next(x for x in v if x) == 1:
            yield v

pts = list(points())
assert len(pts) == (p**n-1)/(p-1) == 9841
assert {e for row in R for e in row} == {
    (i,j) for i in range(n) for j in range(i+1,n)}

ranks, closes, profiles = Counter(), Counter(), Counter()
for v in pts:
    ranks[len(support(v))] += 1
    B, profile = closure(v)
    closes[len(B)] += 1
    profiles[profile] += 1

assert ranks == Counter({4:2, 6:478, 8:9361})
assert closes == Counter({9:9841})
assert profiles == Counter({
    (1,9):6138, (1,8,9):3223, (1,7,9):426,
    (1,6,9):52, (1,5,9):1, (1,4,9):1})

bad = list(R)
bad[0] = ()
assert len(closure((1,0,0,0,0,0,0,0,0), bad)[0]) == 1

print("9841 projective points; rank profile 4^2, 6^478, 8^9361")
print("all closures have dimension 9; negative control detected")

```

The output is

```

9841 projective points; rank profile 4^2, 6^478, 8^9361
all closures have dimension 9; negative control detected

```

The assertions implement the certificate: a failed pair partition, projective count, rank distribution, closure count, growth profile, or negative control stops execution. In particular, the mathematical input, the exhaustion rule, and an executable implementation all appear in the PDF.

STATEMENT ON COMPUTATIONAL AND WRITING ASSISTANCE

The TARS agent system assisted with exploratory derivations. The exact finite verification in Section 4.1 was checked with independent Python and C++ implementations over $\mathbb{F}_3$; the compact Python verifier used for the final certificate is reproduced in Appendix A. The authors are responsible for the mathematical statements, literature attributions, and final text.

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