

Riemannian Density-Driven Optimal Control: Tangent-Space LQR for Second-Order Multi-Agent Systems on Curved Manifolds

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Abstract—Density-Driven Optimal Control (D²OC) provides an effective framework for steering multi-agent systems toward prescribed spatial distributions. However, existing D²OC formulations are primarily developed for Euclidean domains and do not directly account for intrinsic manifold geometry. This paper extends D²OC to second-order multi-agent systems evolving on Riemannian manifolds. The proposed Riemannian D²OC (*R*-D²OC) constructs a local distribution objective in the tangent space of each agent through logarithmic maps and uses its weighted center as the reference for a finite-horizon LQR. The resulting control is executed on the manifold through intrinsic second-order dynamics and parallel transport within a receding-horizon scheme. We establish local curvature-dependent bounds that quantify the approximation introduced by the tangent-space reduction and characterize the resulting target bias. Furthermore, we derive a conditional discrete-descent result showing that the closed-loop objective decreases when a local velocity-alignment condition is satisfied. Numerical simulations on a 3D ellipsoidal manifold demonstrate distribution-level control and empirically support the proposed approximation and descent results.

I. INTRODUCTION

Steering multi-agent systems toward prescribed spatial distributions while minimizing control effort is a fundamental challenge in robotics [1]. Density-Driven Optimal Control (D²OC) addresses this via a Lagrangian framework penalizing spatial distribution mismatch along control effort [1], [2], [3], [4], [5]. Existing D²OC methods, however, rely on Euclidean spaces where quadratic density objectives yield standard linear quadratic tracking representations. When agent configurations evolve on curved manifolds, these Euclidean formulations break down because linear vector operations violate intrinsic geometric constraints.

To address these geometric limitations in multi-agent control, prior efforts have broadly explored three avenues. First, geometric control on Riemannian manifolds and Lie groups intrinsically resolves rotational singularities for attitude and rigid-body tracking [6], [7], [8]. Yet, these methods address internal system configuration constraints rather than agent navigation over environmentally constrained curved surfaces or collective spatial distribution control. Second, geodesic path-planning methods leverage intrinsic distances to generate valid or shortest paths on curved spaces [9], [10], while primarily focusing on geometric path generation for individual agents without accounting for system dynamics

or collective distribution control. Third, swarm and coverage-control methods shape agent configurations via Voronoi or density-based objectives [11], [12], [13], but are predominantly Euclidean and do not preserve intrinsic manifold geometry. Thus, an optimal distribution-control framework accounting for intrinsic geometry, system dynamics, and computational efficiency remains undeveloped.

To resolve these limitations, we develop a local Riemannian realization of D²OC by constructing the distribution target in the tangent space at the current agent state. The logarithmic map converts local reference samples of target distribution into tangent vectors, whose weighted average defines a virtual target for a finite-horizon LQR. The resulting control is then executed through the intrinsic second-order dynamics using parallel transport, while the receding-horizon update continuously shifts the tangent-space reference with the agent state. This construction retains the fixed-dimensional structure of LQR while providing explicit control over the geometric approximation introduced by the tangent-space reduction. We establish a geometric alignment condition under which a local discrete-time descent property holds, providing a formal criterion for manifold-based tracking control.

The main contributions of this paper are thus threefold: 1) a local Riemannian formulation of D²OC (*R*-D²OC) for second-order agents on curved spaces; 2) a rigorous analysis of approximation errors introduced by the tangent-space reduction under local curvature; and 3) a conditional discrete-descent guarantee for the resulting receding-horizon controller under a local velocity-alignment condition.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Density-Driven Optimal Control Framework

The Euclidean density-driven optimal control framework steers a multi-agent system toward a prescribed spatial distribution represented by weighted reference samples. As described in [1], [2], [3], [5], the process consists of three iterative stages. For a given coverage mission, a reference distribution is represented by sample points $\{q_j\}_{j=1}^N$ with uniform weights $1/N$. In Stage A, each agent identifies a local subset of reference samples and computes a control input driving its motion toward the corresponding weighted spatial target. Stage B decreases the sample weights based on coverage progress, while Stage C exchanges updated weights among neighboring agents within communication range. Repeated execution enables progressive coverage of the reference distribution. Due to space limits, further details are omitted and can be found in the cited literature.

*This work was supported by NSF CAREER Grant CMMI-DCSD-2638508.

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This paper focuses on Stage A. At each time step, an agent selects a local subset of reference samples and constructs a local target represented by their weighted barycenter. The agent then solves a finite-horizon optimal control problem to track this target while minimizing control effort. In Euclidean D^2OC , this problem reduces to standard linear-quadratic control since agent dynamics and targets are represented in Euclidean coordinates. However, this construction relies on linear vector operations that are not intrinsically defined on a general Riemannian manifold. Consequently, local target construction and its LQR realization cannot be directly applied to agents moving on curved spaces. In what follows, we develop an intrinsic counterpart of Stage A using logarithmic maps and tangent-space dynamics.

B. Density Representation on Manifold

To lift the aforementioned Euclidean D^2OC framework onto generic curved domains, we now formulate the density representation and optimal transport metrics on a smooth Riemannian manifold. Consider agent i with configuration $x_i(t) \in \mathcal{M}$, where $(\mathcal{M}, g)$ is a smooth d -dimensional Riemannian manifold. Let $v_i(t) \in T_{x_i(t)}\mathcal{M}$ denote its velocity. At planning step k , the agent considers its local subset of reference points $\mathcal{S}_i^k$. The local desired density for agent i is represented by the finite positive measure $\nu_i^k = \sum_{j \in \mathcal{S}_i^k} \pi_j \delta_{q_j}$, $q_j \in \mathcal{M}$, $\pi_j \geq 0$, with total local mass defined as $\alpha_i = \sum_{j \in \mathcal{S}_i^k} \pi_j = \frac{1}{M_i} > 0$, where π_j represents the mass transport weight associated with reference point q_j , $\delta_{(\cdot)}$ denotes a Dirac measure, and M_i stands for the fixed total agent capacity assigned to agent i . The symbol M_i is determined by the ratio of total operation time to the sampling interval $(T_{\text{total}}/\Delta t)$, reflecting a uniform per-step energy consumption profile.

Extending the Wasserstein distance concept to the Riemannian setting, the optimal transport cost between the agent's current state (modeled as a Dirac measure at x_i with mass α_i) and the local reference measure ν_i^k is given by the squared Riemannian Wasserstein distance:

$$\mathcal{W}_2^2(\alpha_i \delta_{x_i}, \nu_i^k) = \sum_{j \in \mathcal{S}_i^k} \pi_j \text{dist}_g(x_i, q_j)^2, \quad (1)$$

where $\text{dist}_g(x_i, q_j)$ denotes the Riemannian distance between x_i and q_j induced by the metric tensor g on the configuration manifold $\mathcal{M}$. Thus, the density mismatch naturally translates into a weighted sum of squared Riemannian distances over the agent's local sample set.

To characterize the geometric center toward which the agent is steered, the corresponding local Riemannian barycenter for agent i is defined as the minimizer of this transport cost:

$$\bar{q}_{R,i} \in \arg \min_{z \in \mathcal{U}} \frac{1}{2} \sum_{j \in \mathcal{S}_i^k} \pi_j \text{dist}_g(z, q_j)^2, \quad (2)$$

where $\mathcal{U}$ is a strongly convex neighborhood containing the relevant local sample points.

C. Intrinsic Second-Order Dynamics

For notational simplicity, the agent index i is omitted hereafter. Each agent obeys

$$\dot{x} = v, \quad \nabla_{\dot{x}} v = u, \quad u(t) \in T_{x(t)}\mathcal{M}, \quad (3)$$

where ∇ is the Levi-Civita connection, thus u is the covariant acceleration. We consider the Bolza problem

$$\begin{aligned} J &= \int_{t_0}^{t_f} \left[V(x(t)) + \frac{1}{2} \|u(t)\|_{x(t)}^2 \right] dt + \Phi(x(t_f)), \\ V(x) &= \frac{1}{2} \sum_{j \in \mathcal{S}^k} \pi_j \text{dist}_g(x, q_j)^2, \\ \Phi(x) &= \frac{w_f}{2} \sum_{j \in \mathcal{S}^k} \pi_j \text{dist}_g(x, q_j)^2, \quad w_f \geq 0, \end{aligned} \quad (4)$$

subject to (3) and a fixed initial state. The constant $w_f \geq 0$ penalizes the terminal distribution error relative to the running effort.

III. TANGENT-SPACE REDUCTION

A. Normal Coordinates

To map global geometric entities from the Riemannian manifold $\mathcal{M}$ into a local Euclidean vector space suitable for linear control design, we utilize exponential and logarithmic maps. The exponential map $\text{Exp}_{x_k}: T_{x_k}\mathcal{M} \rightarrow \mathcal{M}$ maps a tangent vector from the tangent space at x_k to a point along the geodesic on $\mathcal{M}$. Its inverse, the logarithmic map $\text{Log}_{x_k}: \mathcal{M} \rightarrow T_{x_k}\mathcal{M}$, projects points from the manifold back onto the tangent space $T_{x_k}\mathcal{M}$, where the norm of the tangent vector corresponds to the Riemannian distance along the minimizing geodesic.

At replanning step k , we define the local coordinates in the tangent space via the logarithmic map as

$$\xi = \text{Log}_{x_k}(x), \quad \eta_j = \text{Log}_{x_k}(q_j). \quad (5)$$

By choosing an orthonormal basis of $T_{x_k}\mathcal{M}$, we explicitly identify this tangent space with $\mathbb{R}^d$.

For each x in the local normal neighborhood, let $\mathcal{P}_{x \rightarrow x_k}$ denote parallel transport along the unique radial geodesic

$$s \mapsto \text{Exp}_{x_k}(s\xi), \quad s \in [0, 1].$$

Define the transported velocity $\zeta = \mathcal{P}_{x \rightarrow x_k} v \in T_{x_k}\mathcal{M}$.

Assume the sectional curvature satisfies $|\text{Sec}| \leq \kappa$ on a compact normal neighborhood containing the local trajectory and the relevant target samples. To characterize the linearization error on the manifold, consider the radial geodesic $\gamma(s) = \text{Exp}_{x_k}(s\xi)$ for $s \in [0, 1]$. For any direction $v \in T_{x_k}\mathcal{M}$, the action of the differential $D\text{Exp}_{x_k}(\xi)v$ is represented by a Jacobi field $J(s)$ along $\gamma(s)$ with $J(0) = 0$ and $\nabla_{\dot{\gamma}} J(0) = v$. Integrating the Jacobi equation $\nabla_{\dot{\gamma}}^2 J + R(J, \dot{\gamma})\dot{\gamma} = 0$ yields the integral form

$$J(1) = v - \int_0^1 (1-s) R(J(s), \dot{\gamma}) \dot{\gamma} ds. \quad (6)$$

Noting that $J(1) = D\text{Exp}_{x_k}(\xi)v$, applying parallel transport $\mathcal{P}_{x \rightarrow x_k}$ along γ and subtracting $v = Iv$ gives

$$\begin{aligned} (\mathcal{P}_{x \rightarrow x_k} D\text{Exp}_{x_k}(\xi) - I)v &= \\ &= - \int_0^1 (1-s) \mathcal{P}_{\gamma(s) \rightarrow x_k} R(J(s), \dot{\gamma}) \dot{\gamma} ds. \end{aligned} \quad (7)$$

Taking the norm on both sides and using the sectional curvature bound $|\text{Sec}| \leq \kappa$, the geodesic speed $\|\dot{\gamma}\| = \|\xi\|$, and the Jacobi field estimate $\|J(s)\| \leq C_0 \|v\|$ (where $C_0 >$

0 is a local constant bounded near unity), the integral term is bounded by $C_1 \kappa \|\xi\|^2 \|v\|$ for a constant $C_1 > 0$ depending on C_0 . Taking the supremum over $\|v\| = 1$ leads directly to the operator norm estimate

$$\|\mathcal{P}_{x \rightarrow x_k} \text{DExp}_{x_k}(\xi) - I\| \leq C_1 \kappa \|\xi\|^2. \quad (8)$$

To understand the dynamic behavior in the tangent space $T_{x_k}M$, recall that $\xi = \text{Log}_{x_k}(x)$ and its velocity variable in the tangent space is defined via parallel transport as $\zeta = \mathcal{P}_{x \rightarrow x_k} \dot{x}$. Differentiating ξ with respect to time yields $\dot{\xi} = \text{DLog}_{x_k}(x) \dot{x} = \text{DLog}_{x_k}(x) \mathcal{P}_{x_k \rightarrow x} \zeta$. Since the logarithmic map is the inverse of the exponential map, its differential satisfies $\text{DLog}_{x_k}(x) = (\text{DExp}_{x_k}(\xi))^{-1}$. Applying the operator norm estimate from (8) to this inverse differential and utilizing Neumann series expansion, the leading-order linear mapping reduces to the identity operator I , while the curvature-induced perturbation is collected in the residual term r_ξ . Consequently, after possibly shrinking the neighborhood,

$$\dot{\xi} = \zeta + r_\xi, \quad (9)$$

with $\|r_\xi\| \leq C_2 \kappa \|\xi\|^2 \|\zeta\|$.

Similarly, consider the acceleration dynamics under the intrinsic covariant derivative $\nabla_{\dot{x}} \dot{x} = u$. Differentiating the parallel-transported velocity $\zeta = \mathcal{P}_{x \rightarrow x_k} \dot{x}$ along the state trajectory $x(t)$ requires accounting for the curvature-induced non-commutativity of parallel transport along distinct paths (the path along the trajectory versus the radial geodesic connecting $x(t)$ to x_k). Evaluating this geometric drift via standard Jacobi-field and curvature tensor estimates along the geodesic triangle formed by x_k , $x(t)$, and $x(t + \Delta t)$ yields

$$\dot{\zeta} = u_0 + r_\zeta, \quad (10)$$

where $u_0 = \mathcal{P}_{x \rightarrow x_k} u$ represents the parallel-transported control input, and the residual acceleration error r_ζ is bounded by $\|r_\zeta\| \leq C_3 (\kappa \|\xi\| \|\zeta\|^2 + \kappa \|\xi\|^2 \|u_0\|)$.

The constants C_1, C_2, C_3 depend only on the selected compact neighborhood and uniform bounds on the relevant geometric quantities, confirming that the Euclidean double integrator ($\dot{\xi} = \zeta, \dot{\zeta} = u_0$) serves as the exact leading-order local model in the fixed tangent space $T_{x_k}M$.

B. Local Expansion of the Density Objective

For ξ, η_j in a sufficiently small common normal neighborhood, the squared Riemannian distance admits the local expansion

$$\frac{1}{2} \text{dist}_g(\text{Exp}_{x_k}(\xi), \text{Exp}_{x_k}(\eta_j))^2 = \frac{1}{2} \|\xi - \eta_j\|_{x_k}^2 + R_j(\xi, \eta_j). \quad (11)$$

In normal coordinates centered at x_k , the squared Riemannian distance has a Euclidean quadratic leading term and a fourth-order curvature correction. Thus, for ξ and η_j in a sufficiently small common normal neighborhood, there exists $C_4 > 0$, depending only on the neighborhood, such that

$$|R_j(\xi, \eta_j)| \leq C_4 \kappa (\|\xi\| + \|\eta_j\|)^4, \quad (12)$$

with C_4 depending only on the chosen neighborhood. The remainder vanishes identically when the neighborhood is flat. Therefore,

$$V(x) = \frac{1}{2} \sum_{j \in \mathcal{S}^k} \pi_j \|\xi - \eta_j\|^2 + R_V(\xi), \quad (13)$$

$$\text{where } |R_V(\xi)| \leq C_4 \kappa \sum_{j \in \mathcal{S}^k} \pi_j (\|\xi\| + \|\eta_j\|)^4. \quad (14)$$

Defining the weighted tangent-space center $\eta_{\text{lin}} = \frac{1}{\alpha} \sum_j \pi_j \eta_j$, we have $\frac{1}{2} \sum_j \pi_j \|\xi - \eta_j\|^2 = \frac{\alpha}{2} \|\xi - \eta_{\text{lin}}\|^2 + c_k$, where $c_k = \frac{1}{2} \sum_j \pi_j \|\eta_j - \eta_{\text{lin}}\|^2$ is independent of ξ . The resulting tangent-space virtual target is η_{lin} , which is mapped back to the manifold as

$$\tilde{q}_R = \text{Exp}_{x_k}(\eta_{\text{lin}}). \quad (15)$$

Remark 1. Note that η_{lin} serves as a tangent-space virtual target, which coincides with the true Riemannian barycenter in flat space but differs under nonzero curvature.

IV. TANGENT-SPACE TARGET AGGREGATION AND LQR CONTROL

A. Local LQR Controller

Neglecting the local geometric remainders in (9)–(14) gives the tangent-space approximation

$$\begin{aligned} \min_{u_0(\cdot)} \quad & J_{\text{loc}} = \frac{1}{2} \int_{t_k}^{t_f} \left[\alpha \|\xi - \eta_{\text{lin}}\|^2 + \|u_0\|^2 \right] dt \\ & + \frac{w_f \alpha}{2} \|\xi(t_f) - \eta_{\text{lin}}\|^2, \\ \text{s.t.} \quad & \dot{\xi} = \zeta, \quad \dot{\zeta} = u_0, \end{aligned} \quad (16)$$

where the terminal weight follows directly from (4). Define the tracking error $e = \xi - \eta_{\text{lin}}$ and state vector $y = [e^\top, \zeta^\top]^\top$. Since η_{lin} is fixed during one replanning horizon, the resulting nominal linear error dynamics satisfy $\dot{y} = Ay + Bu_0$, where $A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$. An exact zero-order-hold discretization with step Δt gives $y_{m+1} = A_d y_m + B_d u_m$ with $A_d = \begin{bmatrix} I & \Delta t I \\ 0 & I \end{bmatrix}$, $B_d = \begin{bmatrix} \frac{1}{2} \Delta t^2 I \\ \Delta t I \end{bmatrix}$ at discrete time index m . Approximating the running integral gives

$$J_d = \frac{1}{2} y_T^\top Q_f y_T + \sum_{m=0}^{T-1} \frac{1}{2} (y_m^\top \bar{Q} y_m + u_m^\top \bar{R} u_m), \quad (17)$$

with $\bar{Q} = \Delta t \text{diag}(\alpha I, 0)$, $\bar{R} = \Delta t I$, and $Q_f = \text{diag}(w_f \alpha I, 0)$, thus $P_T = Q_f$. Backward dynamic programming gives $u_m^* = -K_m y_m$ with

$$\begin{aligned} S_m &= \bar{R} + B_d^\top P_{m+1} B_d, & K_m &= S_m^{-1} B_d^\top P_{m+1} A_d, \\ P_m &= \bar{Q} + A_d^\top P_{m+1} A_d - A_d^\top P_{m+1} B_d S_m^{-1} B_d^\top P_{m+1} A_d. \end{aligned} \quad (18)$$

Because $\bar{R} \succ 0$ and $P_{m+1} \succeq 0$, $S_m \succ 0$, the backward recursion is well-defined.

Remark 2 (Fixed-Dimensional Computational Complexity). For a horizon T , solving the discrete Riccati recursion requires $\mathcal{O}(Td^3)$ operations, as the state dimension of y depends solely on the manifold dimension d . Meanwhile, computing the tangent-space target η_{lin} via weighted summation requires $\mathcal{O}(|\mathcal{S}^k|d)$ operations. Consequently, the target

sample size $|S^k|$ affects only the linear vector aggregation, leaving the matrix Riccati state dimension completely invariant to sample size.

B. Receding-Horizon Implementation for R-D²OC

At replanning step k , the local control command is synthesized in the tangent space $T_{x_k}\mathcal{M}$ and updated recursively. Given the current manifold state $x_k \in \mathcal{M}$ and a set of discrete target points $\{q_j\}$, we first compute the logarithmic map coordinates $\eta_j^k = \text{Log}_{x_k}(q_j) \in T_{x_k}\mathcal{M}$. The weighted spatial centroid in $T_{x_k}\mathcal{M}$ is evaluated as $\eta_{\text{lin}}^k = \frac{1}{\alpha} \sum_j \pi_j \eta_j^k$ with total weight $\alpha = \sum_j \pi_j$.

Since x_k serves as the origin of the chosen orthonormal basis for $T_{x_k}\mathcal{M}$, the initial state vector for the local LQR policy is formulated as $y_k = [(-\eta_{\text{lin}}^k)^\top, v_k^\top]^\top$, where $v_k \in T_{x_k}\mathcal{M}$ is the current velocity vector. The feedback acceleration command is then obtained as $u_k = -K_0 y_k \in T_{x_k}\mathcal{M}$.

To execute this control over the interval $t \in [t_k, t_k + \Delta t]$, u_k is continuously pushed forward along the trajectory $x(t)$ via parallel transport, $u(t) = \mathcal{P}_{x_k \rightarrow x(t)} u_k$, driving the intrinsic system dynamics $\dot{x} = v$ and $\nabla_{\dot{x}} v = u(t)$. A local second-order numerical update over the time step Δt is then used to advance the state on $\mathcal{M}$:

$$\begin{aligned} x_{k+1} &= \text{Exp}_{x_k} \left(\Delta t v_k + \frac{1}{2} \Delta t^2 u_k \right), \\ v_{k+1} &= \mathcal{P}_{x_k \rightarrow x_{k+1}} (v_k + \Delta t u_k). \end{aligned} \quad (19)$$

This update provides a local approximation of the intrinsic dynamics over each sampling interval. The tangent-space origin is then shifted from x_k to x_{k+1} at each step to complete the receding-horizon control loop.

V. THEORETICAL ANALYSIS: APPROXIMATION BIAS AND LOCAL CONDITIONAL DESCENT

A. Objective Approximation Error

Define the local objective approximation error at an arbitrary trajectory point

$$E(x; x_k) = V(x) - \frac{1}{2} \sum_j \pi_j \|\text{Log}_{x_k}(x) - \eta_j^k\|_{x_k}^2, \quad (20)$$

where $\eta_j^k = \text{Log}_{x_k}(q_j)$. This is exactly the gap between the true running cost and the quadratic surrogate the local LQR controller actually minimizes.

Proposition 1 (Local objective error). *Suppose x and all target samples belong to a common sufficiently small normal neighborhood of x_k . Then*

$$|E(x; x_k)| \leq C_4 \kappa \alpha (\|\xi\|_{x_k} + \sigma_k^{\text{tar}})^4, \quad (21)$$

where $\xi = \text{Log}_{x_k}(x)$ for the local coordinate of x and $\sigma_k^{\text{tar}} = \max_j \|\eta_j^k\|_{x_k}$.

Proof. From (14), $|E(x; x_k)| \leq C_4 \kappa \sum_j \pi_j (\|\xi\| + \|\eta_j^k\|)^4 \leq C_4 \kappa \alpha (\|\xi\| + \sigma_k^{\text{tar}})^4$. $\square$

This is a pointwise approximation, not a cumulative convergence result. A cumulative bound requires additional assumptions controlling target spread and prediction error along a trajectory.

B. Bias of the Tangent-Space Center

To characterize the Riemannian barycenter $\bar{q}_R$ in (2), define the weighted Fréchet energy [14] as $\mathcal{F}(x) = \frac{1}{2} \sum_j \pi_j \text{dist}_g(x, q_j)^2$, whose critical point satisfies $\text{grad} \mathcal{F}(\bar{q}_R) = 0$. Although $\mathcal{F}$ is identical to the state cost V in (4), we use $\mathcal{F}$ here to emphasize its role in the barycenter analysis. To quantitatively compare $\bar{q}_R$ in (15) with $\bar{q}_R$, we require $\mathcal{F}$ to be locally strongly convex near the barycenter. Otherwise, their intrinsic distance cannot be stably controlled.

Assumption 1 (Nondegenerate local barycenter). *There exists a strongly convex neighborhood $\mathcal{U}_B$ containing the reference point x_0 , all target samples q_j , the tangent-space virtual target $\tilde{q}_R$, and the Riemannian barycenter $\bar{q}_R$, such that the minimizing geodesics used below remain in $\mathcal{U}_B$ and*

$$\text{Hess} \mathcal{F}(x) \succeq \lambda_{\min} I, \quad x \in \mathcal{U}_B, \quad (22)$$

for some $\lambda_{\min} > 0$.

Remark 3. *Assumption 1 is standard in Riemannian optimization [14]. As $\mathcal{F}$ is a weighted sum of squared distance functions, $\text{Hess} \mathcal{F}(x) = \alpha I + \mathcal{O}(\kappa \alpha \rho_{\max}^2)$ with $\alpha = \sum_j \pi_j$. Thus, strong convexity naturally holds within a sufficiently small normal neighborhood.*

Lemma 1 (Local change-of-base-point estimate). *Let x_0 and q belong to a sufficiently small strongly convex normal neighborhood, $\eta = \text{Log}_{x_0}(q)$, $\delta \in T_{x_0}\mathcal{M}$, $\tilde{x} = \text{Exp}_{x_0}(\delta)$. Assume $|\text{Sec}| \leq \kappa$ on the neighborhood. Then, there exists $C_5 > 0$, independent of $\|\eta\| + \|\delta\|$, such that*

$$\begin{aligned} \mathcal{P}_{\tilde{x} \rightarrow x_0} \text{Log}_{\tilde{x}}(q) &= \eta - \delta + r(\eta, \delta), \\ \|r(\eta, \delta)\|_{x_0} &\leq C_5 \kappa (\|\eta\| + \|\delta\|)^3. \end{aligned} \quad (23)$$

Proof. This is the local third-order change-of-base-point expansion for the Riemannian logarithmic map, following from the Jacobi-field expansion of the exponential map and the variation of radial parallel transport. The Euclidean relation $\eta - \delta$ is exact when curvature vanishes. The first nonzero curvature correction is cubic in the local displacement, with uniform remainder on a compact strongly convex neighborhood. $\square$

This local expansion serves as a key technical tool in our framework, enabling us to compare $\text{grad} \mathcal{F}$ evaluated at two distinct base points within the same vector space, up to a controlled cubic error.

Theorem 1 (Local curvature bias). *Let $\rho_{\max} = \max_j \|\text{Log}_{x_0}(q_j)\|_{x_0}$, $\eta_{\text{lin}} = \frac{1}{\alpha} \sum_j \pi_j \text{Log}_{x_0}(q_j)$, $\tilde{q}_R = \text{Exp}_{x_0}(\eta_{\text{lin}})$. Suppose the conditions of Assumption 1 hold with some $\lambda_{\min} > 0$, $|\text{Sec}| \leq \kappa$ on $\mathcal{U}_B$, and $\rho_{\max} \leq \rho_*$ for a sufficiently small $\rho_* > 0$. Then, there exists $C_6 > 0$, independent of $\rho_{\max}$, such that*

$$\text{dist}_g(\tilde{q}_R, \bar{q}_R) \leq C_6 \frac{\kappa \alpha}{\lambda_{\min}} \rho_{\max}^3. \quad (24)$$

Proof. Let $\eta_j = \text{Log}_{x_0}(q_j)$. By definition, $\|\eta_j\| \leq \rho_{\max}$ and $\|\eta_{\text{lin}}\| \leq \rho_{\max}$. Applying Lemma 1 with $\delta = \eta_{\text{lin}}$, $\tilde{x} = \tilde{q}_R$,

for each j , we have

$$\mathcal{P}_{\tilde{q}_R \rightarrow x_0} \text{Log}_{\tilde{q}_R}(q_j) = \eta_j - \eta_{\text{lin}} + r_j, \quad \|r_j\|_{x_0} \leq 8C_5\kappa\rho_{\text{max}}^3. \quad (25)$$

By the same first-variation formula [14] applied at $\tilde{q}_R$, $\text{grad } \mathcal{F}(\tilde{q}_R) = -\sum_j \pi_j \text{Log}_{\tilde{q}_R}(q_j)$. Parallel transporting to $T_{x_0}\mathcal{M}$ and using (25), $\mathcal{P}_{\tilde{q}_R \rightarrow x_0} \text{grad } \mathcal{F}(\tilde{q}_R) = -\sum_j \pi_j (\eta_j - \eta_{\text{lin}} + r_j) = -\sum_j \pi_j r_j$, due to the fact $\sum_j \pi_j (\eta_j - \eta_{\text{lin}}) = 0$. Since the parallel transport preserves the Riemannian norm, we have

$$\|\text{grad } \mathcal{F}(\tilde{q}_R)\|_{\tilde{q}_R} \leq 8C_5\kappa\alpha\rho_{\text{max}}^3. \quad (26)$$

Let $\gamma : [0, 1] \rightarrow \mathcal{M}$ be the minimizing geodesic from $\tilde{q}_R$ to $\tilde{q}_R$ with constant speed, remaining in $\mathcal{U}_B$ by Assumption 1. Since $\text{grad } \mathcal{F}(\tilde{q}_R) = 0$, parallel transport along γ and the fundamental theorem of calculus give

$$\mathcal{P}_{\tilde{q}_R \rightarrow \tilde{q}_R} \text{grad } \mathcal{F}(\tilde{q}_R) = \int_0^1 \mathcal{P}_{\gamma(s) \rightarrow \tilde{q}_R} [\text{Hess } \mathcal{F}(\gamma(s)) \dot{\gamma}(s)] ds. \quad (27)$$

Because $\dot{\gamma}(s)$ is parallel along the geodesic, Assumption 1 gives

$$\begin{aligned} \langle \mathcal{P}_{\tilde{q}_R \rightarrow \tilde{q}_R} \text{grad } \mathcal{F}(\tilde{q}_R), \dot{\gamma}(0) \rangle_{\tilde{q}_R} &\geq \lambda_{\min} \int_0^1 \|\dot{\gamma}(s)\|^2 ds \\ &= \lambda_{\min} \text{dist}_g(\tilde{q}_R, \tilde{q}_R)^2. \end{aligned} \quad (28)$$

By Cauchy–Schwarz, $\lambda_{\min} \text{dist}_g(\tilde{q}_R, \tilde{q}_R) \leq \|\text{grad } \mathcal{F}(\tilde{q}_R)\|$, which combined with (26) gives (24) with $C_6 = 8C_5$. $\square$

C. Local Descent under Tangent-Space LQR

In the receding-horizon implementation, the target $\bar{q}$ is instantiated at each replanning step as the tangent-space target $\tilde{q}_R^k$. The following result directly connects the proposed LQR controller to local descent. In particular, the required alignment condition is imposed on the velocity generated by the tangent-space LQR and explicitly accounts for the local target spread.

For a fixed target $\bar{q} \in \mathcal{M}$, define

$$\Psi(x) = \frac{1}{2} \text{dist}_g(x, \bar{q})^2, \quad r(x) = \text{Log}_x(\bar{q}). \quad (29)$$

The first variation formula [14] gives

$$d\Psi_x[v] = -\langle r(x), v \rangle_x. \quad (30)$$

Assumption 2 (LQR alignment and local boundedness). *For the tangent-space LQR controller in Section IV, there exist constants $c_1, c_2, \Gamma, L_v, L_u > 0$ and a sufficiently small neighborhood such that, at each replanning step,*

$$\langle r_k, v_k \rangle_{x_k} \geq c_1 \|r_k\|_{x_k}^2 - c_2 \kappa \rho_k^3, \quad (31)$$

where $r_k = \text{Log}_{x_k}(\bar{q})$, $\rho_k = \max\{\|r_k\|_{x_k}, \sigma_k^{\text{tar}}\}$, $\sigma_k^{\text{tar}} = \max_j \|\eta_j^k\|_{x_k}$, and

$$\sigma_k^{\text{tar}} \leq \Gamma \|r_k\|_{x_k}. \quad (32)$$

Moreover, along the corresponding LQR closed-loop segment $t \in [t_k, t_k + \Delta t]$,

$$\|v(t)\|_{x(t)} \leq L_v \rho_k, \quad \|u(t)\|_{x(t)} \leq L_u \rho_k. \quad (33)$$

The constants are uniform over the selected local neighborhood.

Theorem 2 (Local descent under tangent-space LQR). *Suppose the trajectory generated by the tangent-space LQR remains in a compact strongly convex neighborhood on which Ψ has uniformly bounded Hessian. Under Assumption 2, there exist $\rho^* > 0$, $\Delta t^* > 0$, and $C_7 > 0$ such that, whenever $\rho_k \leq \rho^*$ and $0 < \Delta t \leq \Delta t^*$,*

$$\Psi(x_{k+1}) - \Psi(x_k) \leq -C_7 \Delta t \|r_k\|_{x_k}^2. \quad (34)$$

Thus, the proposed tangent-space LQR yields local descent whenever its closed-loop velocity satisfies the gain-dependent alignment condition (31).

Proof. Let $\rho_k = \max\{\|r_k\|_{x_k}, \sigma_k^{\text{tar}}\}$. By (32),

$$\rho_k \leq \Gamma_\rho \|r_k\|_{x_k}, \quad \Gamma_\rho := \max\{1, \Gamma\}. \quad (35)$$

Along the LQR closed-loop trajectory,

$$\frac{d^2}{dt^2} \Psi(x(t)) = \text{Hess } \Psi(x(t)) [v(t), v(t)] + \langle \text{grad } \Psi(x(t)), u(t) \rangle_{x(t)}. \quad (36)$$

Since the trajectory remains in a compact strongly convex neighborhood, there exists $C_H > 0$ such that $\|\text{Hess } \Psi\| \leq C_H$. Moreover, $\|\text{grad } \Psi(x(t))\| = \text{dist}_g(x(t), \bar{q})$ is locally bounded by $C_r \rho_k$ for some uniform $C_r > 0$. Using (33) therefore gives

$$\left| \frac{d^2}{dt^2} \Psi(x(t)) \right| \leq (C_H L_v^2 + C_r L_u) \rho_k^2 =: C_8 \rho_k^2. \quad (37)$$

Taylor's formula consequently yields

$$\Psi(x_{k+1}) - \Psi(x_k) \leq -\Delta t \langle r_k, v_k \rangle_{x_k} + \frac{1}{2} C_8 \Delta t^2 \rho_k^2. \quad (38)$$

Using (31) and (35),

$$\begin{aligned} \Psi(x_{k+1}) - \Psi(x_k) &\leq -c_1 \Delta t \|r_k\|_{x_k}^2 + c_2 \kappa \Delta t \rho_k^3 + \frac{1}{2} C_8 \Delta t^2 \rho_k^2 \\ &\leq \left[-c_1 + c_2 \kappa \Gamma_\rho^3 \rho_k + \frac{1}{2} C_8 \Gamma_\rho^2 \Delta t \right] \Delta t \|r_k\|_{x_k}^2. \end{aligned} \quad (39)$$

Choose $\rho^* > 0$ and $\Delta t^* > 0$ such that

$$c_2 \kappa \Gamma_\rho^3 \rho^* \leq \frac{c_1}{4}, \quad \frac{1}{2} C_8 \Gamma_\rho^2 \Delta t^* \leq \frac{c_1}{4}. \quad (40)$$

Then, for $\rho_k \leq \rho^*$ and $0 < \Delta t \leq \Delta t^*$,

$$\Psi(x_{k+1}) - \Psi(x_k) \leq -\frac{c_1}{2} \Delta t \|r_k\|_{x_k}^2. \quad (41)$$

Hence (34) holds with $C_7 = c_1/2$. $\square$

Remark 4. The alignment condition (31) is imposed on the velocity generated by the proposed tangent-space LQR, while (33) captures the corresponding local closed-loop bounds. The additional inequality (32) ensures that the target-spread term introduced by the tangent-space approximation is of the same local scale as the tracking error. Therefore, the result provides a direct sufficient condition under which the proposed LQR inherits local descent on the nonlinear manifold.

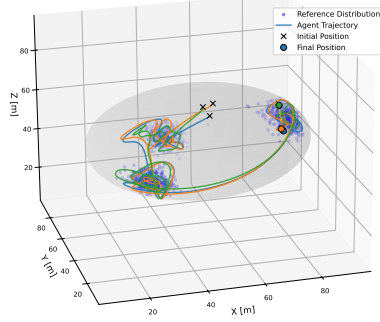


Fig. 1: R -D²OC result: Three-dimensional multi-agent trajectories and reference distribution on the Riemannian ellipsoid manifold.

VI. SIMULATION RESULTS

To validate the theoretical guarantees, we implement the proposed R -D²OC framework on a 3D ellipsoidal manifold, where multiple second-order agents coordinate to match a target distribution over the curved geometry.

Fig. 1 depicts the overall system behavior and spatial trajectories. The agents successfully track the reference distribution along the Riemannian manifold while maintaining smooth coordination paths. Originating from tightly clustered initial positions (black crosses), trajectories evolve smoothly across the surface to match their time-averaged distribution with the targets (blue dots). This confirms that the second-order tangent-space LQR formulation respects geometric constraints while driving the team toward the desired distribution.

To evaluate the geometric approximation properties and empirically assess Theorem 1, we analyze the log-log scaling relationship in Fig. 2(a). The empirical approximation bias exhibits a fitted slope of 2.73, which is consistent with the theoretical cubic-order $\mathcal{O}(\rho_{\max}^3)$ upper bound. This observation supports the predicted local scaling of the tangent-space center bias as the target spread varies.

We next evaluate the local descent property in Theorem 2 using Fig. 2(b). The closed-loop trajectories satisfy the alignment condition of Assumption 2 with $c_1 = 1.40$, yielding $C_7 = c_1/2 = 0.70$ in (34). Within the empirical local neighborhood $\rho_k \leq \rho^* \approx 4.1$ (shaded region), the observed objective decrease aligns well with the theoretical bound $-C_7 \Delta t \|r_k\|^2$. Beyond ρ^* , this local bound no longer holds, which is consistent with the conditional and local nature of Theorem 2.

With an average per-agent Riccati solve time of 0.49 ms relative to the 0.1 s replanning interval Δt , the computation remains well within the budget for onboard execution.

VII. CONCLUSION

This paper presented an intrinsic R -D²OC framework for second-order multi-agent systems on Riemannian manifolds. By mapping the intrinsic transport problem to local tangent spaces, the framework reduces online optimization to linear-quadratic subproblems with computational complexity de-

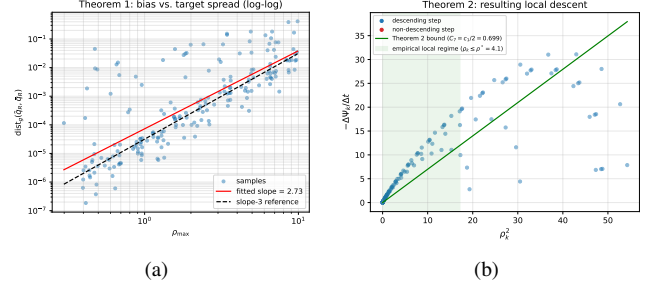


Fig. 2: Empirical verification of main theoretical results. (a) Approximation bias $\text{dist}_g(\tilde{q}_R, \bar{q}_R)$ versus target spread, validating the third-order scaling in Theorem 1. (b) Empirical descent rate $-\Delta\Psi_k/\Delta t$ versus ρ_k^2 , with the shaded region marking the neighborhood $\rho_k \leq \rho^* \approx 4.1$ where Theorem 2 holds.

pending strictly on the manifold dimension. We provided rigorous theoretical guarantees, including third-order barycentric bias bounds and a conditional discrete-time descent property. Numerical simulations on the sphere validated these theoretical results, confirming efficient distribution control on curved spaces.

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