Time and Supply Fairness in Electricity Distribution using k-times bin packing *

Dinesh Kumar Baghel[1,2][0000−0001−8518−7908], Alex Ravsky[3], and Erel Segal-Halevi[1][0000−0002−7497−5834]

[1] Ariel University, Ariel 40700, Israel
{dinkubag21, erelsgl}@gmail.com
[2] UPES, Dehradun, Uttarakhand, India
[3] Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine, Lviv, Ukraine
alexander.ravsky@uni-wuerzburg.de

Abstract. Given items of different sizes and a fixed bin capacity, the bin-packing problem is to pack these items into a minimum number of bins such that the sum of item sizes in a bin does not exceed the capacity. We define a new variant called *k-times bin-packing (kBP)*, where the goal is to pack the items such that each item appears exactly k times, in k different bins. We generalize some existing approximation algorithms for bin-packing to solve kBP, and analyze their performance ratio.

The study of kBP is motivated by the problem of *fair electricity distribution*. In many developing countries, the total electricity demand is higher than the supply capacity. Our goal is to allocate the available supply among the households according to the egalitarian principle, which aims to maximize the smallest amount of time a household is connected to electricity.

We prove that every electricity division problem can be solved by k-times bin-packing for some finite k, that depends only on the number of households. We also show that k-times bin-packing can be used in practice to distribute the electricity in a fair and efficient way. Particularly, we implement generalizations of the First-Fit and First-Fit Decreasing bin-packing algorithms to solve kBP, and apply the generalizations to real electricity demand data. We show that our generalizations outperform existing heuristic solutions to the same problem in terms of the egalitarian allocation of connection time.

We then study another variant of the egalitarian allocation problem, in which the goal is to maximize the smallest amount of watts allocated to a household. For this variant, we prove an impossibility result: there does not exist such a k that depends only on the number of agents. This impossibility result motivates us to develop four different heuristic algorithms to solve the egalitarian allocation of watts problem. We evaluate the heuristics using the sum of the minimum watts allocated to any household in each hour, providing a fairness metric that reflects the lowest watt allocation in all hours. A higher total minimum of watts indicates a more equitable distribution. Thus we establish new benchmarks for fair allocation of watts.

Keywords: Approximation algorithms · bin-packing · First-Fit · First-Fit Decreasing · Next-Fit · fair division · Karmarkar-Karp algorithms · Fernandez de la Vega-Lueker algorithm · electricity distribution · utilitarian metric · egalitarian metric · utility difference.

1 Introduction

This work is motivated by the problem of *fair electricity distribution*. In developing countries, the demand for electricity often surpasses the available supply [31]. Such countries have to come up with a fair and efficient method of allocating of the available electricity among the households.

Formally, we consider a power-station that produces a fixed supply S of electricity. The station should provide electricity to n households. The demands of the households in a given period are given by a (multi)set D. Typically, $\sum_i D[i] > S$ (where $D[i]$ is the electricity demand of a household i), so it is not possible to connect all households simultaneously.

* This is an extended version of our paper presented at SAGT-2024 (https://link.springer.com/chapter/10.1007/978-3-031-71033-9_27). This version corrects a bug present in the accepted SAGT paper that affected Section 8.2. In addition, it adds the following over the conference version of the paper We give a lower bound on k that depends on the number of households (see Section 4.2); In Section 7, we study another version of the egalitarian allocation, that is, the egalitarian allocation of watts problem. In this direction, we present impossibility results that state that there does not exist a finite k that depends on the number of households. In addition, we develop four heuristic algorithms to solve the egalitarian allocation of watts problem.

Egalitarian allocation of connection time: In the basic variant of the problem, the goal is to ensure that each household is connected the same amount of time, and that this amount is as large as possible. We assume that an agent gains utility only if the requested demand is fulfilled; otherwise it is zero. Practically it can be understood as follows: Suppose at some time, a household i is running some activity that requires $D[i]$ kilowatt of electricity to operate; in the absence of that amount, the activity will not function. Therefore, an allocation where demands are fractionally fulfilled is not relevant. We also assume that an agent's utility does not increase for receiving electricity more than his/her demand, as large batteries for storing a substantial amount of electricity are still expensive and not very common in developing countries.

While it is not necessary that agents consume electricity equal to their demand. In addition, some agents may consume more electricity than their demand and some agents consume less. It may lead to the cancellation of under and excess electricity consumption. We also assume that any uncertainty in demand is met by emergency power by the grid. Such assumptions have been made in [35]. These assumptions are necessary to cover uncertainties in the demand consumption by some agents.

A simple approach to this problem is to partition the households into some q subsets such that the sum of demands in each subset is at most S, and then connect the agents in each subset for a fraction $1/q$ of the time. To maximize the amount of time each agent is connected, we have to minimize q. This problem is equivalent to the classic problem of *bin-packing*. In this problem, we are given some n items, of sizes given by a multiset of positive numbers D, and a positive number S representing the capacity of a bin. The goal is to pack items in D in the smallest possible number of bins, so that the sum of the item sizes in each bin is at most S. The problem is NP-complete [18], but it has many efficient approximation algorithms.

However, even an optimal solution to the bin-packing problem may provide a suboptimal solution to the electricity division problem. As an example, suppose that we have three households x, y, z with demands $2, 1, 1$, respectively, and an electricity supply $S = 3$. Then, the optimal bin-packing results in 2 bins, for instance, $\{x, y\}$ and $\{z\}$. This means that each agent would be connected $1/2$ of the time. However, it is possible to connect each agents for $2/3$ of the time, by connecting each pair $\{x, y\}, \{x, z\}, \{y, z\}$ for $1/3$ of the time, as each agent appears in 2 different subsets. More generally, suppose that we construct q subsets of agents, such that each agent appears in exactly k different subsets. Then we can connect each subset for $1/q$ of the time, and each agent will be connected for k/q of the time.

1.1 The k-times bin-packing problem

To study this problem more abstractly, we define the *k-times bin-packing problem (or kBP)*. The input to kBP is a set of n items of sizes given by a multiset D, a positive number S representing the capacity of a bin, and an integer $k \geq 1$. The goal is to pack items in D in the smallest possible number of bins, such that the sum of the item sizes in each bin is at most S, and each item appears in k different bins, where each item occurs at most once in a bin. In the above example, $k = 2$. It is easy to see that, in the above example, 2-times bin-packing yields the optimal solution to the electricity division problem.

Our first main contribution **(Section 4)** is to prove ,that for every electricity division problem, there exists some finite k for which the optimal solution to the kBP problem yields the optimal solution to the electricity division problem. Moreover, we give an explicit upper bound on k, as a function of the number of households.

We note that kBP may have other applications beyond the electricity division. For example, it could be used to create a backup of files on different file servers [27]. We would like to store k different copies of each file, with at most one copy of the same file on the same server. This can be solved by solving kBP on the files as items, and the server disk space as the bin capacity.

Motivated by these applications, we would like to find ways to efficiently solve kBP . However, it is well-known that kBP is NP-hard even for $k = 1$. We therefore look for efficient approximation algorithms for kBP .

1.2 Using existing bin-packing algorithms for kBP

Several existing algorithms for bin-packing can be naturally extended to kBP. However, it is not clear whether the extension will have a good approximation ratio.

As an example, consider the simple algorithm called *First-Fit (FF)*: process the items in an arbitrary order; pack each item into the first bin it fits into; if it does not fit into any existing bin, open a new bin for it. In the example $D = [10, 20, 11], S = 31$, the FF would pack two bins: $\{10, 20\}$ and $\{11\}$. This is

clearly optimal. The extension of FF to kBP would process the items as follows: for each item x_r in the list (in order), suppose that the b bins have been used thus far. Let j be the lowest index ($1 \leq j \leq b$) such that (a) the bin j can accommodate x_r and (b) the bin j does not contain any copy of x_r, should such j exist; otherwise open a new bin with index $j = b + 1$. Place x_r in the bin j.

There are two ways to process the input. One way is by processing each item k times in sequence. In the above example, with $k = 2$, FF will process the items in order $[10^1, 10^2, 20^1, 20^2, 11^1, 11^2]$, where the superscript specifies the instance to which an item belongs. This results in four bins: $\{10^1, 20^1\}, \{10^2, 20^2\}, \{11^1\}, \{11^2\}$, which simply repeats k times the solution obtained from FF on D. However, the optimal solution here is 3 bins: $\{10^1, 20^1\}, \{11^1, 10^2\}, \{20^2, 11^2\}$.

Another way is to process the whole sequence D, k times. In the above example, FF will process the sequence $D_2 = DD = [10^1, 20^1, 11^1, 10^2, 20^2, 11^2]$. Applying the FFk algorithm to this input instance will result in three bins $\{10^1, 20^1\}, \{11^1, 10^2\}, \{20^2, 11^2\}$, which is optimal. Thus, while the extension of FF to kBP is simple, it is not trivial, and it is vital to study the approximation ratio of such algorithms in this case.

As another example, consider the approximation schemes of de la Vega and Lueker [41] and Karmarkar and Karp [30]. These algorithms use a linear program that counts the number of bins of each different *configuration* in the packing (see Section 6.1 for the definitions) One way in which these algorithms can be extended, without modifying the linear program, is to give D_k as input. But then a configuration might have more than one copy of an item in D, which violates the kBP constraint. Another approach is to modify the constraint in the configuration linear program, to check that there are k copies of each item in the solution, while keeping the same configurations as for the input D. Doing so will respect the kBP constraint. Again, while the extension of the algorithm is straightforward, it is not clear what the approximation ratio would be; this is the main task of the present paper.

The most trivial way to extend existing algorithms is to run an existing bin-packing algorithm, and duplicate the output k times. However, this will not let us enjoy the benefits of kBP for electricity division (in the above example, this method will yield 4 bins, so each agent will be connected for $2/4 = 1/2$ of the time). Therefore, we present more elaborate extensions, that attain better performance. The algorithms we extend can be classified into two classes:

1. Fast constant-factor approximation algorithms ***(Section 5)***. Examples are First-Fit (FF) and First-Fit-Decreasing (FFD). For bin-packing, these algorithms find a packing with at most $1.7 \cdot OPT(D)$ and $\frac{11}{9} \cdot OPT(D) + \frac{6}{9}$ bins respectively [11,10,12] We adapt these algorithms by running them on an instance made of k copies, $DD \ldots D$ (k copies of D), which we denote by D_k. We show that, for $k > 1$, the extension of FF to kBP (which we call FFk) finds a packing with at most $\left(1.5 + \frac{1}{5k}\right) \cdot OPT(D_k) + 3 \cdot k$ bins. For any fixed $k > 1$, the asymptotic approximation ratio of FFk for large instances (when $OPT(D_k) \to \infty$) is $(1.5 + \frac{1}{5k})$, which is better than that of FF, and improves towards 1.5 when k increases.

We also prove that the lower bound for $FFDk$ (the extension of FFD to kBP) is $\frac{7}{6} \cdot OPT(D_k) + 1$, and conjecture by showing on simulated data that $FFDk$ solves kBP with at most $\frac{11}{9} \cdot OPT(D_k) + \frac{6}{9}$ bins which gives us an asymptotic approximation ratio of at most 11/9.

We also show that the extension of NF (next-fit algorithm) to kBP (we call this extension NFk) has the asymptotic ratio of 2.

2. Polynomial-time approximation schemes ***(Section 6)***. Examples are the algorithms by Fernandez de la Vega and Lueker [41] and Karmarkar and Karp [30]. We show that the algorithm by Fernandez de la Vega and Lueker can be extended to solve kBP using at most $(1 + 2 \cdot \epsilon)OPT(D_k) + k$ bins for any fixed $\epsilon \in (0, 1/2)$. For every $\epsilon > 0$, Algorithm 1 of Karmarkar and Karp [30] solves kBP using at most $(1 + 2 \cdot k \cdot \epsilon)OPT(D_k) + \frac{1}{2 \cdot \epsilon^2} + (2 \cdot k + 1)$ bins, and runs in time $O(n(D_k) \cdot \log n(D_k) + T(\frac{1}{\epsilon^2}, n(D_k))$, where $n(D_k)$ is the number of items in D_k, and T is a polynomially-bounded function. Algorithm 2 of Karmarkar and Karp [30] can be generalized to solve kBP using at most $OPT(D_k) + O(k \cdot \log^2 OPT(D))$ bins, and runs in time $O(T(\frac{n(D)}{2}, n(D_k)) + n(D_k) \cdot \log n(D_k))$.

1.3 Egalitarian allocation of watts (Section 7)

At a given point in time, the equal supply allocation problem concerns the allocation of electricity as equal as possible (in kW) for agents that are not connected to electricity all the time, as these are the agents that do not get their stipulated demand.

Example 1.1. *The input set D contains three agents x, y, z with demands $1, 2,$ and 3 respectively and supply $S = 5$. An equal allocation of time would result in the following possible packing of items in*

D in 3 *bins:* $\{1,2\},\{2,3\},\{3,1\}$. *The allocation of supply in this case would be:* $0.667, 1.33,$ *and* 2 *watts respectively. The solution is fair w.r.t. the equal allocation of time, but unfair w.r.t. the equal allocation of supply. It is unfair to agents with low demand. In this example, the solution is unfair to the agent x with demand* 1. *A better solution in this example could have* 5 *bins in total:* $\{1,2\},\{1,2\},\{1,2\},\{1,3\},\{1,3\}$. *It will yield the allocation of supply* $1, 1.2,$ *and* 1.2 *for the agents $x, y,$ and z respectively. This solution is much better w.r.t to the equal allocation of supply.*

Now, we describe our solution approaches for equal supply allocation problem that distributes the electricity (in kW) as equal as possible:

- We derive an instance from the given input instance. In this derived instance, the sum of the item sizes of all the occurrences of items is as equal as possible. For example, consider the input instance $D = \{x = 1, y = 2, z = 3\}$ and supply $S = 5$. Let $D' = \{1, 2, 3, 1, 2, 3, 1, 2, 1, 1, 1\}$ be the instance derived from D. The sum of item sizes of all the occurrences of items x, y, z is $6, 6, 6$ respectively.
- Now, if we use any algorithm that connects the agents equally in terms of time, it will result in an equal allocation of supply for the instance derived in the above step. For the above example, a possible solution could result in 6 bins: $\{1,2\},\{3,1\},\{2,3\},\{1,2\},\{1\},\{1\},\{1\}$. Connecting each bin $1/6$ of the time will result in $1, 1, 1$ kW of supply for agents x, y, z respectively.

However, the above approach suffers if there are some items with very small item sizes compared to the largest item size. Imagine the situation where the smallest item in the instance is, say, 0.05 and the maximum item size in the instance is, say, 14. In this case, there are 280 copies of this smallest item w.r.t a single copy of the maximum item size. The final derived instance will be a large instance in terms of the number of items. Moreover, in this case, the equal supply allocation is limited by the small item size i.e. agent with max demand 14 will not get electricity more than 0.05(kW).

However, it is possible to improve the solution: we connect the agent with demand $x = 1$ all the time. Now, the remaining agents are packed in the bin whose capacity is $S' = 5 - 1 = 4$ and the structure of each such bin is $\{1,\}$. After packing the remaining items, a possible final solution could be $\{1,2\},\{1,3\}$. Since the agent $x = 1$ is connected all the time, connecting each bin $1/2$ of the time will result in the allocation of supply w.r.t the remaining agents as $1, 1.5$ which better than the previously obtained solution. Overall, the allocation vector of supply $(1, 1, 1.5)$ is preferred over $(1, 1, 1)$.

More generally, suppose we find a set of agents that are connected all the time, then for the remaining agents and the remaining supply we can determine a better allocation vector of supply. In Section 7, we discuss four heuristic approaches to solve the egalitarian allocation of Watts problem.

Simulation experiments ***(Section 8.1 and Section 8.3)*** We complement our theoretical analysis of the approximation ratios of FFk and $FFDk$ bin-packing algorithms with simulation experiments, which validate our conjectures in Section 5.1 and Section 5.2 in Section 8.1. Further, more results that shows the effect of different values of k and uncertainty in the agent's demand has on the number of connection hours, comfort and electricity supplied have been discussed in Section J.

The main objective of heuristics algorithms (HA1, HA2, HA3, and HA4) is to distribute electricity as equally as possible. The computational results of these heuristics algorithms in terms of electricity allocated in watts have been discussed in Section 8.3. More results that compare these heuristics in terms of hours of connection to supply, and comfort delivered have been discussed in Section J.1.

Electricity distribution ***(Section 8)*** The fair electricity division problem was introduced by Oluwasuji, Malik, Zhang and Ramchurn [34,35] under the name of "fair load-shedding". They presented several heuristic as well as ILP-based algorithms, and tested them on a dataset of 367 households from Nigeria. We implement the FFk and $FFDk$ algorithms for finding approximate solutions to kBP, and use the solutions to determine a fair electricity allocation. We test the performance of our allocations on the same dataset of Oluwasuji, Malik, Zhang and Ramchurn [34]. We compare our results on the same metrics used by Oluwasuji, Malik, Zhang and Ramchurn [34]. These metrics are utilitarian and egalitarian social welfare and the maximum utility difference between agents. We compare our results in terms of hours of connection to supply on average, utility delivered to an agent on average, and electricity supplied on average, along with their standard deviation. We find that our results surmount their results in terms of the egalitarian allocation of connection time to the electricity FFk and $FFDk$ run in time that is nearly linear in the number of agents. We conclude that using kBP can provide a practical, fair and efficient solution to the electricity division problem where the objective is to connect each agent as much as possible.

In the same Section 8, we present computational results of heuristic algorithms developed for egalitarian allocation of watts. We find that heuristic algorithm HA1 with $FFDk$ outperforms all other heuristics in supplying more electricity (in terms of watts) to the worst-off agent.

In **Section 9**, we conclude with a summary and directions for future work. We defer most of the technical proofs and algorithms to the Appendix.

2 Related Literature

First-Fit. We have already defined the working of FF in Section 1.2. Denote by FF the number of bins used by the First-Fit algorithm, and by OPT the number of bins in an optimal solution for a multiset D. An upper bound of $FF \leq 1.7OPT + 3$ was first proved by Ullman in 1971 [40]. The additive term was first improved to 2 by Garey, Graham and Ullman [17] in 1972. In 1976, Garey, Graham, Johnson and Yao [19] improved the bound further to $FF \leq \lceil 1.7OPT \rceil$, equivalent to $FF \leq 1.7OPT + 0.9$ due to the integrality of FF and OPT. This additive term was further lowered to $FF \leq 1.7OPT + 0.7$ by Xia and Tan [45]. Finally, in 2013 Dosa and Sgall [11] settled this open problem and proved that $FF \leq \lfloor 1.7OPT \rfloor$, which is tight.

First-Fit Decreasing. Algorithm *First-Fit Decreasing* (FFD) first sorts the items in non-increasing order, and then implements FF on them. In 1973, in his doctoral thesis [29], D. S. Johnson proved that $FFD \leq \frac{11}{9}OPT + 4$. Unfortunately, his proof spanned more than 100 pages. In 1985, Baker [3] simplified their proof and improved the additive term to 3. In 1991 Minyi [46] further simplified the proof and showed that the additive term is 1. Then, in 1997, Li and Yue [32] narrowed the additive constant to 7/9 without formal proof. Finally, in 2007 Dosa [10] proved that the additive constant is 6/9. They also gave an example which achieves this bound.

Next-fit. The algorithm next-fit works as follows: It keeps the current bin (initially empty) to pack the current item. If the current item does not pack into the currently open bin then it closes the current bin and opens a new bin to pack the current item. Johnson in his doctoral thesis [29] proved that the asymptotic performance ratio of next-fit is 2.

Efficient approximation schemes. In 1981, Fernandez de la Vega and Lueker [41] presented a polynomial time approximation scheme to solve bin-packing. Their algorithm accepts as input an $\epsilon > 0$ and produces a packing of the items in D of size at most $(1 + \epsilon)\, OPT + 1$. Their running time is polynomial in the size of D and depends on $1/\epsilon$. They invented the *adaptive rounding* method to reduce the problem size. In adaptive rounding, they initially organize the items into groups and then round them up to the maximum value in the group. This results in a problem with a small number of different item sizes, which can be solved optimally using the linear configuration program. Later, Karmarkar and Karp [30] devised several PTAS for the bin-packing problem. One of the Karmarkar-Karp algorithms solves bin-packing using at most $OPT + O(\log^2 OPT)$ bins. Other Karmarkar–Karp algorithms have different additive approximation guarantees, and they all run in polynomial time. This additive approximation was further improved to $O(\log OPT \cdot \log \log OPT)$ by Rothvoss [36]. They used a "glueing" technique wherein they glued small items to get a single big item. In 2017, Hoberg and Rothvoss [25] further improved the additive approximation to a logarithmic term $O(\log OPT)$.

Bin-packing with constraints. Jansen [27] has proposed a FPTAS for the generalization of the bin-packing problem called *bin-packing with conflicts*. The input instance for their algorithm is the conflict graph. Its vertices are the items and any two items are adjacent provided they cannot be packed into the same bin. In particular, kBP can be considered as the bin-packing with conflicts, where the conflict graph D_k is a disjoint union of copies of a complete graph K_k. Their bin-packing problem with conflicts is restricted to $q-$inductive graphs. In a $q-$inductive graph the vertices are ordered from $1, \ldots, n$. Each vertex in the graph has at most q adjacent lower numbered vertices. Since the degree of each vertex of D_k equals $k-1$, D_k is a $k-1$-inductive graph. In their method first they obtain an instance of large items from the given input instance. Let this instance be J_k. They apply the linear grouping method of Fernandez de la Vega and Lueker [41] to obtain a constant number of different item sizes. Next they apply the Karmarkar and Karp algorithm [30] to obtain an approximate packing of the large items. The bins in this approximate packing may have conflicts, so they use the procedure called COLOR which places each conflicted item into a new bin. In the worst case it may happen that all the items in each bin have conflict and hence each one of them is packed into a separate bin. Finally, after removing the

conflicts, they pack the small items into the existing bins, respecting conflicts among items. In doing so, new bins are opened if necessary.

For the input instance I that consists of a $q-$inductive graph their algorithm packs the items using at most $(1+2\cdot\epsilon)OPT(I)+\frac{2\cdot q+1}{\epsilon^2}+3\cdot q+4$ bins. In case of kBP , I equals D_k and q is $k-1$.

Their algorithm solves the kBP using at most $(1+2\cdot\epsilon)OPT(D_k)+\frac{2\cdot k-1}{\epsilon^2}+3\cdot k+1$ bins.

In this paper we focus on a special kind of conflicts, and for this special case, we present a better approximation ratio: our extension to Algorithm 1 and 2 of Karmarkar-Karp algorithms solves the kBP using at most $(1+2\cdot\epsilon)OPT(D_k)+\frac{1}{2\cdot\epsilon^2}+2\cdot k+1$ and $OPT(D_k)+O(k\cdot\log^2 OPT(D))$ bins respectively.

Gendreau, Laporte and Semet [20] propose six heuristics named H1 to H6 for bin-packing with item-conflicts, where the input instance is represented by a general conflict graph. The heuristic H1 is a variant of FFD which incorporates the conflicts, whereas H6 is a combination of a maximum-clique procedure and FFD. They show that H6 is better than H1 for conflict graphs with high density, whereas H1 performs marginally better for low density conflict graphs (where density is defined as the ratio of the number of edges to the number of possible edges). kBP can be represented by duplicating each item k times, and constructing a conflict graph in which there are edges between each two copies of the same item. The density of this graph is $(k-1)/(kn-1)$, which becomes smaller for large n. This suggests that H1 is a better fit for kBP .

But if we use their method to solve kBP , then the vertices of the conflict graphs are ordered as blocks corresponding to the items, in such a way that their sizes are non-decreasing. We have already seen in Section 1.2 that when we change such item order we can obtain a better packing.

Ekici [14] has given a heuristic solution based on linear programming for the bin packing problem with item-conflicts (BPPC) and item fragmentation (BPPIF [6][7]). In their problem an item can be fragmented, and also an item can have size larger than the bin capacity. They have also represented the problem by a general conflict graph. In contrast to their problem, our problem have item sizes at most the bin capacity and also we do not allow an item to be fragmented. Hence, their solutions are not directly applicable in our case. Moreover, they do not consider the number of fragments an item can have in their solution.

Gupta and Ho [23], have developed the MBS (minimum bin slack) procedure that determines the total possible subset of items that can fill the bin and that leave the minimum space remaining in the bin. Their heuristic algorithm finds an optimal solution where the total sum of all items is at most twice the bin capacity. The complexity of their procedure is exponential in input size. Fleszar and Hindi [16], suggested heuristic algorithms based on MBS to save computation. These improvements first pack the large item in a bin leaving small space for other items to pack, therefore, resulting in a small exponential-time search for the remaining items to fill the bin. In electricity distribution problem, the supply is very large as compared to the demand of the households. Therefore, adoption of such heuristic will be computationally expensive.

Recently, Doron-Arad, Kulik and Shachnai [9] have solved in polynomial time a more general variant of bin-packing, with partition matroid constraints. Their algorithm packs the items in $OPT+O\left(\frac{OPT}{(\ln\ln OPT)^{1/17}}\right)$ bins. Their algorithm can be used to solve the kBP: for each item in D, define a category that contains k items with the same size. Then, solve the bin-packing with the constraint that each bin can contain at most one item from each category (it is a special case of a partition-matroid constraint). However, in the present paper we focus on the special case of kBP. This allows us to attain a better running-time (with FFk and $FFDk$), and a better approximation ratio (with the de la Vega–Lueker and Karmarkar–Karp algorithms).

Cake cutting and electricity division The electricity division problem can also be modeled as a classic resource allocation problem known as 'cake cutting'. The problem was first proposed by Steinhaus [39]. A number of cake-cutting protocols have been discussed in [4,42]. In cake cutting, a cake is a metaphor for the resource. Like previous approaches, a time interval can be treated as a resource. A cake-cutting protocol then allocates this divisible resource among agents who have different valuation functions (or preferences) according to some fairness criteria. The solution to this problem differs from the classic cake-cutting problem in the sense that at any point in time, t, the sum of the demands of all the agents, should respect the supply constraint, and several agents may share the same piece.

Other solutions to fair load shedding. Load-shedding strategy at the bus level has been discussed in [37]. They have dealt with the issue of cascading failure, which may occur in line contingencies. So when a line contingency happens, first they do load-shedding to avoid cascading failure. This load-shedding may reduce the loads beyond what is necessary (let's call this unnecessary load) to prevent cascading

failure. Then, they recover as much unnecessary load as possible in the second step while maintaining the system's stability. Shi and Liu [37] presented a distributed algorithm for compensating agents for load-shedding based on the proportional-fairness criterion. In contrast, we present a centralized algorithm for computing load-shedding that attains a high egalitarian welfare.

In a smart grid, total distributed electricity combines utility supply and distributed renewable sources (DERs). [26] proposes a fair, starvation-free mechanism allocating energy among microgrids so each receives a portion of its demand at lower cost. The method uses two phases: the first proportionally distributes DER energy; the second uses a Vickrey auction for remaining DER energy. If microgrid demand persists, and DER energy is fully allocated, any unmet need is addressed by purchasing from the utility grid. However, for households, when demand exceeds supply, fractional allotments may hinder critical activities. Bus-level load shedding can be inequitable. An equity-aware model using the grid Gini coefficient ensures fairer load curtailment across buses, but may increase total load shed and cost [15]. Each time an agent connects, recalculating the Gini coefficient is computationally expensive for large groups.

Gerding et. al. [21] proposed a model-free mechanism for coordinating electric vehicle charging that ensures truthfulness by leaving some units unallocated, which return to the grid but are wasted when the purpose is to distribute electricity to households. In [38], the mechanism pre-commits to fulfill each selected agent's exact demand, assumes no additive value for extra resource, and allows allocations below the reported maximum, in contrast to our mandatory full-demand model.

Buermann et. al. [5] studied variable-energy allocation over time with agents' linear satiable valuations, allowing partial fulfillment, while our model uses piecewise constant utility so agents gain only if their full demand is met; partial allocation is worthless.

Mechanisms in which a load is shifted from peak hours to non-peak hours has been studied in [2,1] However, in this research, we assumed that an agent's utility for fractional allocation of their demand is 0, and also shifting an agent's demand to other time-intervals may result in inhibiting an agent from performing some critical activities.

3 Definitions and Notation

3.1 Electricity division problem

The input to the Electricity Division problem consists of:

- A number $S > 0$ denoting the total amount of available supply (e.g. in kW);
- A number n of households, and a list $D = D[1], \ldots, D[n]$ of positive numbers, where $D[i]$ represents the demand of households i (in kW);
- An interval $[0, T]$ representing the time in which electricity should be supplied to the households.

The desired output consists of:

- A partition $\mathcal{I}$ of the interval $[0, T]$ into sub-intervals, $I_1, \ldots, I_p$;
- For each interval $l \in [p]$, a set $A_l \subseteq [n]$ denoting the set of agents that are connected to electricity during interval l, such that $\sum_{i \in A_l} D[i] \leq S$ (the total demand is at most the total supply).

In terms of connection time to electricity, the utility of agent i equals the total time agent i is connected: $u_i^t(\mathcal{I}) = \sum_{l:i \in A_l} |I_l|$ (we will consider other utility functions in Section 8).

The optimization objective is

$$\max_{\mathcal{I}} \min_{i \in [n]} u_i^t(\mathcal{I}),$$

where the maximum is over all partitions that satisfy the demand constraints. This max-min value is called the *egalitarian connection-time* of the given instance.

In terms of allocation of electricity supply, utility of an agent i is defined as: $u_i^s(\mathcal{I}) = \sum_{l:i \in A_l} D[i] \cdot |I_l|$. The optimization objective is:

$$\max_{\mathcal{I}} \min_{i \in [n]} u_i^s(\mathcal{I})$$

3.2 k-times bin packing

We denote the bin capacity by $S > 0$ and the multiset of n items by D.

Let $n(D)$ and $m(D)$ denote the number of items and the number of different item sizes in D, respectively. We denote these sizes by $c[1], \ldots, c[m(D)]$. Moreover, for each natural $i \leq m(D)$ let $n[i]$ be the number of items of size $c[i]$. The *size* of a bin is defined as the sum of all the item sizes in that bin. Given a multiset B of items, we denote by $V(B)$ its *size*, defined asthe sum of the sizes of all items of B.

We denote k copies of D by $D_k := DD \ldots D$. We denote the number of bins used to pack the items in D_k by the optimal and the considered algorithm by $OPT(D_k)$ and $A(D_k)$, respectively, where A is the algorithm used.

Note that each item in D_k is present at most once in each bin, so it is present in exactly k distinct bins. Consider the example in Section 1. There are three items x, y, z with demand $2, 1, 1$ respectively. Let $k = 2$ and $S = 3$. Then, $\{x, y\}, \{y, z\}, \{z, x\}$ is a valid bin-packing. Note that each item is present twice overall, but at most once in each bin. In contrast, the bin-packing $\{x, y\}, \{y, z, z\}, \{x\}$ is not valid, because there are two copies of z in the same bin.

In the next section, we will show that for every electricity division problem, a finite value of k can be found such that solving the kBP problem optimally will also produce the optimal solution for the electricity division problem.

4 On optimal k for k-times bin-packing

In this section we prove that, for every electricity division instance, there exists an integer k such that kBP yields the optimal electricity division. Moreover, we give an upper bound on k as a function of the number of agents.

4.1 Upper bound

We start with some useful definitions and lemmas from linear algebra.

Let X be a nonempty set. We denote by $\mathbb{R}^X$ the linear space of all functions from X to the real numbers $\mathbb{R}$. So elements of $\mathbb{R}^X$ have the form $(w_\alpha)_{\alpha \in X}$, where for each element $\alpha \in X$, w_α is the corresponding real number.

For each nonempty subset Y of X, let $\pi_Y : \mathbb{R}^X \to \mathbb{R}^Y$ be the natural projection, which maps each element $(w_\alpha)_{\alpha \in X} \in \mathbb{R}^X$ to the element $(w_\alpha)_{\alpha \in Y} \in \mathbb{R}^Y$.

Let $W \subseteq \mathbb{R}^X$ be a linearly independent set, and $Y \subseteq X$ a subset of the indices. In general, the projection $\pi_Y(W)$ might not be linearly independent. For example, if $X = \{1, 2, 3, 4, 5\}$ and $W = \{[1, 2, 0, 0, 0], [2, 4, 1, 0, 0], [3, 6, 0, 0, 1]\}$ and $Y = \{1, 2\}$, then $\pi_Y(W) = \{[1, 2], [2, 4], [3, 6]\}$, which is linearly dependent.

We shall need the following lemmas.

Lemma 4.1. *Let $W \subseteq \mathbb{R}^X$ be a nonempty finite linearly independent set. Then there exists a subset Y of X with $|Y| = |W|$ such that the set $\pi_Y(W)$ is linearly independent.*

In the above example, as $|W| = 3$, the lemma says that there exists a subset Y containing 3 indices, such that the projection of W on these indices is still linearly independent. Indeed, in this case we can take $Y = \{1, 3, 5\}$, as $\pi_Y(W) = \{[1, 0, 0], [2, 1, 0], [3, 0, 1]\}$, which is linearly independent.

Proof. Let $q = |W|$. We prove the required claim by induction on q. The base case is $q = 1$. Then W consists of a single nonzero vector w. Therefore there exists $\alpha \in X$ such that the αth entry of w is non-zero. Put $Y = \{\alpha\}$. Then the vector $\pi_Y(w)$ is non-zero and so the set $\{\pi_Y(w)\}$ is linearly independent.

Now suppose that the required claim holds for $q - 1$. Pick any vector $w' \in W$ and put $W' = W \setminus \{w'\}$. By the induction assumption, there exists a subset Y' of X with $|Y'| = q - 1$ such that the set $\pi_{Y'}(W')$ is linearly independent. As $|W'| = q - 1$, adding a single vector $\pi_{Y'}(w')$ to $\pi_{Y'}(W')$ makes it linearly dependent. Therefore there exists a unique vector of coefficients $(\lambda_v)_{v \in W'}$ such that $\pi_{Y'}(w') = \sum_{v \in W'} \lambda_v \pi_{Y'}(v)$. Since the set W is linearly independent, $\sum_{v \in W'} \lambda_v v \neq w'$, so there exists $\alpha \in X \setminus Y'$ such that $w'_\alpha \neq \sum_{v \in W'} \lambda_v v_\alpha$. Put $Y = Y' \cup \{\alpha\}$.

We claim that the set $\pi_Y(W)$ is linearly independent. Indeed, suppose for a contradiction that there exist coefficients $(\lambda'_v)_{v \in W}$ which are not all zeroes such that $\sum_{v \in W} \lambda'_v \pi_Y(v) = 0$. Since the set $\pi_{Y'}(W')$ is linearly independent, the set $\pi_Y(W')$ is linearly independent too, so $\lambda'_{w'} \neq 0$. Then $\pi_Y(w') = \sum_{v \in W'} (-\lambda'_v / \lambda'_{w'}) \pi_Y(v)$, and so $\pi_{Y'}(w') = \sum_{v \in W'} (-\lambda'_v / \lambda'_{w'}) \pi_{Y'}(v)$. The uniqueness of

$(\lambda_v)_{v\in W'}$ ensures that $-\lambda'_v/\lambda'_{w'} = \lambda_v$ for each $v \in W'$. But $w'_\alpha \neq \sum_{v\in W'} \lambda_v v_\alpha = \sum_{v\in W'}(-\lambda'_v/\lambda'_{w'})v_\alpha$, a contradiction.

Thus the required claim holds for q. □

Let X be a nonempty set and let $Z \subset \{0,1\}^X$ be a nonempty finite linearly-dependent set of nonzero vectors. This means that there exist real coefficients $(x_w)_{w\in Z}$, not all zeros, such that $\sum_{w\in Z} x_w \cdot w = 0$. The following lemma shows that we can choose these coefficients to be integers with a bounded magnitude.

We recall the following facts from the OEIS [33].

For each natural n, let $a(n)$ be the maximal determinant of a matrix of order n whose entries are 0 or 1. The values of $a(n)$ are known up to $n = 21$: $1, 1, 2, 3, 5, 9, 32, 56, 144, 320, 1458, 3645, 9477, 25515, 131072, 327680, 1114112, 3411968, 19531250, 56640625, 195312500, \ldots$. Hadamard proved that $a(n) \leq 2^{-n}(n+1)^{\frac{n+1}{2}}$, with equality iff a Hadamard matrix of order $n+1$ exists. It is believed that the latter holds iff $n+1 = 1, 2$ or a multiple of 4. [8] provide a lower bound $a(n) > 2^{-n}(\frac{3}{4}(n+1))^{\frac{n+1}{2}}$.

Here are two examples of 6×6 matrices that attain the upper bound $a(6) = 9$:[4]

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{1}$$

Lemma 4.2. *Let $Z \subset \{0,1\}^X$ be a nonempty finite linearly dependent set of nonzero vectors and $p = |Z|-1$. Then there exist integers $(\Delta_w)_{w\in Z}$ which are not all zeros such that $|\Delta_w| \leq a(p)$ for each $w \in Z$ and $\sum_{w\in Z} \Delta_w \cdot w = 0$. That is if* some *nontrivial linear combination of Z equals 0, then there exists such a linear combination in which the coefficients are all integers, and are all bounded by $a(p)$.*

Proof. Let W be a maximal linearly independent subset of Z (note that $|W| \leq |Z|-1 = p$). By Lemma 4.1, there exists a subset Y of X with $|Y| = |W|$ such that the set $\pi_Y(W) \subset \mathbb{R}^Y$ is linearly independent.

Pick any vector $z \in Z \setminus W$. By the maximality of W, the set $W \cup \{z\}$ is linearly dependent, and so the set $\pi_Y(W) \cup \{\pi_Y(z)\}$ is linearly dependent too. Therefore, the system $\sum_{v\ \in W} x_v \cdot \pi_Y(v) = \pi_Y(z)$ has a solution $\mathbf{x}$. Note that this is a square system, with $|Y|$ equations in $|W| = |Y|$ unknowns.

The solution $\mathbf{x}$ is unique, because $\pi_Y(W)$ is linearly independent. Therefore, $\mathbf{x}$ can be computed by Cramer's rule. It gives $x_v = \Delta_v/\Delta_z$ for each $v \in W$, where Δ_z is the determinant of the matrix with columns $\pi_Y(v)$, and Δ_v is the determinant of the same matrix where column v is replaced with $\pi_Y(z)$. All these matrices are square matrices with binary entries and size at most $|Y| \leq p$, so their determinants are integers with absolute value at most $a(p)$.

Since the set $W \cup \{z\}$ is linearly dependent and the system $\sum_{v\in W} x_v \cdot \pi_Y(v) = \pi_Y(z)$ has a unique solution, the system $\sum_{v\in W} x_v \cdot v = z$ has (the same) unique solution. Multiplying by Δ_z gives $\sum_{v\in W} \Delta_v \cdot v - \Delta_z \cdot z = 0$ Putting $\Delta_v = 0$ for each $v \in Z \setminus (W \cup \{z\})$ yields the desired linear combination. □

We are now ready to prove the main theorem of this subsection.

Given an input set of items D, let $OPT(D_k)$ denote the optimal number of bins in k-times bin-packing of the items in D.

Theorem 4.1. *For any electricity division instance with demand-vector D and supply S, there exists an integer $k \leq a(n)$ such that $\frac{k}{OPT(D_k)}$ is the egalitarian connection-time per agent. This time can be attained by solving kBP on D and allocating a fraction $\frac{1}{OPT(D_k)}$ of the time to each bin in the optimal solution.*

Proof. The set $W = \{(w_1, \ldots, w_n) \in \{0,1\}^n : \sum_{i=1}^n d_i w_i \leq S\}$ naturally represents all admissible ways to pack the agents into bins (all feasible "configurations"). The convex hull $\mathrm{CH}(W)$ of W naturally represents the set of all possible schedules, where a schedule is represented by the fraction of time allocated to each configuration.

Denote $e = (1, \ldots, 1) \in \mathbb{R}^n$. Let $r_{\max}$ be the egalitarian connection time, defined by $r_{\max} = \sup\{r \in [0,1] : r \cdot e \in \mathrm{CH}(W)\}$. Since the set $\mathrm{CH}(W)$ is compact, $r_{\max}$ is attained, that is $r_{\max} \cdot e \in \mathrm{CH}(W)$. The electricity division problem has a solution, so $r_{\max} > 0$.

[4] The first is from OEIS; the second is from Dietrich Burde in `https://math.stackexchange.com/a/4965827`.

Since $r_{\max} \cdot e \in \mathrm{CH}(W)$, by Caratheodory's theorem [44], there exists a simplex $\mathrm{CH}(W')$ with the set $W' \subset W$ of vertices such that $r_{\max} \cdot e \in \mathrm{CH}(W')$.

Consider the following illustrating example. Let the supply S is 25 and there are three agents x, y, and z with the demands $d_x = 11$, $d_y = 12$, and $d_z = 13$, respectively. It is possible to connect each agent per $r_{\max} = 2/3$ of the time by connecting each of $\{x, y\}, \{y, z\}, \{x, z\}$ per $1/3$ of the time. This can be visually represented as shown in Figure 1. The three blue points $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ represent

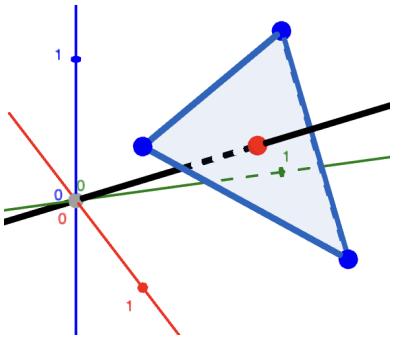

Fig. 1: The three blue points are (1,1,0), (1,0,1) and (0,1,1). The black line that originates from the origin intersects with the triangle (convex hull of the three blue points) at red point (2/3,2/3,2/3).

elements of W', and the white triangle with these vertices represents $\mathrm{CH}(W')$. The black ray originating from the origin $(0, 0, 0)$ is $e \cdot \{r : r \geq 0\}$. The red point $(2/3, 2/3, 2/3)$ represents the intersection of the ray with $\mathrm{CH}(W')$ and corresponds to $r_{\max} = 2/3$.

Let W'' be the set of the vertices of the face of $\mathrm{CH}(W')$ of minimal dimension containing $r_{\max} \cdot e$. Clearly, $r_{\max} \cdot e$ is a boundary point of $\mathrm{CH}(W')$, so $|W''| \leq n$. In other words, there exists an optimal electricity division schedule with at most n different configurations. In the above example $W'' = W'$ and $|W''| = 3 = n$.

Let $(\lambda_w)_{w \in W''}$ be the barycentric coordinates of $r_{\max} \cdot e$, such that $\lambda_w > 0$ for each $w \in W''$ and $\sum_{w \in W''} \lambda_w = 1$, and

$$r_{\max} \cdot e = \sum_{w \in W''} \lambda_w w. \tag{2}$$

In the above example $\lambda_w = 1/3$ for each $w \in W''$.

We claim that the set W'' is linearly independent. Indeed, suppose for a contradiction that there exist disjoint nonempty subsets W_1'' and W_2'' of W'' and positive numbers $(\mu_w)_{w \in W_1'' \cup W_2''}$ such that $\sum_{w \in W_1''} \mu_w w = \sum_{w \in W_2''} \mu_w w$. Assume w.l.o.g. that $\sum_{w \in W_1''} \mu_w \leq \sum_{w \in W_2''} \mu_w$. Let $\nu = \min_{w \in W_2''} \lambda_w / \mu_w$. Then,

$$r_{\max} \cdot e = \sum_{w \in W''} \lambda_w w = \sum_{w \in W''} \lambda_w w + \nu \left(\sum_{w \in W_1''} \mu_w w - \sum_{w \in W_2''} \mu_w w \right) = \sum_{w \in W''} \lambda'_w w,$$

where

$$\lambda'_w = \begin{cases} \lambda_w + \nu \mu_w & \text{if } w \in W_1'' \\ \lambda_w - \nu \mu_w & \text{if } w \in W_2'' \\ \lambda_w & \text{if } w \in W'' \setminus (W_1'' \cup W_2'') \end{cases} \tag{3}$$

Then, $s := \sum_{w \in W''} \lambda'_w = \sum_{w \in W''} \lambda_w + \nu(\sum_{w \in W_1''} \mu_w - \sum_{w \in W_2''} \mu_w) \leq 1$, $\lambda'_w \geq 0$ for each $w \in W''$, and $\lambda'_w = 0$ for some $w \in W_2''$. Then $s^{-1} \cdot r_{\max} \cdot e \in \mathrm{CH}(W'' \setminus \{w\})$, that contradicts the minimality of W''.

Let $Z := W'' \cup \{e\}$. As W'' contains a multiple of e, the set Z is linearly dependent. Lemma 4.2 can be applied to it with $p = |W''| \leq n$. The lemma implies that there exist integers $(\Delta_v)_{v \in Z}$ which are not all zeroes such that $|\Delta_v| \leq a(p) \leq a(n)$ for each $v \in Z$ and $\sum_{v \in Z} \Delta_v \cdot v = 0$. The set W'' is linearly independent, so $\Delta_e \neq 0$ and

$$e = \sum_{w \in W''} (-\Delta_w / \Delta_e) w. \tag{4}$$

But by (2), $e = \sum_{w \in W''} (\lambda_w / r_{\max}) \cdot w$. As W'' is linearly independent, the coefficients in the expression for e are unique, so we must have $-\Delta_w/\Delta_e = \lambda_w / r_{\max} > 0$ for each $w \in W''$. Therefore, $-\Delta_w/\Delta_e = |\Delta_w|/|\Delta_e|$.

Construct a packing in which each configuration $w \in W''$ appears $|\Delta_w|$ times. Then the vector $\sum_{w \in W''} |\Delta_w| \cdot w$ represents the number of times each item appears in the packing; but by (4), this sum equals $|\Delta_e| \cdot e$. Therefore, it is a k-times bin-packing with $k := |\Delta_e| \leq a(n)$.

The total number of bins in the packing is $\sum_w |\Delta_w| = \sum_w |\Delta_e| \cdot \lambda_w / r_{\max} = k / r_{\max}$. Therefore, connecting each bin for an equal amount of time yields an allocation in which each agent is connected for a fraction $k/(k/r_{\max}) = r_{\max}$ of the time, as required. □

4.2 Lower bound

For any electricity division instance D, let $K(D)$ denote the smallest k such that the egalitarian connection time equals $\frac{k}{OPT(D_k)}$. For any n, let $K(n)$ denote the maximum $K(D)$ over all instances with n households. Theorem 4.1 implies $K(n) \leq a(n)$, which provides an exponential upper bound on $K(n)$. This raises the question of whether there is a matching lower bound. Currently, we only have two very loose lower bounds.

Proposition 1. *For any $n \geq 2$, there is a lower bound $K(n) \geq n - 1$*

Proof. Consider an instance with n items of size 1, and let $S := n-1$. Then we can construct n bins, each of which contains a different subset of $n-1$ items. This packing is optimal, as all bins are full; it is an $(n-1)$-times bin-packing, as each item appears in exactly $n-1$ different bins. Therefore, $r_{\max} = (n-1)/n$. We show that any $k < n-1$ does not yield an optimal packing. For any k, the sum of all item sizes in D_k is kn, so any k-times bin-packing must contain at least $kn/S = k\frac{n}{n-1}$ bins. But n and $n-1$ are coprime, so for any $k < n-1$, the expression $k\frac{n}{n-1}$ is not an integer, which means that the packing must have non-full bins. □

Proposition 2. $K(6) = 9$.

Proof. As $K(6) \leq a(6) = 9$, it is sufficient to prove the lower bound $K(6) \geq 9$.

Let A be the rightmost 6×6 matrix in (1). Let $S := \det(A) = 9$.

Let $B := \det(A) \cdot A^{-1}$, and note that B is an integer matrix.

Let the vector of demands be $D := B \cdot e = [4, 2, 5, 3, 2, 1]$. By construction, Each row in A corresponds to a configuration with sum exactly 9. These configurations are: $4+5, 4+2+3, 2+5+2', 5+3+1, 4+3+2', 4+2+2'+1$ (where $2'$ denotes the second household with demand 2).

Construct a packing by $\mathbf{x} := B^T \cdot e = [1, 2, 3, 5, 2, 4]$. Every element x_i represents the number of times that configuration i appears in the packing. In particular, there are —

- 1 bin with $4+5$;
- 2 bins with $4+2+3$;
- 3 bins with $2+5+2'$;
- 5 bins with $5+3+1$;
- 2 bins with $4+3+2'$;
- 4 bins with $4+2+2'+1$.

By construction, each item appears exactly 9 times, so it is a valid 9-times bin-packing. It has 17 bins, and it is optimal since all bins are full. Therefore, $r_{\max} = 9/17$.

We now show that any $k < 9$ does not yield an optimal packing. For any k, the sum of all bins in kBP is $17k$, so any kBP must contain at least $17k/9$ bins. But 17 and 9 are coprime, Therefore, for any $k < 9$, the expression $17k/9$ is not an integer, which means that the packing must have non-full bins. □

It is open whether Proposition 2 can be generalized. The following is our conjectured generalization of Proposition 2.

Conjecture 1. Let A be an $n \times n$ binary matrix, let $B := \det(A) \cdot A^{-1}$, and let g be the sum of all elements in B (the "grand sum" of B). If g and $\det(A)$ are coprime, then $K(n) \geq \det(A)$.

Note that Proposition 2 is a special case with $\det(A) = 9$ and $g = 17$.

5 Fast Approximation Algorithms

The results of the previous section are not immediately applicable to fair electricity division, as kBP is known to be an NP-hard problem. However, they do hint that good approximation algorithms for kBP can provide good approximation for electricity division. Therefore, in this section, we study several fast approximation algorithms for kBP.

5.1 *FFk* — First-Fit for *k*BP

The k-times version of the First-Fit bin-packing algorithm packs each item of D_k in order into the first bin where it fits and does not violate the constraint that each item should appear in a bin at most once. If the item to pack does not fit into any currently open bin, FFk opens a new bin and packs the item into it. For example: consider $D = \{10, 20, 11\}, k = 2, S = 31$. FFk will result the bin-packing $\{10, 20\}, \{11, 10\}, \{20, 11\}$. It is known that the asymptotic approximation ratio of FF is 1.7 [11]. Below, we prove that, for any fixed $k > 1$, the asymptotic approximation ratio of FFk for large instances (when $n \to \infty$) is better, and it improves when k increases.

Theorem 5.1. *For every input D and $k \geq 1$, $FFk(D_k) \leq \left(1.5 + \frac{1}{5k}\right) \cdot OPT(D_k) + 3 \cdot k$.*

Proof. At a very high level, the proof works as follows:

- We define a *weight* for each item, which depends on the item size, and may also depend on the instance to which the item belongs.
- We prove that the average weight of each bin in an optimal packing is at most some real number Z, so the total weight of all items is at most $Z \cdot OPT(D_k)$.
- We prove that the average weight of a bin in the FFk packing (except some $3k$ bins that we will exclude from the analysis) is at least some real number Y, so the total weight of all items is at least $Y \cdot (FFk(D_k) - 3 \cdot k)$.
- Since the total weight of all items is fixed, we get $FFk(D_k) - 3 \cdot k \leq (Z/Y) \cdot OPT(D_k)$, which gives an asymptotic approximation ratio of Z/Y.

Basic weighting scheme The basic weighting scheme we use follows [11]. The weight of any item of size v is defined as

$$w(v) := v/S + r(v),$$

where r is a *reward function*, computed as follows:

$$r(v) := \begin{cases} 0 & \text{if } v/S \leq \frac{1}{6}, \\ \frac{1}{2}(v/S - \frac{1}{6}) & \text{if } v/S \in (\frac{1}{6}, \frac{1}{3}), \\ 1/12 & \text{if } v/S \in [\frac{1}{3}, \frac{1}{2}], \\ 4/12 & \text{if } v/S > \frac{1}{2}. \end{cases}$$

For $k > 1$, we use a modified weighting scheme, which gives different weights to items that belong to different instances. This allows us to get a better asymptotic ratio. We describe the modified weighting scheme below.

Associating bins with instances Recall that FFk processes one instance of D completely, and then starts to process the next instance. We associate each bin in the FFk packing with the instance in which it was opened. So for every $j \in \{1, \ldots, k\}$, the *bins of instance j* are all the bins, whose first allocated item comes from the j-th instance of D. Note that bins of instance j do not contain items of instances $1, \ldots, j-1$, but may contain items of any instance $j, \ldots, k$.

For the analysis, we also need to associate some bins in the optimal packing with specific instances. We consider some fixed optimal packing. For each bin B in that packing:

- If B contains an item x with $V(x) > S/2$, and x belongs to instance j, then we associate bin B with instance j. Clearly, there can be at most one item of size larger than $S/2$ in any feasible bin, so there is no ambiguity. Moreover, we ensure that all other items in B belong to instance j too: if some other item $x' \in B \setminus \{x\}$ belongs to a different instance j', then we replace x' with its copy from instance j. Note that the other copy of x' must be located in a bin different than B, due to the restrictions of kBP. Since both copies of x' have the same size, the size of all bins remains the same.
- If B contains no item of size larger than $S/2$, then we do not associate B with any instance.

Partitioning FFk bins into groups For the analysis, we partition the bins of each instance $j \in [k]$ in the FFk packing into five groups.

Group 1. Bins with a single item of instance j, whose size is at most $S/2$. There is at most one such bin. This is because, if there is one such bin B of instance j, it means that all later items of instance j do not fit into B, so their size must be greater than $S/2$. Therefore, any later bin B' of instance j must have size larger than $S/2$.

Group 2. Bins with two or more items of instance j, whose total size is at most $2S/3$. There is at most one such bin. This is because, if there is one such bin B of instance j, it means that all later items of instance j do not fit into B, so their size must be greater than $S/3$. Therefore, in any later bin B' with two or more items of instance j, their total size is larger than $2S/3$.

Group 3. Bins with a single item of instance j, whose size is larger than $S/2$.

Group 4. Bins with two or more items of instance j, whose total size is at least $10S/12$.

Group 5. Bins with two or more items of instance j, whose total size is in $(2S/3, 10S/12)$.

In the upcoming analysis, we will exclude from each instance, the at most one bin of group 1, at most one bin of group 2, and at most one bin of group 5. All in all, we will exclude at most $3k$ bins of the FFk packing from the analysis.

Analysis using the basic weighting scheme As a warm up we prove that, with the basic weighting scheme, the total weight of each optimal bin B is at most $17/12$. Since the total size of B is at most S, we have $w(B) \leq 1 + r(B)$, so it is sufficient to prove that the reward $r(B) \leq 5/12$. Indeed:

- If B contains an item larger than $S/2$, then this item gives B a reward of $4/12$; the remaining room in B is smaller than $S/2$. This can accommodate either a single item of size at least $S/3$ and some items smaller than $S/6$, or two items of size between $S/6$ and $S/3$; in both cases, the total reward is at most $1/12$.
- If B does not contain an item larger than $S/2$, then there are at most 5 items larger than $S/6$, so at most 5 items with a positive reward. The reward of each item of size at most $S/2$ is at most $1/12$.

In both cases, the total reward is at most $5/12$.

We now prove that, with the basic weighting scheme, the average weight of each FFk bin is at least $10/12$. In fact, we prove a stronger claim: we prove that, for each instance j, the average weight of the items *of instance j only* in bins of instance j (ignoring items of instances $j+1, \ldots k$ if any) is at least $10/12$.

We use the above partition of the bins into 5 groups, excluding at most one bin of group 1 and at most one bin of group 2.

The bins of group 3 (bins with a single item of instance j, which is larger than $S/2$) have a reward of $4/12$, so their weight is at least $1/2 + 4/12 = 10/12$.

The bins of group 4 (bins with two or more items of instance j, which have a total size larger than $10S/12$) already have weight at least $10/12$, regardless of their reward.

We now consider the bins of group 5 (bins with two or more items of instance j, which have a total size in $(2S/3, 10S/12)$). Denote the bins in group 5 of instance j, in the order they are opened, by $B_1, \ldots, B_Q$. Consider a pair of consecutive bins, for instance, B_t, B_{t+1}. Let $x :=$ the free space in bin B_t during instance j. Note that $x \in (S/6, S/3)$. Let c_1, c_2 be some two items of instance j which are packed into B_{t+1}.

For each $c_i \in \{c_1, c_2\}$, as FFk did not pack c_i into B_t during the processing of instance j, there are two options: either c_i is too large (its size is larger than x), or B_t already contains other copies of c_i from other instances. But the second option cannot happen, because B_t belongs to instance j, so it contains no items of instances $1, \ldots, j-1$; and while c_i of instance j was processed by FFk, items of instance $j+1, \ldots, k$ were not processed yet. Therefore, necessarily $V(c_i) > x$.

Since $S/6 < x < S/3$, the reward of each of c_1, c_2 is at least $(x/S - 1/6)/2$, and the reward of both of them is at least $x/S - 1/6$. Therefore, during instance j,

$$V(B_t)/S + r(B_{t+1}) \geq (S - x)/S + (x/S - 1/6) = 1 - 1/6 = 10/12.$$

In the sequence $B_1, \ldots, B_Q$, there are $Q - 1$ consecutive pairs. Using the above inequality, we get that the total weight of these bins is at least $10/12 \cdot (Q - 1)$, which is equivalent to excluding one bin (in addition to the two bins excluded in groups 1 and 2).

Summing up the weight of all non-excluded bins gives at least $(10/12) \cdot (FFk(D_k) - 3k)$. Meanwhile, the total weight of optimal bins is at most $(17/12) \cdot OPT(D_k)$. This yields $FFk(D_k) \leq (17/10) \cdot OPT(D_k) + 3k$, which corresponds to the known asymptotic approximation ratio of 1.7 for $k = 1$.

We now present an improved approximation ratio for $k > 1$. To do this, we give different weights to items of different copies of D. We do it in a way that the average weight of the FFk bins (except the $3k$ excluded bins) will remain at least 10/12. So the total weight of all items is at least $(10/12)(FFk(D_k) - 3k) \leq V(D)$. Meanwhile, the average weight of the optimal bins in some $k-1$ instances (as well as the unassociated bins) will decrease to 15/12, whereas the average weight of the optimal bins in a single instance will remain 17/12. So the total weight of all items is at most $V(D) \leq \frac{(17/12)+(15/12)\cdot(k-1)}{k} \cdot OPT(D_k)$. Therefore,

$$FFk(D_k) \leq (1.5 + \frac{1}{5k}) \cdot OPT(D_k) + 3k.$$

This would lead to an asymptotic ratio of at most $1.5 + \frac{1}{5k}$.

Warm-up: $k = 2$ To explain the main idea of our analysis, we will analyze the case $k = 2$, and prove an asymptotic approximation ratio of $1.5 + \frac{1}{2\cdot 5} = 1.6$. We give a detailed proof of the general case of any $k \geq 1$ in Section A.

We call a bin in the FFk packing *underfull* if it has only one item, and this item is larger than $S/2$ and smaller than $2S/3$. Note that all underfull bins belong to group 3. We consider two cases.

Case 1 No FFk bin of instance 1 is underfull (in instance 2 there may or may not be underfull bins). In this case, we modify the weight of items in instance 1 only: we reduce the reward of each item of size $v > S/2$ in instance 1 from 4/12 to 2/12.

- For any bin B in the optimal packing, if B does not contain an item larger than $S/2$, then its maximum possible reward is still 3/12 as with the basic weights. If B contains an item larger than $S/2$, then this item gives B a reward of 2/12 if it belongs to instance 1, or 4/12 if it belongs to instance 2. The remaining room in B is smaller than $S/2$, and the maximum total reward of items that can fit into this space is 1/12 (as the remaining space can accommodate either one item of size at least $S/3$ and some items of size smaller than $S/6$, or two items of sizes in $(S/6, S/3)$). Overall, the reward of B is at most 5/12 if it contains an item larger than $S/2$ from instance 2, and at most 3/12 otherwise. As at most half the bins in the optimal packing contain an item larger than $S/2$ from instance 2, at most half these bins have reward 5/12, while the remaining bins have reward at most 3/12, so the average reward is at most 4/12. Adding the size of at most 1 per bin leads to an average weight of at most 16/12.
- In the FFk packing, we note that the change in the reward affects only the bins of group 3 (bins with a single item and $V(B) > S/2$). These bins now have a reward of at least 2/12. Since none of them is underfull by assumption, their weight is at least $2/3 + 2/12 = 10/12$. The bins of groups 4 and 5 are not affected by the reduced reward: the same analysis as above can be used to deduce that their average weight (except one bin per instance excluded in group 5) is at least 10/12.

Case 2 At least one FFk bin of instance 1 is underfull. In this case, we modify the weight of items in instance 2 only, as follows (note that we reduce the *weights* of these items, and not only their rewards):

$$w(v) = \begin{cases} 0 & \text{if } v/S < \frac{1}{3}, \\ 5/12 & \text{if } v/S \in [\frac{1}{3}, \frac{1}{2}], \\ 10/12 & \text{if } v/S > \frac{1}{2}. \end{cases}$$

Note that all these weights are not higher than the basic weights of the same items.

- In the optimal packing, every bin that contains an item larger than $S/2$ from instance 2 is associated with instance 2, and therefore contains only items of instance 2 by construction. Therefore, the weight of any such bin is at most 15/12 (the remaining space in such bin can pack only one item of weight in the range $[S/3, S/2]$). The weight of every bin that contains no item larger than $S/2$ is still at most 15/12, since its reward is at most 3/12 as with the basic weights. The only bins that may have a larger weight (up to 17/12) are those that contain an item larger than $S/2$ from instance 1; at most half the bins in the optimal packing contain such an item. Therefore, the average weight of a bin in the optimal packing is at most 16/12.
- In the FFk packing, some bin B^1 in instance 1 is underfull, that is, B^1 contains a single item of size $v \in (S/2, 2S/3)$, and the free space in B^1 is $S - v \in (S/3, S/2)$. This means that, in the bins of instance 2, there is no item of size at most $S/3$, since every such item would fit into B^1 (it is smaller than the available space in B^1, and it is not a copy of the single item in B^1). So for every bin B^2 in instance 2 (except the excluded ones), there are only two options:

- B^2 contains one item larger than $S/2$ — so its weight is 10/12;
- B^2 contains two items, each of which is larger than $S/3$ — so its weight is at least $2{\cdot}5/12 = 10/12$.

The weight of items in instance 1 does not change. So the average weight of bins in the FFk packing (except the $3k$ excluded bins) remains at least 10/12.

In all cases, for $k = 2$, we get that the average weight of bins in the optimal packing is at most 16/12, and the average weight of non-excluded bins in the FFk packing is at least 10/12. Therefore, $FFk - 3k \leq (16/10) \cdot OPT = 1.6 \cdot OPT$. □

Approximation ratio lower bound For $k = 1$, the approximation ratio of 1.7 is tight; an example is given in [13] and on page 306 in [28]. In Section A.1, we analyze these examples for $k = 2$, and show that FFk achieves a ratio of 1.35. For the example given in [28] we observed that as k increases, the approximation ratio continues to decrease. This raises the question of whether the bound of Theorem 5.1 is tight.

Although we do not have a complete answer for this question, we show below an example in which FFk attains an approximation ratio of 1.375 for $k = 2$, which is higher than the example of [28].

Consider the example $D = \{371, 659, 113, 47, 485, 3, 228, 419, 468, 581, 626\}$ and bin capacity $S = 1000$. Then, FFk for $k = 2$ will result in 11 bins $[371, 113, 47, 3, 228]$, $[659, 113, 47, 3]$, $[485, 419]$, $[468, 371]$, $[581, 228]$, $[626]$, $[659]$, $[485, 419]$, $[468]$, $[581]$, $[626]$. The optimal packing for the items in D_2 is $[626, 371, 3]$, $[626, 371, 3]$, $[659, 228, 113]$, $[659, 228, 113]$, $[419, 581]$, $[419, 581]$, $[468, 485, 47]$, $[468, 485, 47]$. Clearly, $\frac{FFk(D_2)}{OPT(D_2)} = \frac{11}{8} = 1.375$. We observed that as k increases, the approximation ratio continues to decrease. Experimental results in support of this ratio are given in Section 8.1. Based on the example for which we have achieved a ratio of 1.375 and the simulation of the algorithm on different datasets, we conjecture that for $k > 1$ the *absolute* approximation ratio for FFk is 1.375.

5.2 *FFDk*

The k-time version of the First-Fit Decreasing bin-packing algorithm first sorts D in non-increasing order. Then it constructs D_k using k consecutive copies of the sorted D, and then implements FFk on D_k. In contrast to FFk, we could not prove an upper bound for $FFDk$ that is better than the upper bound for FFD; we only have a lower bound.

Lemma 5.1. $FFDk(D_k) \geq \frac{7}{6} \cdot OPT(D_k) + 1$.

Proof. We use the following example from page 2 of [10]. Let δ be a sufficiently small positive number, and let $S = 1$. Let $D = \{\frac{1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{4} + 2\delta, \frac{1}{4} + 2\delta, \frac{1}{4} + 2\delta, \frac{1}{4} + 2\delta, \frac{1}{4} + \delta, \frac{1}{4} + \delta, \frac{1}{4} + \delta, \frac{1}{4} + \delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta\}$.

An optimal packing for D contains 4 bins of type $\{\frac{1}{2} + \delta, \frac{1}{4} + \delta, \frac{1}{4} - 2\delta\}$ and 2 bins of type $\{\frac{1}{4} + 2\delta, \frac{1}{4} + 2\delta, \frac{1}{4} - 2\delta, \frac{1}{4} - 2\delta\}$. As all bin sizes are exactly 1, this pattern is clearly optimal for any k. Therefore, for all $k \geq 1$, $OPT(D_k) = 6k$.

On applying $FFDk$ on D_k the resulting number of bins are $8 + 7(k - 1)$. The detailed $FFDk$ packing of the items in D_k is given in Section B. This gives us a lower bound of $\frac{7}{6} \cdot OPT(D_k) + 1$. □

We discuss the challenges in extending the existing proof for the FFD in Section B. Based on the simulation of $FFDk$ on different datasets, we conjecture that the upper bound for $FFDk$ is $\frac{11}{9} \cdot OPT(D_k) + \frac{6}{9}$. Experimental results supporting this conjecture are provided in Section 8.1.

5.3 *NFk*

Given the input D_k, the algorithm NFk works as follows: like NF, NFk always keeps a single bin open to pack items. If the current item does not pack into the currently open bin then NFk closes the current bin and opens a new bin to pack the item.

Theorem 5.2. *For every input D_k and $k \geq 1$, the asymptotic ratio of $NFk(D_k)$ is 2.*

We give a detailed proof of the above Theorem in Section C.

6 Polynomial-time Approximation Schemes

6.1 Some general concepts and techniques

The basic idea behind generalizing Fernandez de la Vega-Lueker and all the Karmarkar-Karp algorithms to solve kBP is similar. It consists of three steps: 1. Keeping aside the small items, 2. Packing the remaining large items, and 3. Packing the small items in the bins that we get from step 2 (opening new bins if necessary) to get a solution to the original problem.

In step 3, the main difference from previous work is that, in kBP, we cannot pack two copies of the same small item into the same bin, so we may have to open a new bin even though there is still remaining room in some bins. The following lemma analyzes the approximation ratio of this step.

Lemma 6.1. *Let D_k be an instance of the kBP problem, and $0 < \epsilon \leq 1/2$. We say that the item is* large*, if its size is bigger than $\epsilon \cdot S$ and* small *otherwise. Assume that the large items are packed into L bins. Consider an algorithm which starts adding the small items into the L bins respecting the constraint of kBP, but whenever required, the algorithm opens a new bin. Then the number of bins required for the algorithm to pack the items in D_k is at most* $\max\{L, (1+2\cdot\epsilon)\cdot OPT(D_k)+k\}$.

Proof. Let I be the set of all small items in D and I_k be the k copies of I. Let $A(D_k)$ be the number of bins in the packing of D_k. If adding small items do not require a new bin, then $A(D_k) = L$. Otherwise, since the first item in the last bin cannot be packed to $A(D_k) - k$ previous bins, each of these bins has less than $(1-\epsilon)\cdot S$ free size. Thus

$$\begin{aligned}
\sum D_k[i] &\geq (S - \epsilon\cdot S)(A(D_k)-k)\\
\sum D_k[i] &\leq OPT(D_k)\cdot S\\
A(D_k) &\leq \frac{1}{1-\epsilon}\cdot OPT(D_k)+k\\
A(D_k) &\leq (1+2\cdot\epsilon)\cdot OPT(D_k)+k \qquad (5)
\end{aligned}$$

Therefore,

$$A(D_k) \leq \max\{L, (1+2\cdot\epsilon)\cdot OPT(D_k)+k\} \qquad (6)$$

□

Step 2 is done using a linear program based on *configurations*.

Definition 6.1. *A* configuration *(or a* bin type*) is a collection of item sizes which sums to, at most, the bin capacity S.*

For example [43]: suppose there are 7 items of size 3, 6 items of size 4, and $S = 12$. Then, the possible configurations are $[3,3,3,3], [3,3,3], [3,3], [3], [4,4,4], [4,4], [4], [3,3,4], [3,4,4], [3,4]$.

Enumerate all possible configurations by the natural numbers from 1 to t. Let $A = \|a_{ij}\|$ be a $m(D)\times t$ matrix, such that for each natural $i \leq m(D)$ and $j \leq t$ the entry a_{ij} is the number of items of size $c[i]$ in the configuration j. Let $\mathbf{n}$ be a $m(D)$-dimensional vector such that for each natural $i \leq m(D)$ its ith entry is $n[i]$ (the number of items of size $c[i]$). Let $\mathbf{x}$ be a t-dimensional vector such that for each natural $j \leq t$ we have that $x[j]$ is the number of bins filled with configuration j, and $\mathbf{1}$ be a t-dimensional vector whose each entry is 1. Consider the following linear program

$$(C_1) \qquad \begin{aligned} \min \quad & \mathbf{1}\cdot\mathbf{x}\\ \text{such that} \quad & A\mathbf{x} = \mathbf{n}\\ & \mathbf{x} \geq 0 \end{aligned}$$

When $\mathbf{x}$ is restricted to integer entries ($\mathbf{x} \in \mathbb{Z}^t$), the solution of this linear program defines a feasible bin-packing. We denote by F_1 the fractional relaxation of the above program, where $\mathbf{x} \in \mathbb{R}^t$.

Recall that in kBP, each item of D has to appear in k distinct bins. *One can observe that kBP uses the same configurations as in the bin-packing, to ensure that each bin contains at most one copy of each item.* Therefore, the configuration linear program C_k for kBP is as follows, where A, $\mathbf{n}$, and $\mathbf{x}$ are the same as in C_1 above (for $k = 1$ it is the same as in [41]):

$$(C_k) \qquad \begin{aligned} \min \quad & \mathbf{1}\cdot\mathbf{x} && (7)\\ \text{such that} \quad & A\mathbf{x} = k\mathbf{n} && (8)\\ & \mathbf{x} \geq 0 && (9) \end{aligned}$$

Lemma 6.2. *Every integral solution of C_k can be realised as a feasible solution of kBP.*

Proof. Since each bin can contain at most one copy of each item of D, for each natural $j \leq t$ if the configuration j can be realized then $a_{ij} \leq n[i]$ for each natural $i \leq m(D)$. We shall call such configurations *feasible*. We can realise every sequence of feasible configurations as a solution of the kBP problem as follows. For each natural $i \leq m(D)$, let $d_1, \ldots, d_{n[i]}$ be the items from D of size $c[i]$. Let the queue Q_i be arranged of k copies of these items, beginning from the first copies of $d_1, \ldots, d_{n[i]}$ in this order, then of the second copies of these items in the same order, and so forth. To realize the sequence, consider the first configuration, say, j, from the sequence, and for each natural $i \leq m(D)$ move a_{ij} items of size $c[i]$ from the queue Q_i into the first bin (or just do nothing when Q_i is already empty), then similarly process the second configuration from the sequence and so forth. Since all configurations are feasible, we never put two copies of the same item in the same bin, so the above procedure constructs a feasible solution to kBP.

A detailed example to illustrate the realisation, as a feasible solution to kBP , of the integral solution of C_k is given in Section D. □

Let F_k be the fractional bin-packing problem corresponding to C_k. Step 2 involves grouping. Grouping reduces the number of different item sizes, and thus reduces the number of constraints and configurations in the fractional linear program F_k.

To solve the configuration linear program efficiently, both Fernandez de la Vega-Lueker algorithm and Algorithm 1 of Karmarkar Karp use a *linear grouping* technique. In linear grouping, items are divided into groups (of fixed cardinality, except possibly the last group), and each item size (in each group) increases to the maximum item size in that group. See Section E for more detail.

Our extension of the Fernandez de la Vega-Lueker and the Karmarkar-Karp algorithms to kBP differs from their original counterparts in mainly two directions. First, in the configuration linear program (see the constraint Equation (8) in C_k), and hence the obtained solution to this configuration linear program is not necessarily the k times copy of the original solution of BP. Second, in greedily adding the small items, see Lemma 6.1. In extension of Karmarkar-Karp algorithm 1 to kBP we have also shown that getting an integer solution from $\mathbf{x}$ by rounding method may require at most $(k-1)/2$ additional bins. We discuss extensions to the Fernandez de la Vega-Lueker and Karmarkar-Karp algorithms and their analyses in Section 6.2 and Section 6.3, respectively.

The inputs to the extension of the algorithms by Fernandez de la Vega-Lueker and Algorithm 1 and Algorithm 2 of Karmarkar-Karp are an input set of items D, a natural number k, and an approximation parameter $\epsilon \in (0, 1/2]$. Algorithm 2 of Karmarkar-Karp, in addition, accepts an integer parameter $g > 0$.

6.2 Fernandez de la Vega-Lueker algorithm to kBP

Fernandez de la Vega and Lueker [41] published a PTAS which, given an input instance D and $\epsilon \in (0, 1/2]$, solves a bin-packing problem with, at most, $(1+\epsilon) \cdot OPT(D) + 1$ bins. They devised a method called "adaptive rounding" for this algorithm. In this method, the given items are put into groups and rounded to the largest item size in that group. This resulting instance will have fewer different item-sizes. This resulting instance can be solved efficiently using a configuration linear program C_k (see Section E for more detail).

A high-level description of the extension of Fernandez de la Vega-Lueker algorithm to kBP. Let I and J be multisets of small and large items in D, respectively. After applying linear grouping in J, let U be the resulting instance and C_k be the corresponding configuration linear program. An optimal solution to C_k will give us an optimal solution to the corresponding kBP instance U_k. Ungrouping the items in U_k will give us a solution to kBP instance J_k. Finally, adding the items in I_k, by respecting the constraints of kBP, to the solution of J_k (and possibly opening new bins if required) will give us a packing of D_k.

Lemma E.2 (see Section E for more detail) bounds the number of bins in an optimal packing of U_k. The extension of the algorithm by Fernandez de la Vega-Lueker to kBP is given in Section E.

Theorem 6.1. *Let A be the generalization of the Fernandez de la Vega-Lueker algorithm to solve kBP . Then, $A(D_k) \leq (1 + 2 \cdot \epsilon) \cdot OPT(D_k) + k$.*

We give a detailed proof of Theorem 6.1, along with the runtime analysis of the Fernandez de la Vega-Lueker algorithm to kBP, in Section E.

6.3 Karmarkar-Karp Algorithms to kBP

Karmarkar and Karp [30] improved the work done by Fernandez de la Vega and Lueker [41] mainly in two directions: (1) Solving the linear programming relaxation of C_1 using a variant of the GLS method [22] and (2) using a different grouping technique. These improvements led to the development of three algorithms. Their algorithm 3 is a particular case of the algorithm 2; we will discuss the generalization of algorithms 1 and 2 of Karmarkar-Karp algorithms to solve kBP. Let $LIN(F_k)$ denote the optimal solution to the fractional linear program F_k for kBP. We will discuss helpful results relevant to analyzing the generalized version of their algorithms. These results are an extension of the results in [30]. Lemma G.1 bounds from above the number of bins needed to pack the items in an optimal packing of some instance D_k. Lemma G.2 concerns obtaining an integer solution from a basic feasible solution of the fractional linear program. We discuss Lemma G.1 and Lemma G.2 in appendix Section G.

Before moving further, we would like to mention that if we use some instance (or group) without subscript k, we are talking about the instance when $k = 1$.

All Karmarkar-Karp algorithms use a variant of the ellipsoid method to solve the fractional linear program. So, we will talk about adapting this method to kBP.

Solving the fractional linear program: Solving the fractional linear program F_k involves a variable for each configuration. This results in a large number of variables. The fractional linear program F_k has the following dual D_F.

$$\max \quad k \cdot \mathbf{n} \cdot \mathbf{y} \tag{10}$$

$$\text{such that} \quad A^T \mathbf{y} \leq \mathbf{1} \tag{11}$$

$$\mathbf{y} \geq 0 \tag{12}$$

The above dual linear program can be solved to any given tolerance h by using a variant of the ellipsoid method that uses an approximate separation oracle [30]. The running time of the algorithm is $T(m(D_k), n(D_k)) = O\left(m(D)^8 \cdot \ln m(D) \cdot \ln^2\left(\frac{m(D)\cdot n(D)}{\epsilon \cdot S \cdot h}\right) + \frac{m(D)^4 \cdot k \cdot n(D) \cdot \ln m(D)}{h} \ln \frac{m(D)\cdot n(D)}{\epsilon \cdot S \cdot h}\right)$. We give a description of this variant of the ellipsoid method and its running time in Section H.1

A high-level description of the extensions of the Karmarkar-Karp algorithms. We will give a high-level description behind the extension of Karmarkar-Karp algorithms to solve kBP. Let I and J be multisets of small and large items in D, respectively. Let U'' be the instance constructed from J by applying the grouping technique. Construct the configuration linear program C_k for U''_k and solve the corresponding fractional linear program F_k. Let $\mathbf{x}$ be the resulting solution. From $\mathbf{x}$, obtain an integral solution for U''_k. From this solution, get a solution for J_k by ungrouping the items. Add the items in I_k by respecting the constraints of kBP to get a solution for D_k.

Karmarkar-Karp Algorithm 1 extension to kBP: Algorithm 1 of the Karmarkar-Karp algorithms uses the linear grouping technique as illustrated in Section 6.1. We give the extension of the Karmarkar-Karp Algorithm 1 to kBP in Section H.2

Theorem 6.2. *Let $A(D_k)$ denote the number of bins produced by Karmarkar-Karp Algorithm 1 extension to solve kBP . Then, $A(D_k) \leq (1 + 2 \cdot k \cdot \epsilon)OPT(D_k) + \frac{1}{2\cdot\epsilon^2} + (2 \cdot k + 1)$.*

We give the proof of the above Theorem and the running time of the Karmarkar-Karp Algorithm 1 extension to kBP in Section H.2.

Karmarkar-Karp Algorithm 2 extension to kBP. Algorithm 2 of Karmarkar and Karp uses the *alternative geometric grouping technique*. Let J be some instance and $g > 1$ be some integer parameter, then, alternative geometric grouping partitions the items in J into groups such that each group contains the necessary number of items so that the size of each group but the last (i.e. the sum of the item sizes in that group) is at least $g \cdot S$. See Section H.3 for more details on alternative geometric grouping and the extension of the Karmarkar-karp Algorithm 2 to kBP .

Theorem 6.3. *Let $A(D_k)$ denote the number of bins produced by Karmarkar-Karp Algorithm 2 extension to solve kBP . Then, $A(D_k) \leq OPT(D_k) + O(k \cdot \log^2 OPT(D))$.*

We give the proof of the above Theorem and the running time of the Karmarkar-Karp Algorithm 2 extension to kBP in Section H.3.

7 Egalitarian allocation of watts

So far, our aim was to maximize the connection time during which a household is connected to electricity. Another possible direction of research in the egalitarian allocation of the electricity distribution problem is to maximize the minimum allocation of electricity in watts. In terms of allocation of electricity in watts, the utility of an agent i is defined as

$$u_i(\mathcal{I}) = \sum_{l:i\in A_l} D[i] \cdot |I_l|$$

Then, the optimization objective is

$$\max_{\mathcal{I}} \min_{i\in[n]} u_i(\mathcal{I})$$

where the maximum is over all partitions that satisfy the demand constraints. This max-min value is called the *egalitarian watts allocation* of the given instance.

7.1 Examples

We illustrate the difference between the two variants of the egalitarian allocation problem using several examples. Throughout the examples, the demands are given in kW, and the time-interval for allocation is 1 hour.

Example 7.1. *There are two agents, with demands $D[1] = 3$ and $D[2] = 5$ kW. The supply is 5 kW. Egalitarian allocation of time (as discussed in the previous section) would connect each agent exactly $1/2$ of an hour; the agents get $3/2$ and $5/2$ kWh respectively. Egalitarian allocation of watts would connect agent 1 for $5/8$ of an hour and agent 2 for $3/8$ of an hour, so that each agent gets $15/8$ kWh.*

With egalitarian watts allocation, agents with small demands will necessarily get small allocation of watts, even if they are connected all the time. If we look only on the smallest amount of watt allocation, the problem might become uninteresting.

Example 7.2. *There are three agents with demand $D[1] = M, D[2] = M-1, D[3] = 1$ for some large integer $M > 1$. The supply is $S = M + 1$. Agent 3 can get at most 1 kWh. Hence, an algorithm that only consider the minimum amount can connect agents $\{1, 3\}$ for $1/M$ of the time and agents $\{2, 3\}$ for $(M-1)/M$ of the time; then agents 1 and 3 get 1 kWh each, and agent 2 gets $(M-1)^2/M$ kWh. This allocation is clearly unfair towards agent 1.*

A fairer solution would be to connect agents $\{1, 3\}$ for a fraction $\frac{M-1}{2M-1}$ of the time, and the agents $\{2, 3\}$ for a fraction $\frac{M}{2M-1}$ of the time. Then, agent 3 gets 1 kWh, whereas agents 1 and 2 both get $\frac{M(M-1)}{2M-1}$ kWh.

One way to get to the fairer solution is to use the *leximin criterion*. A simple way to apply this criterion is using the following two-step allocation process:

1. Choose an integer $g \geq 0$, and choose a subset G of the g agents with smallest demand; these agents will be connected all the time.
2. Compute an egalitarian watts allocation for a reduced instance, in which the agents are $D \setminus G$ and the supply is $S - V(G)$ (the original supply minus the total demand of agents in G).

Run this procedure for different values of g, and pick the solution with the highest *leximin vector*.

In Example 7.1 above, the optimal g is 0; in Example 7.2, the optimal g is 1.

In Section 4, we proved that egalitarian time allocation can always be solved using kBP . Formally, there exists a function $K(n)$ such that, any egalitarian time allocation problem with n agents can be solved by computing an optimal kBP solution for some $k \leq K(n)$, and connecting each bin an equal amount of time.

This procedure clearly does not yield an egalitarian watts allocation, as agents with a smaller demand should be connected for a longer time. One could hope that this approach could be extended by having a different k for different agents. For example, in Example 7.1 we could choose $k_1 = 5$ and $k_2 = 3$. The solution to the generalized kBP instance would have 8 bins: 5 bins with agent 1 alone and 3 bins with agent 2 alone. Connecting each bin for $1/8$ of the time would yield the egalitarian watts allocation.

One could think that setting k_i inversely proportional to $D[i]$, such that $k_i \cdot D[i]$ would be a constant for all agents, would lead to an egalitarian watts allocation. But this is not the case:

Example 7.3. *There are three agents with demands* $D[1] = 1, D[2] = 2, D[3] = 3$*, and* $S = 5$*. To make the product* $k_i \cdot D[i]$ *constant, we need* $k_1 = 6, k_2 = 3, k_3 = 2$*. The optimal bin allocation satisfying these requirements has 6 bins:* $\{1,2\}, \{1,2\}, \{1,2\}, \{1,3\}, \{1,3\}, \{1\}$*. Connecting each bin* $1/6$ *of an hour results in each agent receiving* 1 *kWh.*

But if we consider only the following 5 *bins:* $\{1,2\}, \{1,2\}, \{1,2\}, \{1,3\}, \{1,3\}$*, and connect each bin for* $1/5$ *of the time, agent 1 still gets 1 kWh, whereas agents 2 and 3 get more: each of them gets* $6/5$ *kWh.*

This issue can be solved by the same procedure mentioned after Example 7.2: taking $g = 1$ leads to connecting agent 1 all the time, and solving the remaining instance with $k_2 = 3$ and $k_3 = 2$.

Unfortunately, we cannot get a universal bound on the required k_i's (that depends only on n).

Example 7.4. *There are two agents with demand* $D[1]$ *and* $D[2]$*, which are co-prime integers. Suppose* $S < D[1] + D[2]$ *but* $S \geq \max\{D[1], D[2]\}$*. Then, the only feasible configurations are* $\{D[1]\}$ *and* $\{D[2]\}$*. For an egalitarian allocation,* $D[1]$ *should be connected* $\frac{D[2]}{D[1]+D[2]}$ *fraction of the time, and* $D[2]$ *should be connected* $\frac{D[1]}{D[1]+D[2]}$ *fraction of the time. This means that, if each bin is connected the same amount of time, then we should have* $D[2]$ *bins with configuration* $\{D[1]\}$ *and* $D[1]$ *bins with configuration* $\{D[2]\}$*. So we have* $k_1 = D[2]$ *and* $k_2 = D[1]$*. These numbers can be arbitrarily high, even though* $n = 2$ *is fixed.*

Another potential problem with the above approach arises when there are some items with very small item sizes compared to the largest item size. Imagine the situation where the smallest item in the instance is, say, 0.05 and the maximum item size in the instance is, say, 14. In this case, there are 280 copies of this smallest item w.r.t a single copy of the maximum item size. The final derived instance will be a large instance in terms of the number of items. Moreover, in this case, the equal watts allocation is limited by the small item size, i.e., the agent with max demand 14 will not get electricity more than 0.05(kW). We can determine such items $D[i]$, and g is initialized with the count of such items.

Assumptions: — We assume that $V(D) > S$. Otherwise, the problem is trivial, as we can connect all agents simultaneously, resulting in allocating the watts equal to their demand.
— When we say "original" kBP it means that we are talking about the kBP problem where each item appears exactly k times in an instance (as mentioned in Sections 1.1 and 3.2).

Below, we present a number of heuristic algorithms to tackle the problem of the electricity distribution of watts as equally as possible.

In all heuristics, we first determine a set of g agents with smallest demands that will be connected all the time. Then, we apply heuristics to the remaining supply and the remaining agents.

We denote $d_{\max} := \max(D) =$ the largest demand in D. Here, we use the term "item" and "demand" interchangeably.

7.2 Heuristic Algorithm 1 (HA1)

Let $D^{\uparrow}$ be the instance D sorted in non-decreasing order. Let d_l be the maximum demand in $D^{\uparrow}$ such that the sum of all the item sizes $\leq d_l$ is at most $S - d_{\max}$ i.e. they all can be packed in a bin whose capacity is $S - d_{\max}$, and adding the next item larger than d_l violates the bin capacity (with bin size $S - d_{\max}$) constraint. Clearly, such an item exists; otherwise, all the items can be packed into a single bin and connected all the time.

Example 7.5. *Let* $D = \{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7, 3, 3.2, 6.5, 6.7, 14, 14.2\}$ *and supply is* $S = 21$*. Then,* $d_{\max} = 14.2$*.* $D^{\uparrow}$ *will be* $D^{\uparrow} = \{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7, 3, 3.2, 6.5, 6.7, 14, 14.2\}$*. Note that* $S - d_{\max} = 21 - 14.2 = 6.8$*.* d_l *will be* 1.7 *because the sum of all the items* $\leq d_l = 1.7$ *(that is the items in the set* $\{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7\}$*) will be* 6.26 *that is* $\leq S - d_{\max} = 6.8$*. Clearly, the next item larger than* d_l *cannot be added to it, as then their sum will be* 9.26 *that is,* $> S - d_{\max} = 6.8$*.*

Let $G \subseteq D$ be some subset of items such that all the items in G have sizes $\leq d_l$. The updated set of remaining agents is $D' = D \setminus G$, and the updated supply is $S' = S - V(G)$. We determine G as follows:

– For some natural g, G contains the first g items in the set $D^{\uparrow}$. In Example 7.5, there are no smallest items for which the number of copies is too large w.r.t a single copy of the maximum item size. Hence, initially g will be 0. such items. Now, there are 9 possible values of g viz, $0, 1, 2, 3, 4, 5, 6, 7, 8$. As an example, for $g = 2$, the set G will be $G = \{0.2, 0.22\}$.

Our heuristic algorithm 1 connects agents in G all the time. Hence, these agents get their required demand. We derive an instance D'' as follows:

– for each item $D'[i]$ in D' we compute the number of copies that needs to be created for (nearly) equal allocation of watts as: $\frac{k \cdot d_{\max}}{D'[i]}$ rounded to the nearest integer, where k denotes the number of copies of $d_{\max}$.
– Now,
  1. for each item $D'[i]$ in D', if the number of copies of demand $D'[i]$ are non-zero then take a copy of it and insert it into the instance D''. Reduce the number of copies of demand $D'[i]$ by 1.
  2. repeat the above procedure until the number of copies of each item becomes 0.

Now, we can apply FFk or $FFDk$ to the derived instance D''. Let B be the resulting solution. Solution B is *incomplete* in the sense that they do not contain items in G. To each bin $B_i \in B$, we add the items in G, these are the agents that are connected all the time. Doing so will *complete* the solution B. After that, compute the watts allocation vector for this computed *complete* solution B. Different such solutions can be computed for all possible values of g. Among these solutions, the one that is leximin-preferred over other solutions (this is done by comparing the watts allocation vector) is chosen as the best solution. Ties are broken arbitrarily.

The above approach can be time-consuming as there can be different possible values of g for each instance. To run it faster, we use the ternary search [5] to determine the possible solution that is leximin-preferred over other computed solutions.

The rationale behind using the ternary search is as follows: if G contains too few demands then the watts allocation may be low as the remaining agents may contain small demands; if G contains too many agents then again the watts allocation for the remaining agents may be low because the remaining capacity may not be sufficient to pack as many demands as possible from the remaining set of demands. Therefore, the peak lies somewhere in between them. Examples to support this claim have been given in Section I.

Example 7.6. *First, we consider Example 7.2 In this example there are three agents with demands* $M, M-1, 1$ *respectively, and the supply is* $M+1$. *There are two possibilities for G: for* $g = 0$ *we get* $G = \{\}$, *and for* $g = 1$ *we get* $G = \{1\}$. *In both of the cases for g there is enough remaining space to pack item* $d_{\max} = M$. *However, for* $g = 2$ *we get* $G = \{1, M-1\}$ *and there is not enough remaining space to pack item* $d_{\max} = M$.

– *When* $G = \{\}$, *HA1 will have* $\frac{k \cdot M}{1}$ *copies of item* 1, *and* $\frac{k \cdot M}{M-1}$ *(rounded to* 0 *decimal points)copies of item* $M-1$ *where k is the number of copies of item M. Then, the rest of the algorithm computes the egalitarian allocation of watts.*

To determine the final allocation let's assume that $\frac{k \cdot M}{M-1}$ *is an integer. Since,* $k \cdot M > \frac{k \cdot M}{M-1} + k = \frac{k \cdot (2M-1)}{M-1}$ *for large* $M > 1$ *and also, the item* $M-1$ *and M can not be packed into the same bin, we can say that the number of bins that contain item* 1 *are sufficient to pack the items* $M-1$ *and M. In fact, to pack these items we require no more than* $k \cdot M$ *bins. Therefore,*

– *Item* 1 *will get* $1 \cdot \frac{k \cdot M}{k \cdot M} = 1kW$, *and*
– *item* $M-1$ *will get* $(M-1) \cdot \frac{\frac{k \cdot M}{M-1}}{k \cdot M} = 1kW$, *and*
– *item M will get* $M \cdot \frac{k}{k \cdot M} = 1kW$.

The allocation vector of watts in this case is $(1, 1, 1)$.

– *When* $G = \{1\}$, *HA1 connects the agent* 1 *all the time. For the remaining agents* $\{M-1, M\}$, *HA1 will have* $\frac{k \cdot M}{M-1}$ *copies of item* $M-1$ *where k is the number of copies of item M. Note that the items* $M-1$ *and M can not fit into a single bin. We require* $\frac{k \cdot M}{M-1} + k = \frac{k \cdot (2M-1)}{M-1}$ *bins to pack the item* $M-1$ *and M, and each of these bins can accommodate item* 1. *Therefore,*

– *Item* 1 *will get* $1 \cdot \frac{\frac{k \cdot (2M-1)}{M-1}}{\frac{k \cdot (2M-1)}{M-1}} = 1kW$ *of electricity, and*
– *item* $M-1$ *will get* $(M-1) \cdot \frac{\frac{k \cdot M}{M-1}}{\frac{k \cdot (2M-1)}{M-1}} = \frac{(M-1) \cdot M}{2M-1} > 1kW$ *of electricity, and*
– *item M will get* $M \cdot \frac{k}{\frac{k \cdot (2M-1)}{M-1}} = \frac{M \cdot (M-1)}{2M-1} > 1kW$ *of electricity.*

The allocation vector of watts in this case is $1, \frac{M \cdot (M-1)}{2M-1}, \frac{M \cdot (M-1)}{2M-1}$ *which is better than the allocation vector* $(1, 1, 1)$. *HA1 outputs this leximin preferred solution.*

We can understand above description with the help of an example. Let's take $M = 5$. *The three agents with demand* $M, M-1, 1$ *are* $5, 4, 1$ *respectively, and the supply is* $S = M + 1 = 6$. *Now, the feasible*

[5] We are thankful for the following stackoverflow answer `https://stackoverflow.com/a/40695871`.

values for g are $0, 1$ (Note that we have already argued why $g = 2$ is not feasible).
– For $g = 0$ we get $G = \{\}$. Assume that $k = 4$, so there are 4 copies of the item $d_{\max} = 5$. In this case HA1 will have $\frac{4 \cdot 5 = 10}{1} = 20$ copies of item 1, and $\frac{4 \cdot 5 = 10}{4} = 5$ (after rounding to 0 decimal points) copies of item 4. Now the derived instance D'' is $\{5, 4, 1, 5, 4, 1, 5, 4, 1, 5, 4, 1, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$ and the updated supply is $S' = S - V(G) = 6 - 0 = 6$. Applying FFk to instance D'' results in the following packing:

$$\{5, 1\}, \{4, 1\}, \{5, 1\}, \{4, 1\}, \{5, 1\}, \{4, 1\}, \{5, 1\}, \{4, 1\},$$
$$\{4, 1\}, \{1\}, \{1\}, \{1\}, \{1\}, \{1\}, \{1\}, \{1\}, \{1\}, \{1\}, \{1\}$$

To each bin in this solution we add all the items in G. Here G is empty so the final solution is same as computed above. The final allocation of supply in this will be $\frac{4}{20} \cdot 5 = 1kW$ for agent with demand $5kW$ and $\frac{5}{20} \cdot 4 = 1kW$ for agent with demand $4kW$ and $1kW$ for agent with demand $1kW$. Final allocation vector in this case is $(1, 1, 1)$
– For $g = 1$ we get $G = \{1\}$. Assume that $k = 4$, so there are 4 copies of the item $d_{\max} = 5$. In this case HA1 will have $\frac{4 \cdot 5}{4} = 5$ (after rounding to 0 decimal points) copies of item 4. Now the derived instance D'' is $\{5, 4, 5, 4, 5, 4, 5, 4, 4\}$ and the updated supply is $S' = S - V(G) = 6 - 1 = 5$. Applying FFk to instance D'' results in the following packing: $\{5\}, \{4\}, \{5\}, \{4\}, \{5\}, \{4\}, \{5\}, \{4\}, \{4\}$. To each bin in this solution we add all the items in G. After adding, the final solution is $\{5, 1\}, \{4, 1\}, \{5, 1\}, \{4, 1\}, \{5, 1\}, \{4, 1\}, \{5, 1\}, \{4, 1\}, \{4, 1\}$. The final allocation of supply in this will be $\frac{4}{9} \cdot 5 = \approx 2.2kW$ for agent with demand $5kW$ and $\frac{5}{9} \cdot 4 \approx 2.2kW$ for agent with demand $4kW$ and $1kW$ for agent with demand $1kW$. Final allocation vector in this case will be $(1, 2.2, 2.2)$.
– Clearly, allocation $(1, 2.2, 2.2)$ is preferred over allocation $(1, 1, 1)$.

Example 7.7. *$D = \{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7, 3, 3.2, 6.5, 6.7, 14, 14.2\}, S = 21$.*

– The feasible values for g in this case are: $0, 1, 2, 3, 4, 5, 6, 7, 8$ (9 is not a feasible value for g. See explanation of the HA1 algorithm).

– For $g = 0$ we get $G = \{\}$. Assume that $k = 3$ so there are 3 copies of the item $d_{\max} = 14.2$. In this case HA1 will have $\frac{3 \cdot 14.2 = 42.6}{0.2} = 213$ copies of the item with size 0.2. Similarly, the number of copies of other items can be computed after rounding to 0 decimal points. The below table shows the item along with their number of copies:

item	*0.2*	*0.22*	*0.4*	*0.42*	*0.8*	*0.82*	*1.7*	*1.7*	*3*	*3.2*	*6.5*	*6.7*	*14*	*14.2*
#copies	*213*	*194*	*106*	*101*	*53*	*52*	*25*	*25*	*14*	*13*	*7*	*6*	*3*	*3*

HA1 creates the instance D'' as follows:

* *first it picks the first item in D' and checks if the number of copies are non-zero. Then picks up this item and inserts it into D''. Then it picks the second item in D and does the same. Following this, D'' contains the 3 copies of the instance $\{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7, 3, 3.2, 6.5, 6.7, 14, 14.2\}$,*
* *After doing this the updated list of items and their corresponding number of copies are:*

item	*0.2*	*0.22*	*0.4*	*0.42*	*0.8*	*0.82*	*1.7*	*1.7*	*3*	*3.2*	*6.5*	*6.7*	*14*	*14.2*
#copies	*210*	*191*	*103*	*98*	*50*	*49*	*22*	*22*	*11*	*10*	*4*	*3*	*0*	*0*

  Note that the item 14 and 14.2 do not have any copies left. Upon repeating the same procedure as explained in above step, the instance D'' will contain 3 copies of the following instance: $\{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7, 3, 3.2, 6.5, 6.7\}$.
* *HA1 repeats the above procedure until the number of copies of each item becomes 0. After that it is not possible to make any addition to the instance D'' since we are left with no items.*

Now, HA1 applies kBP to instance D''. To each bin in the resulting solution HA1 adds all the items in G to get the final solution. The minimum supply delivered in this case is $0.18613kW$.

– For $g = 8$ we get $G = \{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7\}$. These are the set of items that are connected all the time. The remaining set of items is $D' = \{3, 3.2, 6.5, 6.7, 14, 14.2\}$. There are $k = 3$ copies of item $d_{\max} = 14.2$. HA1 will have $\frac{14.2 \cdot 3 = 42.6}{3} \simeq 14$ copies of item 3. Similarly, HA1 computes the number of copies of other items in the remaining set of items. The below table shows the item along with their number of copies:

item	*3*	*3.2*	*6.5*	*6.7*	*14*	*14.2*
#copies	*14*	*13*	*7*	*6*	*3*	*3*

HA1 creates the instance D'' as follows:

* *first it picks the first item in D' whose remaining number of copies are non-zero. It then picks up this item and inserts it into D''. In doing so HA1 reduces the number of copies of this item by 1. Then, it picks the second item in D' with non-zero number of copies and inserts it at the end of D''. Again, HA1 reduces by 1 the numbere of copies of this item. Following this procedure D'' contains the consecutive 3 copies of the following instance: $\{3, 3.2, 6.5, 6.7, 14, 14.2\}$. The updated table showing the items and their corresponding number of copies is:*

item	*3*	*3.2*	*6.5*	*6.7*	*14*	*14.2*
#copies	*11*	*10*	*4*	*3*	*0*	*0*

* *HA1 repeats the above procedure until the remaining number of copies of each item becomes 0. After which no further insertion of items is possible.*

Now, HA1 applies kBP to this created instance D'', and to each bin in the resulting solution HA1 adds all the items in G. The minimum supply delivered in this case is $1.74783kW$.

– HA1 computes different solutions in this way for different values of g (HA1 uses ternary search Algorithm 7 to select only a few values of g for which the solution is to be computed). Finally, it outputs a solution that is leximin preferred over other solutions. In this example, HA1 outputs the solution corresponding to $g = 8$ in which items in set $\{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7\}$ are connected all the time.

7.3 Heuristic Algorithm 2 (HA2)

Our next heuristic algorithm uses the geometric grouping technique of [30]. The idea behind this heuristic is as follows:

First, we determine the set of small items using the same technique described in Section 7.2 (G is initialized to this set of items; For more details, see Section 7.2). Let D' be the set of remaining demands and S' be the updated supply. We pack the items in D' using the approach described below, and to each bin in this solution, we add the items in G.

Recall that the geometric grouping works as follows: Let D' be some instance to pack the items. Let $d_{\min}$ and $d_{\max}$ be the minimum and maximum item in D'. We define $r := \lfloor \log_2 \frac{d_{\max}}{d_{\min}} \rfloor$. Now, we divide the instance into groups L_i such that each group L_i contains all the items whose sizes lie in the interval $(d_{\max} \cdot 2^{-(i+1)}, d_{\max} \cdot 2^{-i}]$ for $i = 0, \ldots, r$. Let L be the set of all groups formed using geometric grouping. Now, depending on the number of groups formed using geometric grouping, we do the following computation:

– If there is only one group $L_1 = D'$, then we compute the "original" kBP solution for this group for some value of k. The watts allocation for each agent in L_1 then is $\frac{k}{B(L_1)}$ where $B(L_1)$ is the number of bins in the "original" kBP solution to L_1.

– **Warm-up case:** Let us assume that there are only two groups L_0, L_1. Note that, $L_1[i] \in L_1 \geq \frac{d_{\max}}{2^2} = \frac{1}{2} \cdot \frac{d_{\max}}{2^1}$, and $L_0[j] \in L_0 \geq \frac{d_{\max}}{2^1}$ Therefore, connecting the agents in group L_1 twice the time allocated to group L_0 will result in a near equal allocation of supply to the agents in group L_1, L_0.
General case: Let $\#L$ be the number of groups resulting from geometric grouping. For each group $L_i \in L$ we compute the "original" kBP solution and let B_i denote both the number of bins and also the set of bins in the "original" kBP solution to group L_i. To compute the equal watts allocation we generalize the idea as mentioned in the warm-up case to compute the time that each bin is connected to so as to result in the equal watts allocation.
Let us denote by T_{L_i} as the final allocation of time to each bin in group L_i. Then,

$$\sum_{i=0}^{\#L-1} T_{L_i} \cdot B_i = 1 \tag{13}$$

Let us denote by τ_{L_i} the final time allocated to each agent in the group L_i. It may happen that k is different for different groups. Let k_{L_i} be the k for group L_i. Since there are k_{L_i} copies of each agent, therefore the total time allocated to each agent in group L_i is

$$\tau_{L_i} = T_{L_i} \cdot k_{L_i} \tag{14}$$

for group L_i we compute the time allocated to each agent in L_i as

$$\tau_{L_i} = 2 \cdot \tau_{L_{i-1}} \tag{15}$$

substituting the value of τ_{L_i} and $\tau_{L_{i-1}}$ from Equation (14) in Equation (15) we get that

$$\begin{aligned} T_{L_i} \cdot k_{L_i} &= 2 \cdot T_{L_{i-1}} \cdot k_{L_{i-1}} \\ T_{L_i} &= 2 \cdot T_{L_{i-1}} \cdot \frac{k_{L_{i-1}}}{k_{L_i}} \end{aligned} \tag{16}$$

from Equation (13) and Equation (16) we get that

$$T_{L_0} = \frac{1}{\sum_{i=0}^{\#L-1} B_i \cdot 2^i \cdot \frac{k_{L_0}}{k_{L_i}}} \tag{17}$$

From this, we can compute the time for each bin in other groups using Equation (16) and hence the time for each agent using Equation (14).

Consider the following example:

Example 7.8. *D = {0.2, 0.4, 0.8, 1.7, 3, 6.5, 14}, S = 21*

In the above example, applying geometric grouping will result in 7 groups $L_0 = \{14\}, L_1 = \{6.5\}, L_2 = \{3\}, L_3 = \{1.7\}, L_4 = \{0.8\}, L_5 = \{0.4\}, L_6 = \{0.2\}$, one item in each group. Following the above procedure for computing the resulting allocation of time will result in nearly ~ 0.1kW of supply to each agent, which is very small.

To alleviate the problem highlighted by the above example, we use the same approach of Section 7.2 in which items are grouped together to connect all the time (G is augmented with this set), and for the remaining items, we compute the equal allocation of supply. Here, we determine the different groups that can be connected all the time and for the remaining groups we compute the (possible) equal allocation of watts. Following this, Example 7.8 results in always connecting the groups L_2, L_3, L_4, L_5, L_6, and agents in the group L_1, L_0 will get the supply 4.33 kW and 4.67 kW, respectively.

This allocation of watts to agents in the group L_0, L_1 is as follows:

– The remaining space in bin to pack the items in group L_0, L_1 is 14.9.

– For each group, only one bin is needed to pack the items.

– Assume that k is the same for each bin. Using Equation (17) and Equation (14) we can determine that the time allocated to each item in L_0 is 1/3.

– Similarly, using Equation (16) and Equation (14), we can determine that the time allocated to each agent in L_1 is 2/3.

– Therefore, the agent in group L_0 will get $14 \cdot \frac{1}{3} \sim 4.67$kW, and the agent in group L_1 will get $6.5 \cdot \frac{2}{3} \sim 4.33$kW of electricity.

In a realistic scenario, it is highly unlikely that each group resulting from the geometric grouping contains a single item as the demand of an agent is very small compared to the supply. There will be less different possible values of g (to determine the groups that are connected all the time). Hence, keeping this in mind, we are not using the ternary search here.

Example 7.9. *First, we consider Example 7.2. To repeat, in that example there are three agents with demand $1, M-1, M$ and supply $M+1$. The geometric grouping of HA2 forms two groups: $L_0 = \{M-1, M\}$, and $L_l = \{1\}$ where l is such that agent with demand 1 lies in $(M \cdot 2^{-(l+1)}, M \cdot 2^{-l}]$. It then applies "original" kBP solution to pack the items in each of the groups.*

Note that two bins are needed to pack the two items in group L_0. Also, it holds that $2^l \leq M < 2^{l+1}$.

– Now, there are two possibilities, either no group is connected all the time, or group L_l is connected all the time.

– In the first case, HA2 computes the final time allocated to each bin in the packing of items in L_0 using Equation (17) (from that we can easily determine the time for each agent using Equation (14)).

For the sake of simplicity, we assume that $k = 1$. In the first case, the computation is as follows: using Equation (17), Equation (16) and Equation (14), the time allocated to item $M, M-1$ and 1 is $\frac{1}{2+2^l}, \frac{1}{2+2^l}$ and $\frac{2^l}{2+2^l}$ respectively. Agents $M, M-1, 1$ are allocated $\frac{M}{2+2^l}, \frac{M-1}{2+2^l}$ and $\frac{2^l}{2+2^l}$ kW of electricity respectively. The minimum allocation of watts in this case will always be $\frac{2^l}{2+2^l} < 1$.

– In the second case, HA2 determines only the time allocated to each bin in group L_0 (from that we can easily determine the time for each agent using Equation (14)). Finally, to each bin in B_0, HA2 adds the agents in L_l.

Using Equation (17) and Equation (14), the time allocated to each agent in $\{M, M-1\}$ is $\frac{1}{2}$. Therefore, the watts allocation for agents $M, M-1, 1$ are $\frac{M}{2}, \frac{M-1}{2}, 1$ respectively. It is easy to observe that the minimum allocation of watts in this case is 1.

– Finally, from among the above two solutions, one can observe that the allocation of watts in the second case is always leximin-preferred over the allocation of watts in the first case.

Example 7.10. *Applying geometric grouping on the same Example 7.7 results in 7 groups $L_0 = \{14, 14.2\}, L_1 = \{6.5, 6.7\}, L_2 = \{3, 3.2\}, L_3 = \{1.7, 1.7\}, L_4 = \{0.8, 0.82\}, L_5 = \{0.4, 0.42\}, L_6 = \{0.2, 0.22\}$, two items in each group. The groups L_6, L_5, L_4, L_3 are connected all the time.*

The remaining bin capacity is 14.74. Since kBP is applied to each group with this remaining bin capacity, one can see that the items in L_1, L_2 can be packed in a single bin whereas packing the items in L_0 requires two bins. The time allocated to each bin in group L_0 is 0.125, and the time allocated to group L_1 is 0.25, and the time allocated to group L_2 is 0.5. Therefore, the resulting egalitarian allocation is 1.5 kW.

7.4 Heuristic Algorithm 3 (HA3)

Our next heuristic uses the alternative geometric grouping as in [30]. Among all the groups $V_{\max}$ denotes the maximum group size. The idea behind the alternative geometric grouping adopted for our algorithm is as follows:

First, the input instance D is sorted in a non-increasing order of item sizes. Then, starting from the largest item, items are grouped such that the sum of the item sizes in the group is sufficient to equal or exceed $\mathfrak{u} \cdot d_{\max}$, for some positive $\mathfrak{u}$. In [30], $\mathfrak{u}$ required to be a positive integer; but in our case, $\mathfrak{u}$ can be a positive real number.

Note that the last group may have a size less than $\mathfrak{u} \cdot d_{\max}$. In this case, we treat the group as if its (pseudo) group size is $\mathfrak{u} \cdot d_{\max}$, the reason being it may be the case that this last group may contain a very small item and may result in the creation of a large number of copies of this group (creating several copies of the group will be apparent in the later part of this algorithm). Let E be the set of all the groups and $\#E$ the cardinality of this set. We treat each group as if it is a single agent with demand equal to the group size (and equal to the pseudo group size for the last group or alternatively, we only consider the pseudo group size of each group. The pseudo group size of a group can be defined as the actual group size if it is not the last group. For the last group, it is computed as explained earlier). Now, we determine the groups that are connected all the time. For that we use the same approach as discussed in Section 7.2.

Now, for each remaining group $E_i \in E$ (that are not connected all the time) we compute the number of copies that need to be created for as equally as possible allocation of watts (to the group) as: $\frac{k \cdot V_{\max}}{V(E_i)}$, where k denotes the number of copies of the group with group size $V_{\max}$.

After that we create an instance D_{group} as follows:

1. for each group E_i in E (such that the number of remaining copies of E_i is a positive non-zero integer) create a copy of an agent whose demand is $V(E_i)$, and insert it into the instance D_{group}. Reduce the number of copies of demand $V(E_i)$ by one.
2. repeat the above procedure till the number of copies of each demand becomes 0.

Now that we have an instance to work with, we can apply the FFk or $FFDk$ on D_{group}. Let B is the resulting solution. Each item $V(E_i)$ in solution B will be replaced by the corresponding group. Also, to each bin in solution B we add the group(s) that are connected all the time. To speed up the process, we use ternary search, and determine the solution that is leximin-preferred over other computed solutions.

Example 7.11. *We illustrate the working of HA3 on the same data of Example 7.2: three agents with demand $M, M-1, 1$ and supply capacity of $M+1$. For simplicity, we take $\mathfrak{u}$ as 1. The alternative geometric grouping in HA3 groups the agents in $\{M, M-1, 1\}$ such that in each group there are sufficient items to have size, $V(\cdot)$, of the group $\geq \mathfrak{u} \cdot d_{\max} = M$. The alternative geometric grouping then results in $\{M\}, \{M-1, 1\}$ groups. Now, we determine the groups that can be connected all the time. Note that in this direction there is no group that can be connected all the time because if we choose any of the group to connect all the time, then the remaining supply will be insufficient to pack other groups (HA3 treats*

each group as a single agent with demand equal to the aggregate demand of agents in the group). HA3 then creates $\frac{k\cdot(V_{\max}=M)}{M}$ *copies of the group with group size* $V(\{M-1,1\}) = M$ *where* k *is the number of copies of the group with group size* $\max\{\{M\},\{M-1,1\}\}$. *After that, the algorithm creates the instance* D_{group} *that contains* k *copies of the instance* $\{M, M\}$. *Applying* FFk *on instance* D_{group} *will place each item in a separate bin. Overall,* $2k$ *bins are required to pack the items in* D_{group}, *and each item will appear in* k *bins. Replacing an item in each bin in the final solution with the corresponding group, the final supply vector is* $\{0.5, \frac{M-1}{2}, \frac{M}{2}\}$.

Example 7.12. *On applying the alternative geometric grouping on the same Example 7.7 with* $\mathfrak{u} = 0.25$ *results in the following groups:* $\{14.2\},\{14\},\{6.7\},\{6.5\},\{3.2,3\},\{1.7,1.7,0.82\},\{0.8,0.42,0.4,0.22,0.2\}$. *Note that the items in each group are sufficient to make group size* V *equal or exceed* $\mathfrak{u}\cdot d_{\max} = 3.55$.

The instance on which algorithm works further is $\{14.2,14,6.7,6.5,6.2,4.22,3.55\}$. *Note that the demand of an item in this instance is equal to the group size of the corresponding group. For example, the demand* 6.2 *is equal to the group size of the group* $\{3.2,3\}$. *Note that the last demand* 3.55 *is the pseudo group size of the group* $\{0.8,0.42,0.4,0.22,0.2\}$ *because the sum of all the item sizes in that group is less than* $\mathfrak{u}\cdot d_{\max} = 3.55$. *The maximum group size is* $V_{\max} = 14.2$.

After that, the algorithms does further grouping and determine the number of copies that need to be created for each item in $\{14.2,14,6.7,6.5,6.2,4.22,3.55\}$. *We take* $k = 3$ *as the number of copies for demand* 14.2. *For the remaining demands, the algorithm determines the number of copies as explained above in the algorithm.*

– *For* $g = 0$, *after applying* FFk *and replacing an item in each bin in the final solution with the corresponding group, the final supply vector is* $\{0.12632,0.13895,0.25263,0.26526,0.43158,0.50526,0.89474,0.89474,1.10526,1.17894,2.11579,2.21046,2.24204,2.39473\}$. *Egalitarian allocation of supply in this case is* 0.12632.

– *For* $g = 1$, *the item* 3.55 *is connected all the time (and therefore all the items in the group* $\{0.2,0.22,0.4,0.42,0.8\}$ *corresponding to this item are connected all the time in the final solution). The algorithm creates the instance* D_{group}. *After applying* FFk *on instance* D_{group} *and replacing an item in each bin in the final solution with the corresponding group, the final supply vector (for all the items that are not connected all the time) is* $\{0.5125,1.0625,1.0625,1.3125,1.4,2.5125,2.625,2.6625,2.84375\}$, *and the egalitarian allocation of watts is* $0.5125kW$. *To each bin in the final solution, the algorithm adds all the items in the group* $\{0.2,0.22,0.4,0.42,0.8\}$ *corresponding to item* 3.55.

– *Finally, the algorithm outputs a solution that is leximin preferred over other solutions. In this example, HA3 outputs the solution corresponding to* $g = 1$ *in which all the items in the group* $\{0.2,0.22,0.4,0.42,0.8\}$ *are connected all the time.*

7.5 Heuristic Algorithm 4 (HA4)

Our last heuristic algorithm uses the "original" kBP solutions to compute a leximin-preferred solution for an input instance. The algorithm works as follows: Let $d_{\max}$ be the maximum demand item in D. Now, we determine the item $d_l \in D$ such that the sum of all the item sizes $\leq d_l$ is at most $S - d_{\max}$, and adding the next demand $d'_l > d_l$ violates the bin capacity constraint, that is, the sum of all the item sizes $\leq d'_l$ is larger than the bin capacity $S - d_{\max}$. Let G be the set of items that are connected all the time as explained and determined in Section 7.1, and each item in G has size $\leq d_l$.

The updated set of demands is $D' = D - G$, and the updated supply is $S' = S - V(G)$. Now, we connect items in G all the time, and for the remaining items, we compute the "original" kBP solution (using either FFk or $FFDk$) and return the solution which is leximin-preferred over other solutions. To make the algorithm run faster, we use Ternary Search (see Algorithm 7).

The main difference with Section 7.2 is that, here the number of copies of each item will remain same whereas in Section 7.2 for an item $D'[i]$ in the remaining set of items the number of copies are computed as $\frac{k\cdot d_{\max}}{D'[i]}$ (where k is the number of copies of item $d_{\max}$) which need not be the same for all the remaining items.

Algorithm 1: Heuristic Algorithm 4

Input: alg:= algorithm to use to compute kBP solution;
D:= the input set of agents' demands;
S:= supply;
k:= as in kBP ;
Output: leximin-preferred best solution;
1 D_{sorted}:= sorted (in non-decreasing order) instance of D
2 $d_{\max} = max(D)$
3 determine d_{large} such that sum of all item sizes $\leq d_{large}$ is at most $S - d_{\max}$ and adding the next largest element after d_{large} violates the bin capacity constraint with bin size $S - d_{\max}$
4 `bestSolution`:= the solution returned by Ternary Search(see Algorithm 7)

Example 7.13. *We illustrate the working of algorithm HA4 on Example 7.2. That example has three agents with demand $M, M-1, 1$, respectively, and the supply capacity is $M+1$. The set of agents that can be connected all the time are: $\{\}, \{1\}$. For the set $\{\}$, HA4 applies "original" kBP to the remaining set of agents $\{M, M-1, 1\}$, and computes the egalitarian allocation of watts. Then, HA4 connects the agents in $\{1\}$ all the time and for the remaining set of agents $\{M, M-1\}$ applies the "original" kBP solution, and computes the egalitarian allocation of watts. Also, to each bin in the solution, HA4 adds the agents in $\{1\}$. Finally, HA4 outputs among the above two solutions the one that is leximin-preferred.*

Example 7.14. *On applying HA4 on the same Example 7.7, HA4 connects the agents $\{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7\}$ all the time. For the remaining agents, it uses "original" kBP , and results in an egalitarian allocation of $0.81819kW$ of electricity as explained below:*

– For $g = g_{begin} = 0$, no item is connected all the time. Applying FFk for $k = 3$ results in the following watts allocation vector (in non-decreasing order): $0.06667, 0.07333, 0.13333, 0.14, 0.26666, 0.27333, 0.56666, 0.56666, 0.99999, 1.06666, 2.16664, 2.23331, 4.66662, 4.73329$. *Egalitarian allocation of watts in this case is 0.06667 kW.*
– Similarly, for $g = g_{end} = 8$, items $\{0.2, 0.22, 0.4, 0.42, 0.8, 0.82, 1.7, 1.7\}$ are connected all the time. Applying FFk for $k = 3$ on the remaining set of elements results in the following allocation of watts: $\{0.81819, 0.87274, 1.77274, 1.82729, 3.81822, 3.87277\}$. *The egalitarian allocation of watts in this case is* 0.81819 *kW. Clearly, this solution is better than the previous one in terms of the egalitarian allocation of watts among agents that are not connected all the time. In addition, the final allocation vector (that is, after adding the agents that are connected all the time) is preferred to the solution (when $g = 0$).*

– After updating g_{begin} and g_{end} the algorithm computes other solutions. Finally, among these solutions, the one that is preferred by leximin is returned. In this example, the algorithm returns the solution for $g = 8$.

8 Experimental Results

8.1 Experiment: approximation ratio of FFk and $FFDk$ algorithms

In this section we describe the second experiment to verify the approximation ratio of FFk and $FFDk$ with respect to the optimal bin-packing. For this purpose we utilize a synthetic dataset. There are datasets available for kBP when $k = 1$, but we do not know their optimal packing when $k > 1$. Next, we describe how the synthetic dataset has been created.

Data generation Since computing the optimal bin-packing of a given instance is NP-hard, we constructed instances whose optimal bin-packing is known. Given the bin capacity S, we use Algorithm 2 to generate a list of item sizes that sum up to the bin capacity S. Algorithm 3 uses Algorithm 2 to generate an instance whose item sizes sum up to OPT times S, and by the construction, we know that the optimal bin-packing for the instance D has exactly OPT bins, and moreover, the optimal bin-packing for D_k has exactly $k \cdot OPT$ bins (all of them full).

Algorithm 2: generate_items

Input: A bin capacity S
Output: A list L of item sizes whose sum is S.

Let L be the list of the randomly generated item sizes. Initially L is empty.
Let s_L be the sum of all the item sizes in the list. Initially $s_L = 0$.
while *True* **do**
  generate a random number $r \in (1, S)$.
  if $s_L + r > S$ **then**
    Append $S - s_L$ to L.
    break
  else
    Append r to L, and update s_L.
  end
end
return L

Our generated dataset contains multiple files. Each file in the dataset contains a fixed number IC of instances. Each instance has several item sizes. These items have an optimal packing into OPT bins. Each bin has capacity S. For example let $[1, 2, 3, 7, 6, 1']$ be an instance in some file where for each instance in the file $OPT = 2$. Bin capacity is $S = 10$. As we can see, there are two groups $\{7, 2, 1\}, \{6, 3, 1'\}$ which sum to 10 and hence an optimal packing requires two bins to pack the items in this instance.

Clearly, this dataset has some limitations in that it uses Algorithm 2 to generate a list of items with a sum exactly equal to the capacity of the bin S. Therefore, the sum of all the items in an instance is a multiple of S. However, in a more general scenario, it is not necessarily true.

Algorithm 3: generate_instance_items

Input: A bin capacity S and the optimal number OPT of bins to pack items in this instance.
Output: A list D of item sizes which sum to $OPT \cdot S$, such that the optimal number of bins is indeed $OPT(D) = OPT$.

Let D be the instance of item sizes. Initially D is empty.
Let s_D be the sum of all item sizes in D. Initially $s_D = 0$.
while *True* **do**
  L = generate_items(S)
  Append item sizes in L to D.
  $s_D = s_D + s_L$.
  if $s_D/S == OPT$ **then**
    Shuffle item sizes in D.
    break
  end
end
return D

Results We ran our experiment for different values of $k > 1$, and for different integer values of OPT, $2 \leq OPT \leq 9$. We computed the conjectured upper bound on the number of bins required to pack items using FFk (for $k > 1$) and $FFDk$ as $1.375 \cdot OPT(D_k)$ and $\frac{11 \cdot OPT(D_k)+6}{9}$, respectively. For each file, we report the maximum number of bins required to pack items. We can see in Figure 2, and Figure 3 that the bins used by FFk (for $k > 1$) and $FFDk$ are at most $1.375 \cdot OPT(D_k)$ and $\frac{11 \cdot OPT(D_k)+6}{9}$, respectively, which supports our conjectures in Section 5.1 and Section 5.2.

8.2 Experiment: FFk and $FFDk$ for egalitarian allocation of connection time

In this section, we describe experimental results checking the performance of the kBP adaptations of FFk and $FFDk$ to our motivating application of fair electricity distribution [6].

[6] All the codes are available in `https://github.com/dinkubag/Electricity-Distribution-Algorithms`.

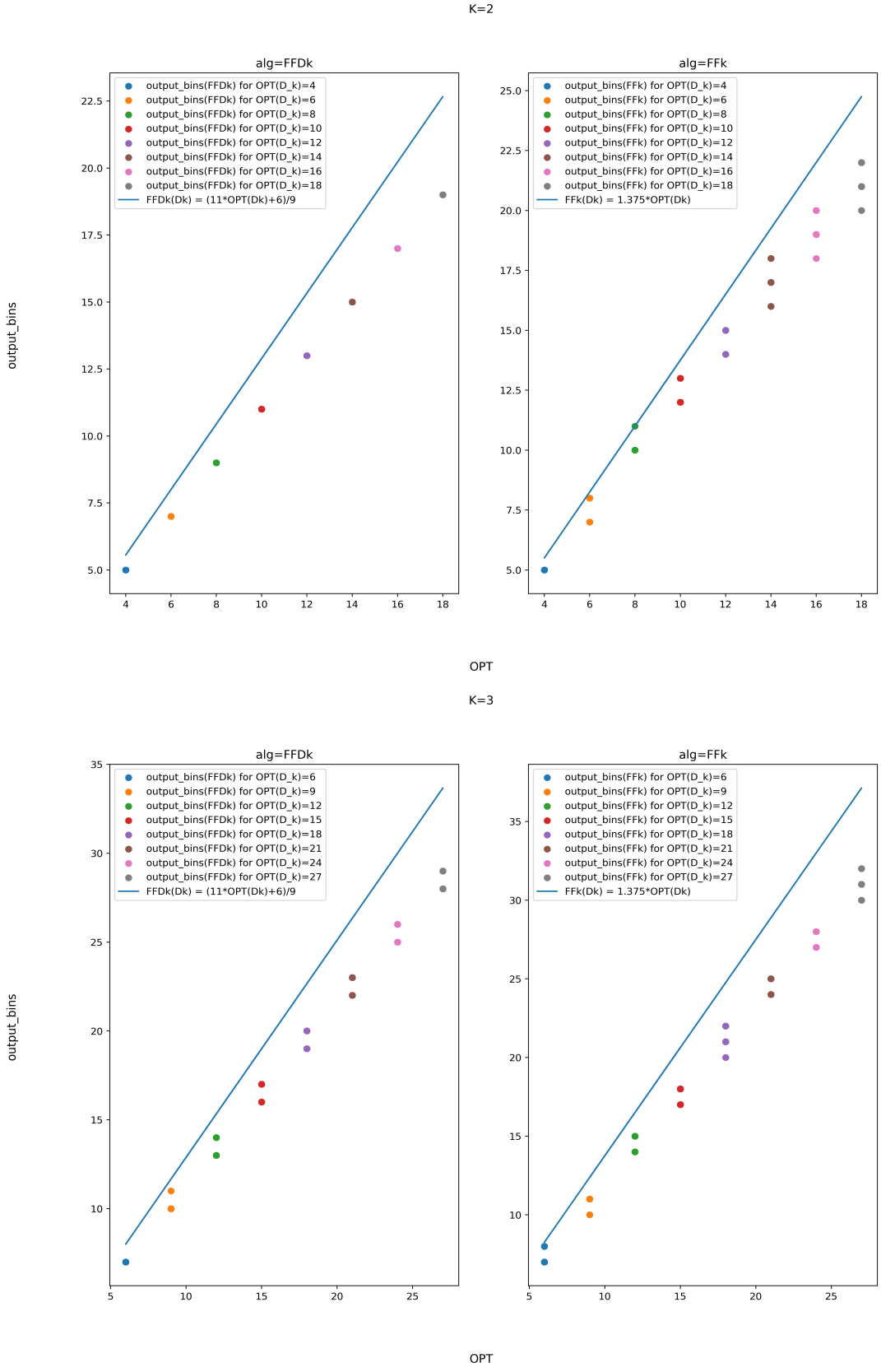

Fig. 2: The optimal numbers $OPT(D_k)$ of bins and numbers of bins used by FFk and $FFDk$ algorithms bins are shown at the x- and y-axis, respectively. The data points in the legend show the upper bound that can be hypothesized for $OPT(D_k) = k \cdot OPT(D)$. Each subplot's data points display the number of different output bins corresponding to each conjectured upper bound. Note that the output bins convey the conjectured upper bounds.

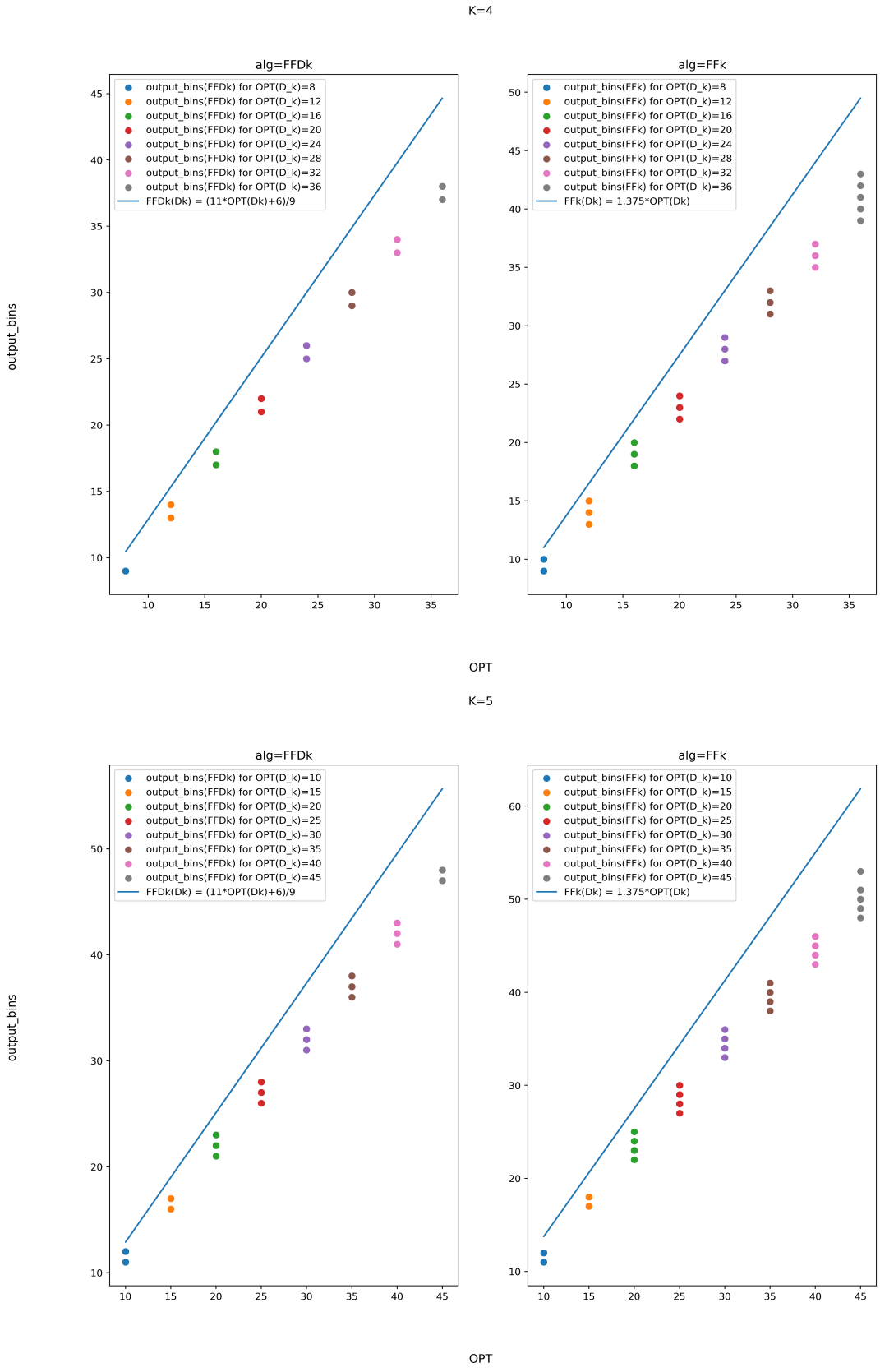

Fig. 3: The optimal numbers $OPT(D_k)$ of bins and numbers of bins used by FFk and $FFDk$ algorithms bins are shown at the x- and y-axis, respectively. The data points in the legend show the upper bound that can be hypothesized for $OPT(D_k) = k \cdot OPT(D)$. Each subplot's data points display the number of different output bins corresponding to each conjectured upper bound. Note that the output bins convey the conjectured upper bounds.

Dataset We use the same dataset of 367 Nigerian households described in [35].[7] This dataset contains the hourly electricity demand for each household for 13 weeks (2184 hours). In addition, they estimate for each agent and hour, the *comfort* of that agent, which is an estimation of the utility the agent gets from being connected to electricity at that hour. For more details about the dataset, readers are encouraged to refer to the papers [34,35]. The electricity demand of agents can vary from hour to hour. We execute our algorithms for each hour separately, which gives us essentially 2184 different instances.

As in [35], we use the demand figures in the dataset as mean values; we determine the actual demand of each agent at random from a normal distribution with a standard deviation of 0.05 (results with a higher standard deviation are presented in Section J).

As in [35], we compute the supply capacity S for each day by averaging the hourly estimates of agents' demand for that day. We run nine independent simulations (with different randomization of agents' demands). Thus, the supply changes in accordance with the average daily demand, but cannot satisfy the maximum hourly demand.

Experiment: For each hour, we execute the FFk and $FFDk$ algorithms on the households' demands for that hour. We then use the resulting packing to allocate electricity: if the packing returns q bins, then each bin is connected for $1/q$ of an hour, which means that each agent is connected for k/q of an hour.

The authors of [35] measure the efficiency and fairness of the resulting allocation, not only by the total time each agent is connected, but also by more complex measures. In particular, they assume that each agent i has a utility function, denoted u_i, that determines the utility that the agent receives from being connected to electricity at a given hour. They consider three different utility models:

1. The simplest model is that u_i equals the amount of time the agent i is connected to electricity (this is the model we mentioned in the introduction).
2. The value u_i can also be equal to the total amount of electricity that the agent i receives. For each hour, the amount of electricity given to i is the amount of time i is connected, times i's demand at that hour.
3. They also measure the "comfort" of the agent i in time t by averaging their demand over the same hour in the past four weeks, and normalizing it by dividing by the maximum value.

For each utility model, they consider three measures of efficiency and fairness:

- Utilitarian: the sum $\sum_i u_i(x)$ (or the average) of all agents' utilities u_i.
- Egalitarian: the minimum utility $\min_i u_i(x)$ of a single agent,
- The maximum difference $\max_{i,j}\{|u_i(x) - u_j(x)|\}$ of utilities between each pair of agents.

Results: The authors of [35] have proposed two models: comfort model (CM) and the supply model. The objective of the CM and the SM model is to maximize the comfort and supply respectively. For the comparison with the results from [35], we show our results for FFk and $FFDk$ for $k = 100$ [8] along with the results in [35] in tables [9] Table 1,Table 2, and Table 3 . We highlight the best results in bold. As can be seen in Table 1, Table 2 and Table 3, FFk and $FFDk$ outperform previous results in terms of the egalitarian allocation of connection time which is the main objective of this paper. Overall, the comparison of FFk and $FFDk$ with their results are as follows:

- FFk and $FFDk$ outperform the previous results in terms of egalitarian allocation of connection time. Maximum Utility difference is also better than all previous results. In terms of utilitarian social welfare metric FFk and $FFDk$ outperforms all previous results except CM.
- FFk and $FFDk$ outperform the previous results in terms of utilitarian allocation of supply. In terms of egalitarian social welfare metric and maximum utility difference, FFk and $FFDk$ outperforms all previous results except SM.
- FFk and $FFDk$ outperform the previous results in terms of utilitarian allocation of comfort except the CM model. In terms of egalitarian allocation of comfort, FFk and $FFDk$ performs better than previous results except the CM and SM model. Their performance is nearly equivalent to the CM model with better standard deviation and maximum utility difference.

[7] We are grateful to Olabambo Oluwasuji for sharing the dataset with us.

[8] We have checked smaller values of k, and found out that the performance increases with k. By the time k reached 100, the performance increase was very slow, so we kept this value.

[9] In Table 1-Table 3 CM, SM, GA, CSA1, RSA, CSA2 stands for: The Comfort Model, The Supply Model, Grouper Algorithm, Consumption-Sorter Algorithm, Random-Selector Algorithm, Cost-Sorter Algorithm respectively [35,34]

In Section J, we have graphs that show how various values of k and varying levels of uncertainty affect the number of connection hours, amount of electricity delivered, and comfort.

Table 1: Comparing results of FFk and $FFDk$ for $k = 100$ with the results in [35] in terms of hours of connection to supply on the average, along with their standard deviation (SD) within parenthesis. In the third column, we have shown the average number of hours an agent is connected to the supply.

Algorithm	Utilitarian: sum(SD)	Utilitarian: average	Egalitarian(SD)	Maximum Utility Difference
FFk	716145.3847 (13.9574)	1951.3498	**1951.3498 (0.0380)**	**0.0(0.0)**
$FFDk$	715891.3137 (13.3774)	1950.6575	**1950.6575 (0.0364)**	**0.0(0.0)**
CM	717031(3950)	1953.7629	1920(3.24)	123(2.09)
SM	709676(3878)	1933.7221	1922(3.41)	71(2.04)
GA	629534(4178)	1715.3515	1609(4.69)	695(3.28)
CSA1	647439(3063)	1764.1389	1764(2.27)	1(0.00)
RSA	643504(4094)	1753.4169	1753(4.33)	1(0.00)
CSA2	641002(3154)	1746.5995	1746(2.38)	1(0.00)

Table 2: Comparing results of FFk and $FFDk$ for $k = 100$ with the results in [35] in terms of electricity supplied on the average, along with their standard deviation (SD) within parenthesis.

Algorithm	Utilitarian(SD)	Egalitarian(SD)	Maximum Utility Difference
FFk	**1364150.4034 (58.6228)**	0.8067 (0.0001)	0.1183 (0.0001)
$FFDk$	**1363494.0885 (48.1455)**	0.8063 (0.0001)	0.1183 (0.0002)
CM	1340015(8299)	0.78(0.01)	0.17(0.02)
SM	1347801(8304)	0.83(0.01)	0.11(0.02)
GA	1297020(11264)	0.35(0.04)	0.58(0.03)
CSA1	1296939(7564)	0.66(0.02)	0.28(0.02)
RSA	1344945(11284)	0.68(0.03)	0.25(0.03)
CSA2	1345537(7388)	0.63(0.03)	0.30(0.02)

In Section J, we discuss the variation in utilitarian, egalitarian, and maximum-utility difference with different values of k and varying levels of uncertainty (standard deviation). The graphs show that the changes appear to saturate as k increases.

8.3 Experiment: Heuristic algorithms for egalitarian allocation of watts

In this section, we evaluate the performance of our heuristic algorithms (HA1, HA2, HA3, HA4) developed to distribute watts as equally as possible (see Section 7). Before comparing the results, we want to highlight some key points that are relevant in interpreting the results of Table 4:

– The computation of supply is dependent on the data. Equal watts allocation computation is necessary for hours during which the aggregate demand is greater than the supply. Therefore, for the hours when aggregate demand is less than supply, we do not consider the allocation of watts in computing the egalitarian allocation of watts and in determining the maximum utility difference. However, they are considered when determining the utilitarian allocation.

Table 3: Comparing results of FFk and $FFDk$ for $k = 100$ with the results in [35] in terms of comfort delivered on the average, along with their standard deviation (SD) within parenthesis.

Algorithm	Utilitarian(SD)	Egalitarian(SD)	Maximum Utility Difference
FFk	296630.3583 (7.3626)	0.8085 (0.00003)	0.1155 (0.00004)
$FFDk$	296493.8100 (7.5569)	0.8081 (0.00003)	0.1156 (0.00003)
CM	303217(3447)	0.81(0.01)	0.13(0.02)
SM	292135(3802)	0.83(0.01)	0.09(0.02)
GA	291021(5198)	0.38(0.04)	0.56(0.03)
CSA1	291909(3201)	0.67(0.02)	0.25(0.02)
RSA	268564(5106)	0.65(0.04)	0.28(0.03)
CSA2	270262(3112)	0.64(0.02)	0.28(0.02)

– For each hour when the aggregate demand is greater than the supply, we compute the egalitarian allocation of watts for the agents that are not connected all the time. The final egalitarian allocation is the sum of the egalitarian allocations computed for each such hour.

We compare our results in terms of the number of hours an agent is connected to supply, the total supply delivered to an agent, and the total comfort delivered to an agent. We make this comparison in terms of the social welfare metrics of utilitarian, egalitarian, and maximum utility difference. We do not compare our results with the results in [34,35], because in their article they have not reported the results in terms of supply allocation.

We highlight the best results in bold. Overall, the comparison of heuristic algorithms (HA1 to HA4) is as follows:

– Our main objective in developing heuristics is to allocate watts to agents as equally as possible. Table 4 shows that HA1 with $FFDk$ outperforms all other heuristics in supplying more electricity to the worst-off agent. HA1 with FFk outperforms other algorithms in distributing electricity as equally as possible (see the maximum utility difference column). On the other hand, HA4 with FFk outperforms other algorithms in terms of total watts delivered. More results are discussed in Section J.1.
– Additionally, we compare our results in terms of hours of connection to supply (see Table 5 in Section J.1). In this case, HA4 with $FFDk$ connects the worst-off agent to more number of hours than to any other heuristic algorithms. It also has the minimum "maximum utility difference." On the other hand, algorithm HA4 with FFK outperforms other algorithms in total connection time delivered.
– When we compare the results in terms of the comfort delivered on average (see Table 6 in Section J.1), HA4 with $FFDk$ performs better than other algorithms in delivering comfort to the worst-off agent. It also has the minimum "maximum utility difference," which means it delivers comfort more uniformly among the agents. On the other hand, algorithm HA4 with FFk outperforms other algorithms in terms of total comfort delivered.

9 Conclusion and Future Directions

We have shown that the existing approximation algorithms, like the First-Fit and the First-Fit Decreasing, can be extended to solve kBP. We have proved that, for any $k \geq 1$, the asymptotic approximation ratio for the FFk algorithm is $\left(1.5 + \frac{1}{5k}\right) \cdot OPT(D_k) + 3 \cdot k$. We have also proved that the asymptotic approximation ratio for the NFk algorithm is 2. We have also demonstrated that the generalization of efficient approximation algorithms like Fernandez de la Vega-Lueker and Karmarkar Karp algorithms solves kBP in $(1 + 2 \cdot \epsilon) \cdot OPT(D_k) + k$ and $OPT(D_k) + O(k \cdot \log^2 OPT(D))$ bins respectively in polynomial time. We have also shown the practical efficacy of FFk and $FFDk$ in solving the fair electricity distribution problem.

Given the usefulness of k-times bin-packing to electricity division, an interesting open question is how to determine the optimal value of k — the k that maximizes the fraction of time each agent is connected

Table 4: Comparing results of heuristic algorithms HA1-HA4 in terms of electricity allocated in watts on average, along with their standard deviation (SD) within parenthesis.

Algorithm	Utilitarian: sum(SD)	Egalitarian(SD)	Maximum Utility Difference
HA-1($k = 5$, alg=FFk)	1319476.193(202.97)	2405.297(10.305)	**187.591(0.622)**
HA-1($k = 5$, alg=$FFDk$)	1319219.591(27.98)	**2412.67(4.026)**	187.611(0.115)
HA-2($k = 50$, alg=FFk)	1305321.696(512.344)	1837.016(4.518)	1773.596(3.978)
HA-2($k = 50$, alg=$FFDk$)	1305310.397(247.794)	1833.378(8.039)	1769.877(5.584)
HA-3($k = 5$, alg=FFk)	1333406.016(555.198)	1088.202(2.844)	3706.019(26.488)
HA-3($k = 5$, alg=$FFDk$)	1332752.563(336.110)	1088.203(4.158)	3718.627(23.409)
HA-4($k = 50$, alg=FFk)	**1363677.72(41.934)**	1886.345(5.524)	11734.702(18.849)
HA-4($k = 50$, alg=$FFDk$)	1363650.312(39.603)	1887.696(3.423)	11730.615(24.72)

— the fraction $\frac{k}{OPT(D_k)}$. Note that this ratio is not necessarily increasing with k. For example, consider the demand vector $D = \{11, 12, 13\}$:

- For $k = 1$, $OPT(D_k) = 2$, so any agent is connected $\frac{1}{2}$ of the time.
- For $k = 2$, $OPT(D_k) = 3$, so any agent is connected $\frac{2}{3}$ of the time.
- For $k = 3$, $OPT(D_k) = 5$, so any agent is connected only $\frac{3}{5} < \frac{2}{3}$ of the time.

We have also shown that, for egalitarian allocation of watts, if each bin is allocated the same amount of time, then there is no upper bound on k that depend only on n. That motivates the development of four heuristic algorithms (HA1, HA2, HA3, HA4) for egalitarian allocation of supply. Among the solutions computed by each of the heuristic algorithms, a leximin preferred solution is returned.

Some other questions left open are

1. To bridge the gap in the approximation ratio of FFk, between the conjectured lower bound 1.375 and the upper bound $\left(1.5 + \frac{1}{5k}\right) \cdot OPT(D_k) + 3 \cdot k$, which converges to 1.5.
2. To prove or disprove that the conjectured bound $\frac{11}{9}OPT + \frac{6}{9}$ is tight for $FFDk$.

Another direction of possible research is to extend other heuristic/approximation algorithms to solve kBP and develop a composite heuristic that randomly selects a heuristic (out of a set of heuristics) for each item and packs accordingly. It then use the 1−Opt procedure to determine the number of bins as follows: Determine the most empty bin. For each of the other bins, check if it can be merged with the most empty bin without violating kBP constraints. One can repeat this 1−Opt procedure for some arbitrary number of times. Such combination of composite heuristic and 1−Opt has been discussed in [24].

Another direction of search could be to generalize the variable neighborhood search schemes (VNS) such as presented in [16]. The VNS scheme presented in [16] used an heuristic algorithm based on minimum bin slack (also called as MBS, it determines and selects a subset of items (from the remaining set of items) that can fill the remaining space in a bin and leaves minimum remaining space). Initial solutions generated from heuristic algorithms like FFD are found to be inferior. A future research direction could be to extend the VNS scheme to solve kBP . Also, it will be quite interesting to see the performance of VNS scheme when the initial solution is generated from FFk and $FFDk$ as we have seen that as k increases the solution quality generated from FFk and $FFDk$ also increases (as shown in the experimental results related to the conjectured upper bound on FFk and $FFDk$).

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A Fast Approximation Algorithms

Theorem 5.1. *For every input D and $k \geq 1$, $FFk(D_k) \leq \left(1.5 + \frac{1}{5k}\right) \cdot OPT(D_k) + 3 \cdot k$.*

In this section we will give the proof of Theorem 5.1 for the general case. Recall that we have already gave the proof for $k = 2$ as a warm up case in Section 5.1.

Proof. Recall that we have a basic weighting scheme, in which each item of size v is given a weight

$$w(v) := v/S + r(v),$$

where r is a *reward function*, computed as follows:

$$r(v) := \begin{cases} 0 & \text{if } v/S \leq \frac{1}{6}, \\ \frac{1}{2}(v/S - \frac{1}{6}) & \text{if } v/S \in (\frac{1}{6}, \frac{1}{3}), \\ 1/12 & \text{if } v/S \in [\frac{1}{3}, \frac{1}{2}], \\ 4/12 & \text{if } v/S > \frac{1}{2}. \end{cases}$$

We modify this scheme later for some items, based on the instances to which they belong.

Denote by u, the first instance in the FFk packing in which there are underfull bins (if there are no underfull bins at all, we set $u = k$). We modify the item weights as follows:

- For instances $1, \ldots, u-1$, we reduce the reward of each item of size $v > S/2$ from $4/12$ to $2/12$.
- For instance u, we keep the weights unchanged.
- For instances $u+1, \ldots, k$, we reduce the weights of items as in case 2 of warm-up case for $k = 2$ in the proof of Theorem 5.1), that is

$$w(v) = \begin{cases} 0 & \text{if } v/S < \frac{1}{3}, \\ 5/12 & \text{if } v/S \in [\frac{1}{3}, \frac{1}{2}], \\ 10/12 & \text{if } v/S > \frac{1}{2}. \end{cases}$$

Now we analyze the effects of these changes.

- In the optimal packing, the maximum reward of every bin that contains an item larger than $S/2$ from instances $1, \ldots, u-1$ drops from at most $5/12$ to at most $3/12$, so their weight is at most $15/12$. Additionally, every bin that contains an item larger than $S/2$ from some instance $j \in \{u+1, \ldots, k\}$, contains only items from instance j. Therefore, its weight can be at most $15/12$ (the remaining space in such bin can pack only one item from $[S/3, S/2]$ of positive weight. Additionally, the weight of every bin that contains no items larger than $S/2$ remains at most $15/12$. The only bins that can have a larger weight (up to $17/12$) are bins with items larger than $S/2$ from instance u. At most $1/k$ bins in the optimal packing can contain such an item. Therefore, the average weight of a bin in the optimal packing is at most $\frac{(17/12)+(15/12)\cdot(k-1)}{k}$.
- In the FFk packing, instances $1, \ldots, u-1$ have no underfull bins. The change in the reward affects only the bins of group 3 (bins with a single item larger than $S/2$): the weight of each bin in group 3 (which must have one item larger than $2S/3$ since it is not underfull) is at least $2/3 + 2/12 = 10/12$. Since we only reduce the reward of the item larger than $S/2$ for the instances $1, \ldots, u-1$; the analysis of bins in groups 4 and 5 is not affected at all by this reduction in reward and therefore leads to their average weight (except excluding one bin per instance in groups 1, 2, and 5) of at least $10/12$.
  In instance u, the weights are unchanged, so the average bin weight (except excluding one bin per instance in groups 1, 2, and 5) is still at least $10/12$.
  Finally, the bins of instances $u+1, \ldots, k$ contain no item of size at most $S/3$ — since any such item would fit into the underfull bin in instance u. Therefore, for each such bin, there are only two options:
  - the bin contains one item larger than $S/2$ — so its weight is $10/12$;
  - the bin contains two items, each of which is larger than $S/3$ — so its weight is at least $2 \cdot 5/12 = 10/12$.

Again, the average bin weight (except excluding one bin per instance in groups 1, 2, and 5) of instances $u+1, \ldots, k$ is at least $10/12$.

We conclude that the average weight of all bins in the FFk packing (except the $3k$ excluded bins) is at least $10/12$. Therefore

$$FFk \leq \left(\frac{\frac{(17/12)+(15/12)\cdot(k-1)}{k}}{10/12} \cdot OPT \right) + 3k = \left(1.5 + \frac{1}{5k}\right) \cdot OPT + 3k.$$

□

A.1 Worst-case example

Lemma A.1. *The approximation ratio of $FFk(D_k)$ for $k = 2$ for the worst case examples given in [13,28] is* 1.35*. For the example given in [28], the approximation ratio continues to decrease as k increases.*

Proof. Dosa and Sgall [13] gave a simple example for the lower bound construction of FF as following: For a very small ϵ, first there are $10n$ *small* items of size approximately 1/6 (smaller and bigger), then comes $10n$ *medium* items of size approximately 1/2 (smaller and bigger), and finally $10n$ *large* items of size $1/2 + \epsilon$ follows. They have proved that for this list D, $OPT(D) = 10n$ and $FF(D) = 17n$. Now, consider the packing of the items in D_2. After packing the first instance of D, FFk packs the *small* and *medium* items into the existing bins. Only each of the *large* items require a separate bin to pack. So it is easy to see that $FFk(D_2) = (17 + 10)n$ while $OPT(D_2) = (10 + 10)n$. This will give us $\frac{FFk(D_2)}{OPT(D_2)} = \frac{27}{20} = 1.35$. Note that here OPT is arbitrarily big.

Johnson, Demers, Ullman, Garey and Graham [28] provide the following example for FF. We now show that, on the same example, FFk achieves an approximation ratio of 1.35.

All the item sizes in this example D are in non-decreasing order and are as follows $D = \{6(7), 10(7), 16(3), 34(10), 51(10)\}$ and bin size $S = 101$. The number in the parenthesis denotes the occurrence of that item in D. In the above example 6(7) expands to $\{6, 6, 6, 6, 6, 6, 6\}$. Before proceeding further, we describe some notation that we will use. For two bins B_i, B_j, $i < j$ implies bin B_i has been created before bin B_j.

Note that OPT for $k = 1$ can result the following bin-packing

$O_1 = O_2 = O_3 = [51, 34, 16]$,

$O_4 = O_5 = O_6 = O_7 = O_8 = O_9 = O_{10} = [51, 34, 10, 6]$.

It is easy to observe that optimal number of bins for packing D_k is $OPT(D_k) = k \cdot OPT(D)$.

In Figure 4 we have shown the packing of D_1, D_2, D_3, D_4, and D_5. Each box in the figure represents a bin. Each bin represents the items packed into the bin along with their count and the instance from which they belong. Each bin also represents the space remaining in the bin. For example, in Figure 4a, $6^1(7)$, inside bin B_1 represents that there are 7 items of size 6 from first instance of D. $R(9)$ in bin B_1 denotes the remaining space in that bin.

Figure 4a and Figure 4b show the FFk packing of the items for instance D_1 and D_2 respectively. Note that there are some bins in which the items from future bins cannot be packed (either there is no space or due to the violation of the kBP constraint). For the lack of space we do not show such bins. For example, in Figure 4c we have the shown the FFk packing of the items for instance D_3. FFk packs the items in the third instance of D after packing the items in the first two instances. However, after packing the second instance, bins B_1, B_2, B_3, B_4 cannot pack any of the items from the third instance (in fact any of the items from all the future instances). Therefore, we do show these bins in Figure 4c. Note that there is a pattern in the FFk packing of D_4 and D_5. It is easy to see that this pattern repeats itself in the packing of $D_6, D_7, \ldots, D_k$. Also note that while packing the items in D_k we add 10 new bins in addition to the bins in the FFk packing of D_{k-1}. Therefore, $\frac{FFk(D_k)}{OPT(D_k)} = \frac{17+10(k-1)}{10k} \leq 1.35$ for $k > 1$. One can observe that as k increases, the approximation ratio continues to decrease.

Lemma A.2. *The lower bound of $FFk(D_k)$ for $k = 2$ is 1.375.*

Proof. Consider the input $D = \{371, 659, 113, 47, 485, 3, 228, 419, 468, 581, 626\}$ and $S = 1000$. For $k = 1$, FFk will result in the following packing:

$B_1 = [371^1, 113^1, 47^1, 3^1, 228^1], sum = 762$,

$B_2 = [659^1], sum = 659$,

$B_3 = [485^1, 419^1], sum = 904$,

$B_4 = [468^1], sum = 468$,

$B_5 = [581^1], sum = 581$,

$B_6 = [626^1], sum = 626$

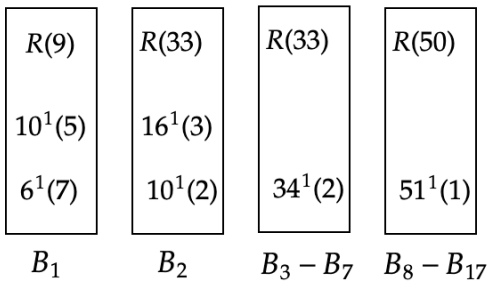
We use the word "SAME" for the content of a bin if a bin cannot pack any of the items from the future instances of D. This happens because every item in a future instance is either too large to fit into that bin, or has the same type as an item already packed into it. Note that bin B_1 is such bin. For $k = 2$, FFk will result in the following packing:

$B_1 =$ SAME,

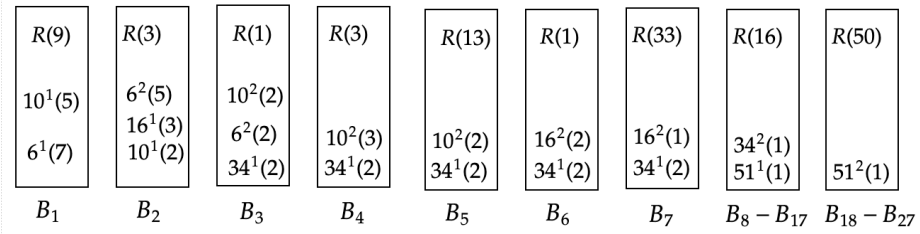
$B_2 = [659^1, 113^2, 47^2, 3^2], sum = 822$,

$B_3 = [485^1, 419^1], sum = 904$,

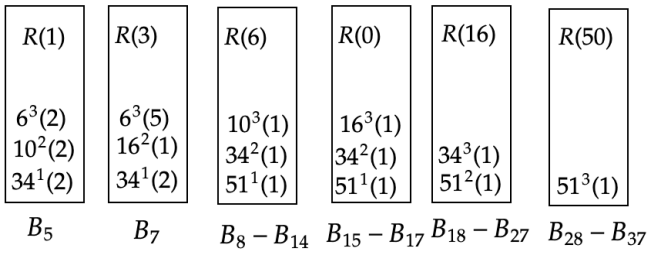
$B_4 = [468^1, 371^2], sum = 839$,



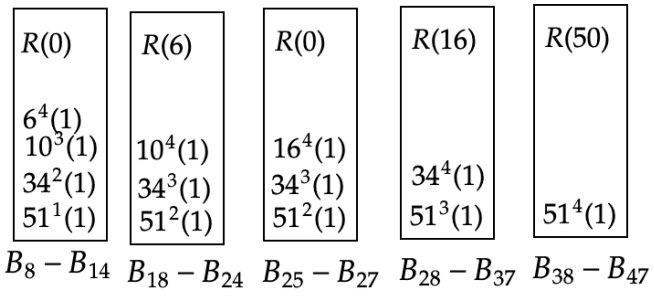
(a) FFk packing for $k = 1$.



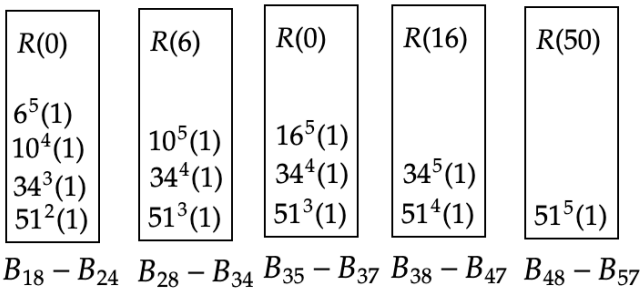
(b) FFk packing for $k = 2$.



(c) FFk packing for $k = 3$. Note that bins from $FFk(D_2)$ packing in which no item from future instances can be packed are not shown.



(d) FFk packing for $k = 4$. Note that bins from $FFk(D_3)$ packing in which no item from future instances can be packed are not shown.



(e) FFk packing for $k = 5$. Note that bins from $FFk(D_4)$ packing in which no item from future instances can be packed are not shown.

Fig. 4: Figure 4a-Figure 4e show the FFk packing for D for different values of k. For the lack of space we are not showing the bins in the packing of D_{k-1} in which no item from future instances can be packed. Note the pattern in packing in Figure 4d and Figure 4e.

$B_5 = [581^1, 228^2], sum = 809,$
$B_6 = [626^1], sum = 626,$
$B_7 = [659^2], sum = 659,$
$B_8 = [485^2, 419^2], sum = 904,$
$B_9 = [468^2], sum = 468,$
$B_{10} = [581^2], sum = 581,$
$B_{11} = [626^2], sum = 626,$

The items in D can be optimally packed in bins $[626, 371, 3], [659, 228, 113], [419, 581], [468, 485, 47]$. The size of each of these bins is exactly 1000. Therefore, the optimal packing of D_k requires $k \cdot OPT(D) = 4k$ bins. Therefore, $\frac{FFk(D_k)}{OPT(D_k} = \frac{11}{8} = 1.375$ for $k = 2$.

Remark A.1. *The approximation ratio for the example instance given in Lemma A.2 continues to decrease as k increases.*

Proof. The packing for $k = 2$ is given in Lemma A.2. We will continue from this packing. Note that bin B_2 cannot pack any of the items from the future instances of D.

For $k = 3$, FFk will result in the following packing:
$B_1 = \text{SAME}, B_2 = \text{SAME}$
$B_3 = [485^1, 419^1, 47^3, 3^3], sum = 954,$
$B_4 = [468^1, 371^2, 113^3], sum = 952,$
$B_5 = [581^1, 228^2], sum = 809,$
$B_6 = [626^1, 371^3], sum = 997,$
$B_7 = [659^2, 228^3], sum = 887,$
$B_8 = [485^2, 419^2], sum = 904,$
$B_9 = [468^2, 485^3], sum = 953,$
$B_{10} = [581^2, 419^3], sum = 1000,$
$B_{11} = [626^2], sum = 626,$
$B_{12} = [659^3], sum = 659,$
$B_{13} = [468^3], sum = 468,$
$B_{14} = [581^3], sum = 581,$
$B_{15} = [626^3], sum = 626$
Note the bin B_3 and B_{10} cannot pack any of the items from the future instances of D.

For $k = 4$, FFk will result in the following packing:
$B_1 = \text{SAME}, B_2 = \text{SAME}, B_3 = \text{SAME}, B_{10} = \text{SAME}$
$B_4 = [468^1, 371^2, 113^3, 47^4], sum = 999,$
$B_5 = [581^1, 228^2, 113^4, 3^4], sum = 925,$
$B_6 = [626^1, 371^3], sum = 997,$
$B_7 = [659^2, 228^3], sum = 887,$
$B_8 = [485^2, 419^2], sum = 904,$
$B_9 = [468^2, 485^3], sum = 953,$
$B_{11} = [626^2, 371^4], sum = 997,$
$B_{12} = [659^3, 228^4], sum = 887,$
$B_{13} = [468^3, 485^4], sum = 953,$
$B_{14} = [581^3, 419^4], sum = 1000,$
$B_{15} = [626^3], sum = 626,$
$B_{16} = [659^4], sum = 659,$
$B_{17} = [468^4], sum = 468,$
$B_{18} = [581^4], sum = 581,$
$B_{19} = [626^4], sum = 626$
Note that bin B_4 and B_{14} cannot pack any of the items from the future instances of D.

For $k = 5$, FFk will result in the following packing:
$B_1 = \text{SAME}, B_2 = \text{SAME}, B_3 = \text{SAME}, B_4 = \text{SAME}, B_{10} = \text{SAME}, B_{14} = \text{SAME}$
$B_5 = [581^1, 228^2, 113^4, 3^4, 47^5], sum = 972,$
$B_6 = [626^1, 371^3, 3^5], sum = 1000,$
$B_7 = [659^2, 228^3, 113^5], sum = 1000,$
$B_8 = [485^2, 419^2], sum = 904,$
$B_9 = [468^2, 485^3], sum = 953,$
$B_{11} = [626^2, 371^4], sum = 997,$

$B_{12} = [659^3, 228^4], sum = 887,$
$B_{13} = [468^3, 485^4], sum = 953,$
$B_{15} = [626^3, 371^5], sum = 997,$
$B_{16} = [659^4, 228^5], sum = 887,$
$B_{17} = [468^4, 485^5], sum = 953,$
$B_{18} = [581^4, 419^5], sum = 1000,$
$B_{19} = [626^4], sum = 626,$
$B_{20} = [659^5], sum = 659,$
$B_{21} = [468^5], sum = 468,$
$B_{22} = [581^5], sum = 581,$
$B_{23} = [626^5], sum = 626$
Note that bin $B_5, B_6, B_7 B_{18}$ cannot pack any of the items from the future instances of D.

For $k = 6$, FFk will result in the following packing:
$B_1 = \text{SAME}, B_2 = \text{SAME}, B_3 = \text{SAME}, B_4 = \text{SAME}, B_5 = \text{SAME}, B_6 = \text{SAME}, B_7 = \text{SAME}, B_{10} = \text{SAME}, B_{14} = \text{SAME}, B_{18} = \text{SAME}$
$B_8 = [485^2, 419^2, 47^6, 3^6], sum = 954,$
$B_9 = [468^2, 485^3], sum = 953,$
$B_{11} = [626^2, 371^4], sum = 997,$
$B_{12} = [659^3, 228^4, 113^6], sum = 1000,$
$B_{13} = [468^3, 485^4], sum = 953,$
$B_{15} = [626^3, 371^5], sum = 997,$
$B_{16} = [659^4, 228^5], sum = 887,$
$B_{17} = [468^4, 485^5], sum = 953,$
$B_{19} = [626^4, 371^6], sum = 997,$
$B_{20} = [659^5, 228^6], sum = 887,$
$B_{21} = [468^5, 485^6], sum = 953,$
$B_{22} = [581^5, 419^6], sum = 1000,$
$B_{23} = [626^5], sum = 626,$
$B_{24} = [659^6], sum = 659,$
$B_{25} = [468^6], sum = 468,$
$B_{26} = [581^6], sum = 581,$
$B_{27} = [626^6], sum = 626$
Note that bins B_8, B_{12}, B_{22} cannot pack any of the items from the future instances of D.

For $k = 7$, FFk will result in the following packing:
$B_1 = \text{SAME}, B_2 = \text{SAME}, B_3 = \text{SAME}, B_4 = \text{SAME}, B_5 = \text{SAME}, B_6 = \text{SAME}, B_7 = \text{SAME}, B_8 = \text{SAME}, B_{10} = \text{SAME}, B_{12} = \text{SAME}, B_{14} = \text{SAME}, B_{18} = \text{SAME}, B_{22} = \text{SAME},$
$B_9 = [468^2, 485^3, 47^7], sum = 1000,$
$B_{11} = [626^2, 371^4, 3^7], sum = 1000,$
$B_{13} = [468^3, 485^4], sum = 953,$
$B_{15} = [626^3, 371^5], sum = 997,$
$B_{16} = [659^4, 228^5, 113^7], sum = 1000,$
$B_{17} = [468^4, 485^5], sum = 953,$
$B_{19} = [626^4, 371^6], sum = 997,$
$B_{20} = [659^5, 228^6], sum = 887,$
$B_{21} = [468^5, 485^6], sum = 953,$
$B_{23} = [626^5, 371^7], sum = 997,$
$B_{24} = [659^6, 228^7], sum = 887,$
$B_{25} = [468^6, 485^7], sum = 953,$
$B_{26} = [581^6, 419^7], sum = 1000,$
$B_{27} = [626^6], sum = 626,$
$B_{28} = [659^7], sum = 659,$
$B_{29} = [468^7], sum = 468,$
$B_{30} = [581^7], sum = 581,$
$B_{31} = [626^7], sum = 626$
Note that bins $B_9, B_{11}, B_{16}, B_{26}$ cannot pack any of the items from the future instances of D.

For $k = 8$, FFk will result in the following packing:
$B_1 = \text{SAME}, B_2 = \text{SAME}, B_3 = \text{SAME}, B_4 = \text{SAME}, B_5 = \text{SAME}, B_6 = \text{SAME}, B_7 = \text{SAME}, B_8 = \text{SAME}, B_9 = \text{SAME}, B_{10} = \text{SAME}, B_{11} = \text{SAME}, B_{12} = \text{SAME}, B_{14} = \text{SAME}, B_{16} = \text{SAME}, B_{18} =$

SAME, B_{22} = SAME, B_{26} = SAME,
$B_{13} = [468^3, 485^4, 47^8], sum = 1000,$
$B_{15} = [626^3, 371^5, 3^8], sum = 1000,$
$B_{17} = [468^4, 485^5], sum = 953,$
$B_{19} = [626^4, 371^6], sum = 997,$
$B_{20} = [659^5, 228^6, 113^8], sum = 1000,$
$B_{21} = [468^5, 485^6], sum = 953,$
$B_{23} = [626^5, 371^7], sum = 997,$
$B_{24} = [659^6, 228^7], sum = 887,$
$B_{25} = [468^6, 485^7], sum = 953,$
$B_{27} = [626^6, 371^8], sum = 997,$
$B_{28} = [659^7, 228^8], sum = 887,$
$B_{29} = [468^7, 485^8], sum = 953,$
$B_{30} = [581^7, 419^8], sum = 1000,$
$B_{31} = [626^7], sum = 626,$
$B_{32} = [659^8], sum = 659,$
$B_{33} = [468^8], sum = 468,$
$B_{34} = [581^8], sum = 581,$
$B_{35} = [626^8], sum = 626$
Note that bins $B_{13}, B_{15}, B_{20}, B_{30}$ cannot pack any of the items from the future instances of D.

At this point, we note a repeating pattern:

- Item $371^7, 371^8$ has been packed in bins B_{23}, B_{27} respectively. These bins have the same content $[626, 371]$ (without superscript). For $k = 9$ it will pack 371^9 in bin B_{31}, and afterwards this bin will also have the same content $[626, 371]$.
- Items $659^7, 659^8$ have been packed in bins B_{28}, B_{32} respectively. Item 659^9 cannot be packed into any of the previous bins and hence will require a new bin B_{36} to pack.
- Items $113^7, 113^8$ have been packed in bins B_{16}, B_{20} respectively. These bins have the same content $[659, 228, 113]$ (without superscript). For $k = 9$ it will pack 113^9 in bin B_{24} and after packing this bin will also have the same content $[659, 228, 113]$.
- Items $47^7, 47^8$ have been packed in bins B_9, B_{13} respectively. These bins have the same content $[468, 485, 47]$ (without superscript). For $k = 9$ it will pack 47^9 in bin B_{17} and after packing this bin will also have the same content $[468, 485, 47]$.
- Items $485^7, 485^8$ have been packed in bins B_{25}, B_{29} respectively. These bins have the same content $[468, 485]$ (without superscript). For $k = 9$ it will pack 485^9 in bin B_{33} and afterwards this bin will also have the same content $[468, 485]$.
- Items $3^7, 3^8$ have been packed in bins B_{11}, B_{15} respectively. These bins have the same content $[626, 371, 3]$ (without superscript). For $k = 9$ it will pack 3^9 in bin B_{19} and after packing this bin will also have the same content $[626, 371, 3]$.
- Items $228^7, 228^8$ have been packed in bins B_{24}, B_{28} respectively. These bins have the same content $[659, 228]$ (without superscript). For $k = 9$ it will pack 228^9 in bin B_{32} and after packing this bin will also have the same content $[659, 228]$.
- Items $419^7, 419^8$ have been packed in bins B_{26}, B_{30} respectively. These bins have the same content $[581, 419]$ (without superscript). For $k = 9$ it will pack 419^9 in bin B_{34} and afterwards this bin will also have the same content $[581, 419]$.
- Items $468^9, 581^9, 626^9$ cannot be packed into any of the previous bins and hence will require new bins B_{37}, B_{38}, B_{39} respectively. Note that for $k = 8$ these items were packed in new bins B_{29}, B_{30}, B_{31} and for $k = 7$ these items were packed in new bins B_{25}, B_{26}, B_{27}.

One can observe that the content of the bins (except the SAME bins) remains same for $k = 7, 8, 9$. This pattern continues for further values of k. After $k \geq 2$, packing of D_k requires 4 new bins than the packing of D_{k-1}. Therefore, $bins(D_k) = 11 + 4 \cdot (k - 2) = 4 \cdot k + 3$.

The items in D can be optimally packed in bins $[626, 371, 3], [659, 228, 113], [419, 581], [468, 485, 47]$. The size of each of these bins is exactly 1000. Therefore, the optimal packing of D_k requires $k \cdot OPT(D) = 4k$ bins. Therefore, $\frac{FFk(D_k)}{OPT(D_k} = \frac{11+4\cdot(k-2)}{4\cdot k} = 1 + \frac{3}{4k} \leq 1.375$ for $k \geq 2$. One can observe that as k increases, the approximation ratio continues to decrease. □

B *FFDk*

Lemma 5.1. $FFDk(D_k) \geq \frac{7}{6} \cdot OPT(D_k) + 1.$

Here, we give in detail the $FFDk$ packing of the items in D_k. Note that D is $\{\frac{1}{2}+\delta, \frac{1}{2}+\delta, \frac{1}{2}+\delta, \frac{1}{2}+\delta, \frac{1}{4}+2\delta, \frac{1}{4}+2\delta, \frac{1}{4}+2\delta, \frac{1}{4}+2\delta, \frac{1}{4}+\delta, \frac{1}{4}+\delta, \frac{1}{4}+\delta, \frac{1}{4}+\delta, \frac{1}{4}-2\delta, \frac{1}{4}-2\delta, \frac{1}{4}-2\delta, \frac{1}{4}-2\delta, \frac{1}{4}-2\delta, \frac{1}{4}-2\delta, \frac{1}{4}-2\delta, \frac{1}{4}-2\delta\}$.

Proof. Note that all items in D are in non-increasing order. We use the following notation: $x \pm c \cdot \delta^i$, where $x \in \{1/2, 1/4\}$ and $c \in \{1, 2\}$, denotes that the item belongs to the instance i. Along with the bins in packing the items in D we also show the sum of all the items in the bin along with an indicator to whether the bin can contain items from the future instances of D or not. We use $\times$ to denote that the bin can not pack items from the future instances of D, otherwise we use symbol ✓. For $k = 1$, $FFDk$ packing for the items in D is:
$B_1 = \{1/2 + \delta^1, 1/4 + 2\delta^1\}, V(B_1) = 3/4 + 3\delta, \times$
$B_2 = \{1/2 + \delta^1, 1/4 + 2\delta^1\}, V(B_2) = 3/4 + 3\delta, \times$
$B_3 = \{1/2 + \delta^1, 1/4 + 2\delta^1\}, V(B_3) = 3/4 + 3\delta, \times$
$B_4 = \{1/2 + \delta^1, 1/4 + 2\delta^1\}, V(B_4) = 3/4 + 3\delta, \times$
$B_5 = \{1/4 + \delta^1, 1/4 + \delta^1, 1/4 + \delta^1\}, V(B_5) = 3/4 + 3\delta, \times$
$B_6 = \{1/4 + \delta^1, 1/4 - 2\delta^1, 1/4 - 2\delta^1, 1/4 - 2\delta^1\}, V(B_1) = 1 - 5\delta, \times$
$B_7 = \{1/4 - 2\delta^1, 1/4 - 2\delta^1, 1/4 - 2\delta^1, 1/4 - 2\delta^1\}, V(B_1) = 1 - 8\delta, \times$
$B_8 = \{1/4 - 2\delta^1\}, V(B_1) = 1/4 - 2\delta, ✓$
Note that the bins $B_1 - B_7$ can not pack any of the items from the future instances of D. Therefore, for $k = 2$, we show only the bins from B_8 onward. Therefore, $FFDk$ packing for the items in D_2 is as follows:
$B_8 = \{1/4 - 2\delta^1, 1/2 + \delta^2, 1/4 + \delta^2\}, V(B_8) = 1, \times$
$B_9 = \{1/2 + \delta^2, 1/4 + 2\delta^2\}, V(B_9) = 3/4 + 3\delta, \times$
$B_{10} = \{1/2 + \delta^2, 1/4 + 2\delta^2\}, V(B_{10}) = 3/4 + 3\delta, \times$
$B_{11} = \{1/2 + \delta^2, 1/4 + 2\delta^2\}, V(B_{11}) = 3/4 + 3\delta, \times$
$B_{12} = \{1/4 + 2\delta^2, 1/4 + \delta^2, 1/4 + \delta^2\}, V(B_{12}) = 3/4 + 4\delta, \times$
$B_{13} = \{1/4 + \delta^2, 1/4 - 2\delta^2, 1/4 - 2\delta^2, 1/4 - 2\delta^2\}, V(B_{13}) = 1 - 5\delta, \times$
$B_{14} = \{1/4 - 2\delta^2, 1/4 - 2\delta^2, 1/4 - 2\delta^2, 1/4 - 2\delta^2\}, V(B_{14}) = 1 - 8\delta, \times$
$B_{15} = \{1/4 - 2\delta^2\}, V(B_{15}) = 1/4 - 2\delta, ✓$
Note that the bins from B_1 to B_{14} can not pack any of the items from the future instances of D, and the bin B_{15} is same as the bin B_8 when $k = 1$. Therefore, it is clear that the packing pattern will repeat for further values of k, and from above it is clearly evident that it will require only 7 more bins. □

Challenges in extending existing proof: Existing proofs for the FFD are based on the assumption that the last bin contains a single item, and no other item(s) are packed after that, i.e. the smallest item is the only item in the last bin of FFD packing. We cannot say the same in the case of kBP when $k > 1$. For example let $D = \{103, 102, 101\}$ and $S = 205$. Then $FFDk$ packing when $k = 1$ is $\{103, 102\}, \{101\}$ whereas for $k = 2$ the packing is $\{103, 102\}, \{101, 103\}, \{102, 101\}$. Hence, the existing proofs for FFD cannot be extended to $FFDk$.

C *NFk*

Theorem 5.2. *For every input D_k and $k \geq 1$, the asymptotic ratio of $NFk(D_k)$ is 2.*

Proof. The idea behind the proof is due to [29].

We can assume that $V(D) > S$, otherwise there is a trivial solution with k bins. While processing input D_k, NFk holds only one open bin, and it cannot contain a copy of each item of D. In fact, the open bin always contains a part of some instance of D, and possibly a part of the next instance of D, with no overlap. Therefore, if the current item x is not packed into the current open bin, the only reason is that x does not fit, as there is no previous copy of x in the current bin (all previous copies, if any, are in already-closed bins).

Let the number of bins in the NFk packing of D_k be $NFk(D_k)$. Let these bins be ordered in the sequence in which they are opened. Then, for any two bins B_{i-1} and B_i where $i \geq 2$,

$$V(B_{i-1}) + V(B_i) > S$$

Therefore,

$$V(D_k) = V(B_1) + \ldots + V(B_{NFk(D_k)}) > \left\lfloor \frac{NFk(D_k)}{2} \right\rfloor \cdot S$$

Since $OPT(D_k) \geq V(D_k)/S$,

$$NFk(D_k) \leq 2 \cdot OPT(D_k) + 1$$

For the lower bound we consider the example given in [47]. Let the bin capacity be 1. Let D be an input instance with $2y$ items (for some large integer y) of sizes $1/2, \epsilon, 1/2, \epsilon, \ldots$ (y pairs overall). Then NFk will pack D_k into $k \cdot y$ bins, whereas an optimal packing of D_k consists of $k \cdot (y/2 + 1)$ bins. This gives an asymptotic ratio of 2. □

D Polynomial-time Approximation Schemes

Lemma 6.2. *Every integral solution of C_k can be realised as a feasible solution of kBP.*

Here, we give an example using which we illustrate how an integral solution of C_k can be realised as a feasible solution of kBP .

Consider the example given after Definition 6.1 In this example, there are $m(D) = 2$ distinct item sizes: $c[1] = 3$ and $c[2] = 4$. There are $n[1] = 7$ items of size $c[1] = 3$, and $n[2] = 6$ items os size $c[2] = 4$. For $k = 2$, an integral solution to C_2 will consist of 2 copies of configuration $[3, 3, 3, 3]$, 2 copies of configuration $[3, 3, 4]$, 2 copies of configuration $[3, 4, 4]$, and 2 copies of configuration $[4, 4, 4]$. Then, this solution sequence can be realised as follows:

– $3, 3, 3, 3, 3, 3, 3$ are the items in D of size $c[1] = 3$. Then, the queue Q_1 will consist of 2 copies of the items $3, 3, 3, 3, 3, 3, 3$. First, there are first copies of these items, then there are second copies of the items. Therefore, finally Q_1 will be $\{3^1, 3^1, 3^1, 3^1, 3^1, 3^1, 3^1, 3^2, 3^2, 3^2, 3^2, 3^2, 3^2, 3^2\}$. Superscript i denotes the ith copy of an item.

– Similarly, Q_2 will be $\{4^1, 4^1, 4^1, 4^1, 4^1, 4^1, 4^2, 4^2, 4^2, 4^2, 4^2, 4^2\}$.

– Now, consider the first configuration $[3, 3, 3, 3]$ in the solution sequence. To realise this configuration we move 4 items of size $c[1] = 3$ from Q_1 into the first bin. Since the next configuration in the solution sequence is same as the first one, we make the same movement of items from Q_1 to the second bin. After these movements, Q_1 will be $\{3^2, 3^2, 3^2, 3^2, 3^2, 3^2\}$, and Q_2 will be $\{4^1, 4^1, 4^1, 4^1, 4^1, 4^1, 4^2, 4^2, 4^2, 4^2, 4^2, 4^2\}$.

– Third configuration in the solution sequence is $[3, 3, 4]$. To realise this configuration we move 2 items of size $c[1] = 3$ from Q_1 to the third bin, and 1 item of size $c[2] = 4$ from Q_2 to the third bin. Since the next configuration in the solution sequence is same as third configuration, we make the same movements of items from Q_1 and Q_2 to the fourth bin. After these movements, Q_1 will be $\{3^2, 3^2\}$, and Q_2 will be $\{4^1, 4^1, 4^1, 4^1, 4^2, 4^2, 4^2, 4^2, 4^2, 4^2\}$.

– Fifth configuration in the solution sequence is $[3, 4, 4]$. To realise this configuration, we move one item of size $c[1] = 3$ from Q_1 to the fifth bin, and two items of size $c[2] = 4$ from Q_2 to this fifth bin. Since the next configuration in the solution sequence is same as fifth configuration, we make the same movements of items from Q_1 and Q_2 to sixth bin. After these movements Q_1 will be empty, and Q_2 will be $\{4^2, 4^2, 4^2, 4^2, 4^2, 4^2\}$.

– Seventh configuration in the solution sequence is $[4, 4, 4]$. To realise this configuration we move three items of size $c[2] = 4$ from Q_2 to seventh bin. Since the next configuration is same as seventh configuration, therefore we make the same movements of items from Q_2 to eighth bin. After, these movement both Q_1 and Q_2 will become empty and also there are no more configurations to realise.

Since all the configuration in the solutions sequence are feasible, the above procedure has constructed a feasible solution to kBP . □

E Fernandez de la Vega-Lueker algorithm to *k*BP

Linear Grouping: Let D be some instance of the bin-packing problem and $g > 1$ be some integer parameter. Order the items in D in a non-increasing order. Let U be the instance obtained by making groups of the items in D of cardinality g and then rounding up the items in each group by the maximum item size in that group. Let U' be the group of the g largest items and U'' be the instance consisting of groups from the second to the last group in U. Then $OPT(U'') \leq OPT(D)$ [30] and $OPT(U') \leq g$, because each item in U' can be packed into a single bin. So

$$OPT(D) \leq OPT(U'' \cup U') \leq OPT(U'') + OPT(U') \leq OPT(U'') + g.$$

It implies the below Lemma E.1 due to [30], where $LIN(D)$ denotes the value of the fractional bin-packing problem F_1 associated with the instance D:

Lemma E.1. *1.* $OPT(U'') \leq OPT(D) \leq OPT(U'') + g$. *2.* $LIN(U'') \leq LIN(D) \leq LIN(U'') + g$. *3.* $V(U'') \leq V(D) \leq V(U'') + g$.

Solving the configuration linear program C_k: Configuration linear program C_k can be solved as follows:

Using exhaustive search: Since there are $n(D)$ items, a configuration can repeat at most $n(D)$ times. Therefore, $(x_\tau)_{\tau \in T} \in \{0, 1, \ldots, n(D)\}^{|T|}$ where T is the set of all possible configurations, τ is some configuration in T, and x_τ is the number of bins filled with the configuration j. Each configuration contains at most $1/\epsilon$ items, and there are $m(D)$ different item sizes. Therefore, the number of possible configurations is at most $m(D)^{1/\epsilon}$. Now, we can check if enough slots are available for each size $c[i]$. Finally, output the solution with the minimum number of bins. Running time then is $m(D) \times n(D)^{m(D)^{1/\epsilon}}$, which is polynomial in $n(D)$ when $m(D)$ and ϵ are fixed.

If the items in D are arranged in a non-decreasing order, then U'' will be the instance consisting of groups from first to the second from the last group, and U' be the last group consisting of the $\leq g$ large items. The above same results hold in this case as well.

We extend the algorithm by de la Vega and Lueker to solve kBP as follows:

Algorithm 4: Fernandez de la Vega-Lueker Algorithm to kBP

Input: A set D of items, an integer k, and $\epsilon \in (0, 1/2]$.
Output: A bin-packing of D_k.

1. Let I and J be multisets of small (of size $\leq \epsilon \cdot S$) and large (of size $> \epsilon \cdot S$) items in D, respectively.
2. Sort the items in J in a non-decreasing order of their sizes.
3. Construct an instance U from J by the linear grouping with $g = n(J) \cdot \epsilon^2$ and round up the sizes in each group to the maximum size in that group.
4. Optimally solve the configuration linear program C_k corresponding to the rounded problem U. This will give us an optimal solution to the kBP instance U_k.
5. To get a solution for J_k, "ungroup" the items, that is replace the items in groups with the original items in that group, as in step 3, prior to rounding up.
6. Greedily add small items in I_k by respecting constraint of kBP to get a solution for D_k.

The instances I_k and J_k consist of k copies of instances I and J, respectively.

Lemma E.2. *Let $OPT(U_k)$ denote the optimal number of bins in solving C_k. Then, $OPT(U_k) < (1 + \epsilon) \cdot OPT(J_k)$.*

Proof. Let L be the problem resulting from rounding down the items in each group but the first by the maximum of the previous group. Then, the instance L_k consists of k copies of instance L. The items in the first group are rounded down to 0. Since bin-packing is monotone,

$$OPT(L_k) \leq OPT(J_k) \leq OPT(U_k) \tag{18}$$

Note that L and U differ by a group of at most $n(J) \cdot \epsilon^2$ items. Therefore, the difference between L_k and U_k is, at most, $k \cdot n(J) \cdot \epsilon^2$. Hence,

$$\begin{aligned} OPT(U_k) - OPT(L_k) &\leq k \cdot n(J) \cdot \epsilon^2 \\ OPT(U_k) &\leq OPT(J_k) + k \cdot n(J) \cdot \epsilon^2 \end{aligned} \tag{19}$$

Since all items in U_k (and hence in J_k) have a size larger than $\epsilon \cdot S$, the number of items in a bin is at most $1/\epsilon$. Hence, the minimum number of bins required to pack the items in J_k are at least $k \cdot n(J) \cdot \epsilon$. Hence,

$$k \cdot n(J) \cdot \epsilon^2 \leq \epsilon \cdot OPT(J_k). \tag{20}$$

By Equation (19) and Equation (20),

$$OPT(U_k) \leq (1 + \epsilon) OPT(J_k) \tag{21}$$

□

Corollary 6. $A(J_k) \leq (1 + \epsilon) \cdot OPT(J_k)$.

Proof. From steps 3 and 5 of the algorithm, it is clear that from a packing of U_k we can obtain a packing of J_k with the same number of bins. Therefore, $A(J_k) \leq (1 + \epsilon) \cdot OPT(J_k)$. □

Theorem 6.1. *Let A be the generalization of the Fernandez de la Vega-Lueker algorithm to solve kBP . Then, $A(D_k) \leq (1 + 2 \cdot \epsilon) \cdot OPT(D_k) + k$.*

Proof. Let I be the set of all items of size at most $\epsilon \cdot S$ in D, and I_k be k copies of the instance I. In the step 6, while packing the items in I_k if fewer than k new bins are opened, then from *Corollary* 6 it is clear that $A(D_k) \leq A(J_k) + k \leq (1+\epsilon) \cdot OPT(J_k) + k$. Since bin-packing is monotone, $OPT(J_k) \leq OPT(D_k)$. Therefore, $A(D_k) \leq (1 + \epsilon) \cdot OPT(D_k) + k$.

Now assume that at least k new bins are opened at the step 6 of the algorithm. Then, From Lemma 6.1,

$$\begin{aligned} A(D_k) &\leq \max\{(1 + \epsilon) \cdot OPT(D_k) + k, (1 + 2 \cdot \epsilon) \cdot OPT(D_k) + k\} \\ &\leq (1 + 2 \cdot \epsilon) \cdot OPT(D_k) + k \end{aligned} \tag{22}$$

□

Running time of Algorithm 4: Time $O(n(D) \log n(D))$ is sufficient for all the steps except the step 4 in Algorithm 4. Step 4 takes time $O\left(m(D) \cdot n(D)^{m(D)^{1/\epsilon}}\right)$. Therefore, the time taken by the algorithm is $O\left(m(D) \cdot n(D)^{m(D)^{1/\epsilon}}\right)$, which is polynomial in $n(D)$ given $m(D)$ and ϵ.

G Karmarkar-Karp Algorithms to *k*BP

Lemma G.1. $OPT(D_k) \leq 2 \cdot V(D_k)/S + k$

Proof. When $k = 1$ then from [30] we know that $OPT(D) \leq \frac{2}{S} \cdot V(D) + 1$. Therefore

$$OPT(D_k) \leq k \cdot OPT(D) \leq \frac{2}{S} \cdot k \cdot V(D) + k = \frac{2}{S} \cdot V(D_k) + k.$$

□

Lemma G.2. $V(D_k) \leq S \cdot LIN(D_k) \leq S \cdot OPT(D_k) \leq S \cdot LIN(D_k) + \frac{S \cdot (m(D_k)+k)}{2}$

Proof. Let $\mathbf{c}$ and $\mathbf{n}$ be the vectors of dimension $m(D)$ such that for each natural $i \leq m(D)$ the ith entry of $\mathbf{c}$ is the size $c[i]$ and the ith entry of $\mathbf{n}$ is the number $n[i]$ of items of size $c[i]$. For each natural $j \leq t$ let the size of the configuration j be the sum of all sizes of the items of a_j, which is equal to the jth entry of the vector $\mathbf{c^T}A$. For instance, let $m(D) = 2$, $n[1] = 2$, $n[2] = 2$, $c[1] = 3$, $c[2] = 4$, and $S = 10$. Then, the number t of feasible configurations is 6, and let the matrix A be $\begin{bmatrix} 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 0 & 0 & 1 & 1 \end{bmatrix}$. Then $\mathbf{c^T}A = \begin{bmatrix} 4 & 8 & 3 & 6 & 7 & 10 \end{bmatrix}$. Note that the size of each configuration is at most the bin size S. So if $\mathbf{S}$ is a vector of dimension t whose each entry equals S then

$$\mathbf{S} \geq \mathbf{c^T}A \tag{23}$$

Let $\mathbf{x}$ be an optimal basic feasible solution of the fractional linear program F_k, which represents the kBP problem, for instance, D_k. Each element $x[j]$ in $\mathbf{x}$ represents the (fractional) number of bins filled with configuration j. Then, $\mathbf{S} \cdot \mathbf{x}$ is the maximum possible sum of all items from all configurations in the solution vector $\mathbf{x}$. Then,

$$\begin{aligned} S \cdot LIN(D_k) = S \cdot LIN(F_k) &= \mathbf{S} \cdot \mathbf{x} && (24) \\ &\geq (\mathbf{c}^T A) \cdot \mathbf{x} && \text{by Equation (23)} \quad (25) \\ &\geq \mathbf{c}^T (k\mathbf{n}) && \text{by Equation (8)} \quad (26) \end{aligned}$$

Since for each natural $i \leq m(D)$, the ith entry of $\mathbf{c^T}$ is the size $c[i]$ and the ith entry of $\mathbf{n}$ is the number of items of size $c[i]$, $\mathbf{c^T n}$ is the sum of sizes of all items of D, so by Equation (26) we have $S \cdot LIN(D_k) = k \cdot V(D) = V(D_k)$.

Since for any natural $j \leq t$ the variable x_j in the problem for $LIN(D_k)$ is the fractional number of occurrences of the configuration j, we have $LIN(D_k) \leq OPT(D_k)$, and the equality holds iff the problem for $LIN(D_k)$ has an integer solution.

It is well-known that the linear program Equation (7) - Equation (9) has an optimal solution $\mathbf{x}$ which is a *basic feasible solution*, where the number of non-zero variables is at most the number $m(D)$ of constraints. For each natural $j \leq t$ such that $x_j > 0$ we have $x_j = \lfloor x_j \rfloor + r_j$, where $\lfloor x_j \rfloor$ is the integer

part of x_j, and r_j is the fractional part of x_j. For each natural $j \leq t$ such that $r_j \in (0,1)$, we take configuration j. Let D' be the resulting constructed instance. Clearly, all items in D' will have a single copy. Then,

$$V(D') \leq S \cdot LIN(D') = S \cdot \sum_j r_j \tag{27}$$

For the residual part, we can construct two packings. First, for each non-zero r_j, let's take a bin of configuration j. We can then remove extra items. Therefore,

$$OPT(D') \leq m(D_k) \text{ (the instances } D \text{ and } D_k \text{ have the same configurations)} \tag{28}$$

Second, from Lemma G.1 we have

$$OPT(D') \leq 2V(D')/S + k \tag{29}$$

By Equation (28) and Equation (29),

$$OPT(D') \leq \min\{m(D_k), 2V(D')/S + k\} \leq \frac{m(D_k) + 2V(D')/S + k}{2} = V(D')/S + \frac{m(D_k) + k}{2}.$$

.

Therefore,

$$\begin{aligned} &OPT(D_k) \leq \text{the principal part} + OPT(D') \\ &OPT(D_k) \leq \text{the principal part} + V(D')/S + \frac{m(D_k) + k}{2} \\ &OPT(D_k) \leq LIN(D_k) + \frac{m(D_k) + k}{2} \qquad \text{by Equation (24)} \end{aligned} \tag{30}$$

□

Corollary 8. *There is an algorithm $\boldsymbol{A}$ that solves the kBP by solving the corresponding fractional bin-packing problem F_k. It produces at most $\mathbf{1} \cdot \mathbf{x} + \frac{m(D_k)+k}{2}$ bins, where $\mathbf{x}$ is an optimal solution of F_k.*

Note that the number of different item sizes in the instance is $m(D_k) = m(D)$.

H.1 Solving the fractional linear program:

Solving the fractional linear program F_k involves a variable for each configuration. This results in a large number of variables. The fractional linear program F_k has the following dual D_F.

$$\begin{aligned} \max \quad & k \cdot \mathbf{n} \cdot \mathbf{y} && (31)\\ \text{such that} \quad & A^T\mathbf{y} \leq \mathbf{1} && (32)\\ & \mathbf{y} \geq 0 && (33) \end{aligned}$$

The above dual linear program can be solved to any given tolerance h by using a variant of the ellipsoid method that uses an approximate separation oracle [30]. This separation oracle accepts as input a vector $\mathbf{y}$ (each element $y[i]$ of $\mathbf{y}$ represents the price of item of size $c[i]$) and determines:

- whether $\mathbf{y}$ is feasible, or
- if not, then return a constraint that is violated, i.e. a vector $\mathbf{a}$ such that $\mathbf{a} \cdot \mathbf{y} > 1$.

Let $\mathbf{c}$ be the $m(D)$-vector of the item sizes. The separation oracle solves the above problem by solving the following knapsack problem:

$$\begin{aligned} \max \quad & \mathbf{a} \cdot \mathbf{y} && (34)\\ \text{such that} \quad & \mathbf{a} \cdot \mathbf{c} \leq S && (35)\\ & \mathbf{a} \geq 0 && (36) \end{aligned}$$

Each entry $a[i]$ of $\mathbf{a}$ represents $a[i]$ pieces of size $c[i]$, and hence $\mathbf{a}$ is an integer vector. If $\mathbf{y}$ is feasible then the optimal value of the above knapsack problem is at most 1. Otherwise, $\mathbf{a}$ correspond to a configuration that violates the constraint $\mathbf{a} \cdot \mathbf{y} \leq 1$.

Suppose we want a solution of D_F within a specified tolerance δ. Then, the above knapsack problem can be solved in polynomial time by rounding down each component of $\mathbf{y}$ to the closest multiple of $\frac{\delta}{n(D)}$. The runtime of the above algorithm is $O\left(\frac{m(D)\cdot n(D)}{\delta}\right)$ [30]. Alternatively, we can round down each component of $\mathbf{y}$ to the nearest multiple of $\frac{\delta}{k\cdot n(D)}$, and then the runtime of the above algorithm is $O\left(\frac{k\cdot m(D)\cdot n(D)}{\delta}\right)$.

The modified ellipsoid method uses the above approximate separation oracle as follows. Given the current ellipsoid center $\mathbf{y_c}$, let $\widetilde{\mathbf{y_c}}$ be the rounded down solution corresponding to $\mathbf{y_c}$. The modified ellipsoid method performs either an *feasibility cut* (if $\widetilde{\mathbf{y_c}}$ is infeasible) or an *optimality cut* (if $\widetilde{\mathbf{y_c}}$ is feasible). Since $\widetilde{\mathbf{y_c}}$ is the rounded down solution corresponding to $\mathbf{y_c}$, if $\widetilde{\mathbf{y_c}}$ is infeasible then $\mathbf{y_c}$ is not feasible too. In a feasibility cut, the modified ellipsoid method cuts from the ellipsoid all points such that $\mathbf{a}\cdot\mathbf{y} > 1$. If $\widetilde{\mathbf{y_c}}$ is feasible, then $\mathbf{y_c}$ may or may not be feasible. If the rounding-down is done to a multiple of $\frac{\delta}{k\cdot n(D)}$, then, by definition of the rounding, we have

$$k\cdot\mathbf{n}\cdot\widetilde{\mathbf{y_c}} \geq k\cdot\mathbf{n}\cdot\mathbf{y_c} - k\cdot\mathbf{n}\cdot\mathbf{1}\cdot\frac{\delta}{k\cdot n}k\cdot\mathbf{n}\cdot\mathbf{y_c} - \delta.$$

In an optimality cut, the method preserves all the vectors whose value exceeds $\widetilde{\mathbf{y_c}}$ by more than δ [30] i.e. it removes all points that satisfy $k\cdot\mathbf{n}\cdot\mathbf{y} < k\cdot\mathbf{n}\cdot\widetilde{\mathbf{y_c}} + \delta$.

The number of iterations in the modified ellipsoid algorithm is the same as that of the modified ellipsoid method in [30] that is, at most, $Q = 4\cdot m(D)^2\cdot\ln\left(\frac{m(D)\cdot n(D)}{\epsilon\cdot S\cdot\delta}\right)$ iterations. The total execution time of the modified algorithm is $O\left(Q\cdot m(D)\cdot\left(Q\cdot m(D)+\frac{k\cdot n(D)}{\delta}\right)\right)$.

Let t be the number of configurations in kBP, which is the same as the number of configurations in BP. There are at most t constraints of the form $\mathbf{a}\cdot\mathbf{y}\leq 1$ during the ellipsoid method. It is well known that in any bounded linear program with $m(D)$ variables, at most $m(D)$ constraints are sufficient to determine the optimal solution. We can use the same constraint elimination procedure as in [30] to come up with this critical set of $m(D)$ constraints. It results in having a reduced dual linear program with $m(D)$ variables and $m(D)$ constraints. Let us call this reduced dual linear program R_{D_F}. By taking the dual of R_{D_F}, we get a reduced primal linear program, say R_F. This reduced primal linear program will have only $m(D)$ variables corresponding to $m(D)$ constraints. Let us denote the optimal solutions of the dual linear program D_F and the reduced dual linear program R_{D_F} as $LIN(D_F)$ and $LIN(R_{D_F})$ respectively. Since R_{D_F} is a relaxation of D_F therefore, $LIN(D_F)\leq LIN(R_{D_F})$. From the modified GLS algorithm, we know that $LIN(R_{D_F})\leq LIN(D_F)+\delta$ if rounding down is done in a multiple of $\frac{\delta}{k\cdot n(D)}$. From the duality theorem of the linear program, both primal and dual have the same optimal solution. Therefore, the optimal solution of the reduced primal linear program, R_F, is at most $LIN(D_F)+\delta$ using the rounding down scheme in the modified GLS method.

The constraint elimination procedure will require to execute the modified GLS algorithm to run at most $(m(D)+1)\left(\left\lceil\frac{R}{m(D)+1}\right\rceil+(m(D)+1)\ln\left(\left\lceil\frac{R}{m(D)+1}\right\rceil\right)+1\right)$ times, where $R\leq Q+2\cdot m(D)$ [30]. In the constraint elimination procedure, in the worst case, one out of every $m(D)+1$ executions succeeds [30]. Therefore, the total accumulated error is $\delta\left(\left\lceil\frac{R}{m(D)+1}\right\rceil+(m(D)+1)\ln\left(\left\lceil\frac{R}{m(D)+1}\right\rceil\right)+1\right)$. If the tolerance δ is set to

$$\frac{h}{\left\lceil\frac{R}{m(D)+1}\right\rceil+(m(D)+1)\ln\left(\left\lceil\frac{R}{m(D)+1}\right\rceil\right)+2}$$

then the optimal solution of the reduced primal linear program is at most $LIN(D_F)+h$ using the rounding down mechanism in the modified GLS method.

The total running time of the algorithm is

$$O\left(\left(Qm(D)\left(Qm(D)+\frac{k\cdot n(D)}{\delta}\right)\right)(m(D)+1)\left(\left\lceil\frac{R}{m(D)+1}\right\rceil+(m(D)+1)\ln\left\lceil\frac{R}{m(D)+1}\right\rceil+1\right)\right)$$

$$\approx O\left(m(D)^8\cdot\ln m(D)\cdot\ln^2\left(\frac{m(D)\cdot n(D)}{\epsilon\cdot S\cdot h}\right)+\frac{m(D)^4\cdot k\cdot n(D)\cdot\ln m(D)}{h}\ln\frac{m(D)\cdot n(D)}{\epsilon\cdot S\cdot h}\right).$$

Since $m(D) = m(D_k)$ and $k\cdot n(D) = n(D_k)$, in our analysis, we denote this running time as $T(m(D_k), n(D_k))$.

H.2 Karmarkar-Karp Algorithm 1 extension to kBP

Recall that the algorithm 1 of Karmarkar-Karp algorithms uses the linear-grouping technique Section 6.1.

Algorithm 5: Karmarkar-Karp Algorithm 1 extension to kBP

Input: A set D of items, a number $\epsilon \in (0, 1/2]$, and an integer k.
Output: A bin-packing of D_k.

1 Let an item of D is *small* if it has size at most $\max\{1/n, \epsilon\} \cdot S$ and *large*, otherwise. Let I and J be multisets of small and large items of D, respectively. Let I_k be the k copies of the instance I.
2 Perform the linear grouping on the instance J by making groups of cardinality $g = \lceil n(J) \cdot \epsilon^2 \rceil$. Denote the resulting instances as U' and U'', as defined in the linear grouping technique in Section E
3 Pack each item in U'_k into a bin. There can be at most $g \cdot k$ bins, because each item of D has k copies in D_k.
4 Solve U''_k using a fractional linear program with tolerance $h = 1$. Denote the resulting solution as $\mathbf{x}$.
5 Construct the integer solution from $\mathbf{x}$ by the rounding method using no more than $1 \cdot \mathbf{x} + \frac{m(U'')+k}{2}$ bins (see Corollary 8).
6 Reduce the item sizes as necessary and obtain a packing for J_k along with the addition of bins as in step 3.
7 Obtain a packing for D_k by adding the items in I_k which are kept aside in step 1 by respecting constraint of kBP, creating new bin(s) when necessary.

Theorem 6.2. *Let $A(D_k)$ denote the number of bins produced by Karmarkar-Karp Algorithm 1 extension to solve kBP . Then, $A(D_k) \leq (1 + 2 \cdot k \cdot \epsilon)OPT(D_k) + \frac{1}{2 \cdot \epsilon^2} + (2 \cdot k + 1)$.*

Proof. Since bin-packing is monotone.

$$OPT(U''_k) \leq OPT(J_k) \leq OPT(D_k) \tag{37}$$

The size of each item in J_k is at least $\frac{\epsilon}{2} \cdot S$. Therefore, from Lemma G.2,

$$S \cdot OPT(J_k) \geq V(J_k) \geq \frac{\epsilon}{2} \cdot S \cdot n(J_k)$$
$$OPT(J_k) \geq \frac{\epsilon}{2} \cdot n(J_k). \tag{38}$$

From algorithm step 2 and from Equation (37) and Equation (38),

$$g = \lceil n(J_k)\epsilon^2 \rceil \leq 2 \cdot \epsilon \cdot OPT(J_k) + 1 \leq 2 \cdot \epsilon \cdot OPT(D_k) + 1$$
$$m(U''_k) = n(U''_k)/g = n(U''_k)/\lceil n(J_k)\epsilon^2 \rceil \leq n(J_k)/\lceil n(J_k)\epsilon^2 \rceil \leq 1/\epsilon^2.$$

We have tolerance $h = 1$, so

$$\mathbf{1} \cdot \mathbf{x} \leq LIN(U''_k) + 1 \leq OPT(U''_k) + 1 \leq OPT(D_k) + 1$$

Step 6 will result in the number of bins which is at most

$$\mathbf{1} \cdot \mathbf{x} + \frac{m(U''_k) + k}{2} + g \cdot k \leq$$

$$OPT(D_k) + 1 + \frac{k + \frac{1}{\epsilon^2}}{2} + k \cdot (2 \cdot \epsilon \cdot OPT(D_k) + 1) \leq$$

$$(1 + 2 \cdot k \cdot \epsilon)OPT(D_k) + \frac{1}{2 \cdot \epsilon^2} + (2k + 1).$$

By Lemma 6.1, the number $bins(D_k)$ of bins required at step 7 of the algorithm is at most

$$\max\left\{(1 + 2 \cdot k \cdot \epsilon)OPT(D_k) + \frac{1}{2 \cdot \epsilon^2} + (2k + 1), (1 + 2 \ \cdot \epsilon)OPT(D_k) + k\right\} =$$

$$(1 + 2 \cdot k \cdot \epsilon)OPT(D_k) + \frac{1}{2 \cdot \epsilon^2} + (2k + 1).$$

Running time of Algorithm 5. The time taken by fractional linear program is at most $T(m(U''_k), n(U''_k)) \leq T\left(\frac{1}{\epsilon^2}, n(D_k)\right)$. The rest of the steps in Algorithm 1 will take at most $O(n(D_k)\log n(D_k))$ time. Therefore, the execution time of above Algorithm 1 is at most $O\left(n(D_k)\log n(D_k) + T(\frac{1}{\epsilon^2}, n(D_k)\right)$.

H.3 Karmarkar-Karp Algorithm 2 extension to *k*BP

Alternative Geometric Grouping. Let J be an instance and $g > 1$ be an integer parameter. Sort the items in J in a non-increasing order of their size. Now, we partition J into groups $G_1, \ldots G_r$ each containing necessary number of items so that the size of each but the last group (the sum of item sizes in that group) is at least $g \cdot S$.

For natural $i \leq r$ let l_j be the cardinality of the group G_i, that is the number of items in the group. Let l_r be the cardinality of G_r. Note that the size of G_r may be less than $g \cdot S$.

Since each item size is less than S, the number of items in each but the last group is at least $g+1$. Let $\epsilon \cdot S$ be the smallest item size in J. Then, the number of items in each but the last group is at most $\frac{g}{\epsilon}$. Therefore, $g+1 \leq l_1 \leq \ldots \leq l_r \leq \frac{g}{\epsilon}$. For each natural $i < r$ let G'_{i+1} be the group obtained by increasing the size of each item in G_{i+1} to the maximum item size in that group except for the smallest $l_{i+1} - l_i$ items in that group. Let $U' = \bigcup_{i=2}^{r} \Delta G_i \cup G_1$ and $U'' = \bigcup_{i=2}^{r} G'_i$, where ΔG_i is the multiset of smallest $l_i - l_{i-1}$ items in G_i for each natural i with $2 \leq i \leq r$.

Since bin-packing is monotone, $OPT(J) \leq OPT(U') + OPT(U'')$. By Lemma G.1, $OPT(U') \leq 2 \cdot V(U')/S + 1$ (note that $k = 1$). It holds that $V(U') \leq S \cdot g \cdot (1 + \ln \frac{1}{\epsilon \cdot S}) + S$ [30, page 315]. Therefore,

$$OPT(J) \leq OPT(U'') + 2 \cdot g \cdot \left(2 + \ln \frac{1}{\epsilon \cdot S}\right) \tag{39}$$

Also, $V(U'') \geq \sum_{i=2}^{r} g \cdot S \cdot \frac{l_{i-1}}{l_i} \geq g \cdot S \cdot \left((r-1) - \ln \frac{1}{\epsilon \cdot S}\right)$. Therefore, by [30],

$$m(U'') = r - 1 \leq \frac{V(U'')}{g \cdot S} + \ln \frac{1}{\epsilon \cdot S}. \tag{40}$$

Algorithm 6: Karmarkar-Karp Algorithm 2 extension to kBP

Input: A set D of items, $\epsilon \in (0, 1/2]$, and integers $g > 1$ and k.
Output: A bin-packing of D_k.

1 Let J be the instance obtained after removing all items of size at most $\epsilon \cdot S$ from D.
2 **while** $V(J) > S \cdot (1 + \frac{g}{g-1} \cdot \ln \frac{1}{\epsilon})$ **do**
3 Do the alternative geometric grouping with the parameter g. Let the resulting instances be U' and U''.
4 Solve U''_k using fractional bin-packing with the tolerance $h = 1$. Denote the resuting basic feasible solution as $\mathbf{x}$.
5 Construct an integer solution from $\mathbf{x}$. Create $\lfloor x_i \rfloor$ bins for each x_i in $\mathbf{x}$. Remove the packed items from J.
6 Pack U'_k in at most $2 \cdot k \cdot g \cdot \left[2 + \ln \frac{1}{\epsilon}\right]$ bins.
7 **end**
8 When $V(J) \leq S \cdot (1 + \frac{g}{g-1} \cdot \ln \frac{1}{\epsilon})$, pack the remaining items greedily in at most $(2 + \frac{2 \cdot g}{g-1} \cdot \ln \frac{1}{\epsilon})$ bins and create k copies of this packing.
9 Greedily pack the k copies of the items of size at most $\epsilon \cdot S$ respecting kBP constraint to get a solution for D_k.

Theorem 6.3. *Let $A(D_k)$ denote the number of bins produced by Karmarkar-Karp Algorithm 2 extension to solve kBP . Then, $A(D_k) \leq OPT(D_k) + O(k \cdot \log^2 OPT(D))$.*

Proof. Let $U''_{k,i}$ be the instance of U''_k at the beginning of the ith iteration of the loop in step 2. Then the fractional part of $\mathbf{x}$ contains at most $m(U''_{k,i})$ non-zero variables x_j for $j \leq t$. Therefore, from (40) we have

$$m(U''_{k,i}) = m(U''_{1,i}) \leq \frac{V(U''_{1,i})}{g \cdot S} + \ln \frac{1}{\epsilon \cdot S} \tag{41}$$

The number of iterations in step 6 of algorithm. As shown in [30] the number of iterations is at most $\frac{\ln V(D)}{\ln g} + 1$.

Number of bins produced using Fractional Bin-Packing solution in steps $3-5$. Let J_i denote the item set J at the beginning of ith iteration. On applying alternative geometric grouping with the parameter g, we get the instances ${U'}_{1,i}$ and ${U''}_{1,i}$. Let ${U'}_{k,i}$ and ${U''}_{k,i}$ be the corresponding kBP instances. Let B_i be the number of bins obtained in solving ${U''}_{k,i}$ by applying fractional bin-packing at the iteration i with the tolerance $h = 1$. Then

$$B_i \leq LIN({U''}_{k,i}) + 1.$$

Summing across all iterations

$$\begin{aligned} \sum B_i &\leq LIN({U''}_k) + \frac{\ln V(D)}{\ln g} + 1 \\ &\leq OPT({U''}_k) + \frac{\ln V(D)}{\ln g} + 1 \end{aligned} \tag{42}$$

The total number of bins used. Let $A(D_k)$ denote the total number of used bins Then $A(D_k)$ is the sum of the bins used in steps $3-5$, in step 6, and in step 8. So

$$\begin{aligned} A(D_k) \leq OPT(D_k) + \left(\frac{\ln V(D)}{\ln g} + 1\right)\left(1 + 4 \cdot k \cdot g + 2 \cdot k \cdot g \ln \frac{1}{\epsilon}\right) + \\ k \cdot \left(2 + \frac{2 \cdot g}{g-1} \ln \frac{1}{\epsilon}\right). \end{aligned} \tag{43}$$

By Lemma 6.1,

$$\begin{aligned} A(D_k) \leq \max\Bigg\{OPT(D_k) + \left(\frac{\ln V(D)}{\ln g} + 1\right)\left(1 + 4 \cdot k \cdot g + 2 \cdot k \cdot g \ln \frac{1}{\epsilon}\right) + \\ k \cdot \left(2 + \frac{2 \cdot g}{g-1} \ln \frac{1}{\epsilon}\right), (1 + 2 \cdot \epsilon) \cdot OPT(D_k) + k\Bigg\} \end{aligned} \tag{44}$$

Choosing $g = 2$ and $\epsilon = \frac{1}{V(D)}$ will result in

$$A(D_k) \leq OPT(D_k) + O(k \cdot \log^2 OPT(D)) \tag{45}$$

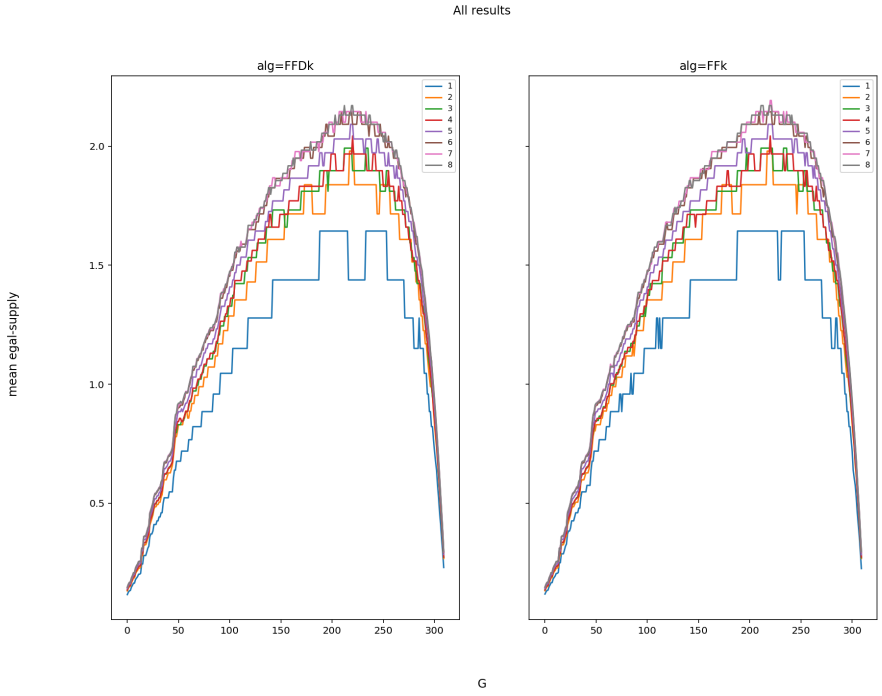
number of bins. □

Running time of Algorithm 6: By Equation (40), at each iteration of the while loop, ${m(U''}_{1,i})$ decreases by a factor of $g \cdot S$. During the ith iteration of the while loop, the time taken by fractional bin-packing is at most $T(m({U''}_{k,i}), n({U''}_{k,i}))$. The remaining steps will take at most $O(n(D_k) \cdot \log n(D_k))$ time. Therefore, the total time taken by the algorithm is $O\left(T\left(\frac{n(D) \cdot S}{g \cdot S} + \ln \frac{1}{\epsilon}, n(D_k)\right) + n(D_k) \cdot \log n(D_k)\right)$. For the choice $g = 2$ and $\epsilon = \frac{1}{V(D)}$ the running time will be $O\left(T\left(\frac{n(D)}{2} + \ln V(D), n(D_k)\right) + n(D_k) \cdot \log n(D_k)\right) \in O\left(T\left(\frac{n(D)}{2}, n(D_k)\right) + n(D_k) \cdot \log n(D_k)\right)$.

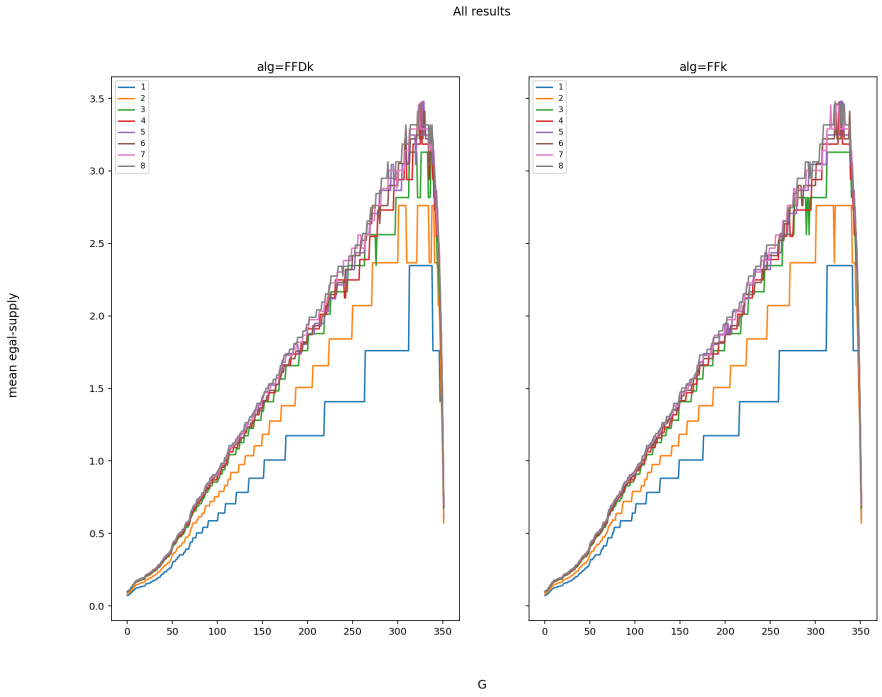
I Example support for the applicability of ternary search

To demonstrate this we have taken some examples from the dataset (same dataset used by [34,35]) used in this paper. Each of these examples has aggregate demand greater than the supply. It makes sense as otherwise every agent gets their stipulated demand. Figure 5 and Figure 6 shows that peak for egalitarian allocation of electricity supply lies between the small and large values of G. This suggests the use of ternary search.

The ternary approach is as follows:

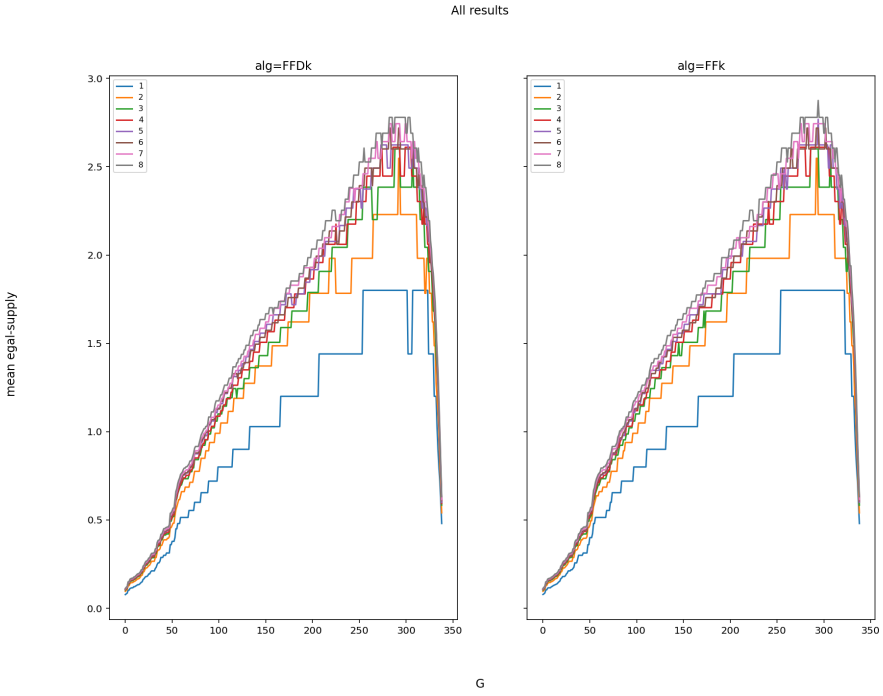


(a)

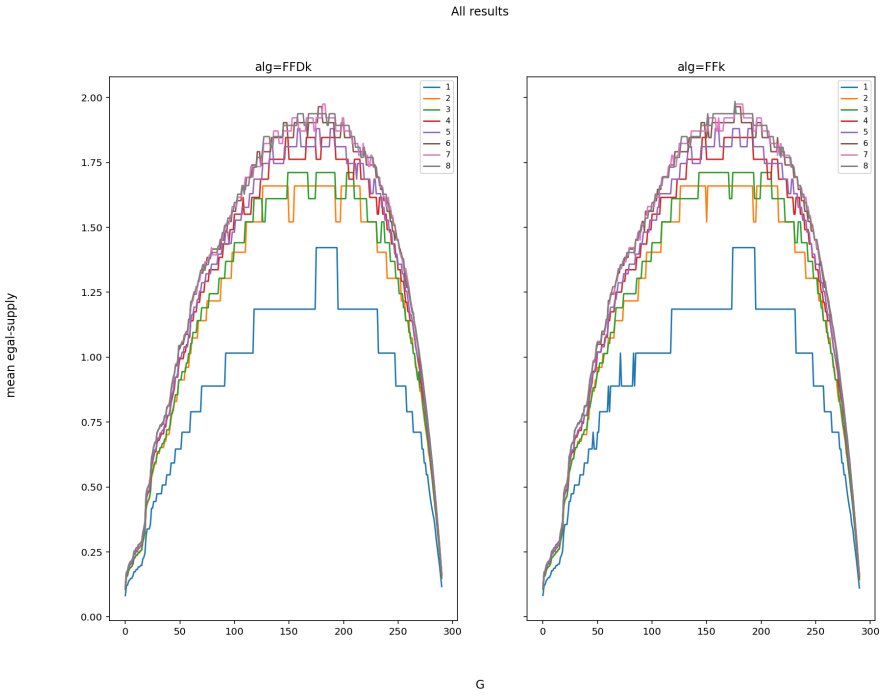


(b)

Fig. 5: Values of G and mean egalitarian value of electricity supply are shown at the x and $y-$ axis respectively. Figure 5a and Figure 5b corresponds to different examples. In each figure we see that the peak lies in between the smallest and highest value of G. Hence, ternary search can be used. Different lines denote a different value of k.



(a)



(b)

Fig. 6: Values of G and mean egalitarian value of electricity supply are shown at the x and $y-$ axis respectively. Figure 6a and Figure 6b corresponds to different examples. In each figure we see that the peak lies in between the smallest and highest value of G. Hence, ternary search can be used. Different lines denote a different value of k.

Algorithm 7: Ternary Search

Input: P: function which computes the solution;
g_{begin}:= an integer parameter. Initially set to 0;
g_{end}:= an integer parameter. Initially set to the cardinilaity of the set that contains all the items $\leq d_l$
`/* NOTE: ... in` P `denotes other parameters that function` P `accepts. */`

`bestsolution` $= None$
while $g_{end} - g_{begin} > 3$ **do**
  Compute `solutionbegin` $= P(\ldots, g_{begin})$
  Compute `solutionend` $= P(\ldots, g_{end})$
  Set `bestsolution` to the one which is best among `bestsolution`, `solutionbegin` and `solutionend`.
  $g_{begin} := round((g_{begin} \cdot 2 + g_{end})/3, 0)$
  $g_{end} := round((g_{begin} + g_{end} \cdot 2)/3, 0)$
end
for $g = g_{begin}$ **to** $g = g_{end}$ **do**
  Compute `solutiong` $= P(\ldots, g)$
  Compare `bestsolution` and `solutiong` and set `bestsolution` accordingly.
end
return `bestsolution`

Function P involves applying algorithm FFk or $FFDk$ (passed as an argument to P) on the instance D''. Finally, after adding the items in G to each bin of the solution obtained, the function P returns this solution.

J More Electricity Distribution Results

In this section, we discuss the variation in hours of connection, comfort, and electricity supplied with different values of k and with varying levels of uncertainty (standard deviation) in the agent's demand.

In Figure 7, we can observe that the sum of comfort the algorithm delivers to agents increase with an increasing value of k. However, that increase appears to saturate as k increases. Similar phenomena occur for other metrics.

Figure 8 shows that the minimum comfort the algorithm delivers to agents individually increases with an increasing value of k. However, that increase appears to saturate as k increases. Similar phenomena occur for other metrics.

In Figure 9, the maximum utility difference decreases with an increasing value of k. Similar phenomena occurs for electricity-supplied social welfare metrics. However, in the case of hours of connection, the maximum utility difference is 0 because the algorithm is executed for each hour.

J.1 Experiment: Heuristic algorithms for egalitarian allocation of watts

Figure 10 shows that maximum watts difference decreases with increasing value of k and with varying levels of uncertainty (standard deviation) in the agent's demand. The below Figure 10 is newly added.

Table 5: Comparing results of heuristic algorithms HA1-HA4 in terms of hours of connection to supply on the average, along with their standard deviation (SD) within parenthesis. In the third column, we have shown the average number of hours an agent is connected to the supply.

Algorithm	Utilitarian: sum(SD)	Egalitarian(SD)	Maximum Utility Difference
HA-1(k = 5, alg=FFk)	750517.7(47.39)	1587.06(0.79)	596.938(0.796)
HA-1(k = 5, alg=FFk)	750441.267(27.508)	1587.1(0.575)	596.903(0.574)
HA-2(k = 50, alg=FFk)	743036.794(124.97)	16454.95(2.424)	538.047(2.434)
HA-2(k = 50, alg=$FFDk$)	743023.081(64.725)	1647.564(1.116)	536.436(1.116)
HA-3(k = 5, alg=FFk)	735908.209(234.535)	1770.334(1.744)	413.666(1.744)
HA-3(k = 5, alg=$FFDk$)	735497.902(148.749)	1771.869(1.415)	412.131(1.415)
HA-4(k = 50, alg=FFk)	**752765.642**(**41.830**)	1828.394(0.941)	355.606(0.941)
HA-4(k = 50, alg=$FFDk$)	752748.122(10.153)	**1828.61(0.654)**	**355.39(0.654)**

Table 6: Comparing results of heuristic algorithms HA1-HA4 in terms of comfort delivered on average, along with their standard deviation (SD) within parenthesis.

Algorithm	Utilitarian: sum(SD)	Egalitarian(SD)	Maximum Utility Difference
HA-1(k = 5, alg=FFk)	310997.434(27.382)	0.629(0.0001)	0.371(0.0001)
HA-1(k = 5, alg=$FFDk$)	310942.697(18.246)	0.629(0.0002)	0.371(0.0002)
HA-2(k = 50, alg=FFk)	306235.496(83.102)	0.662(0.001)	0.338(0.001)
HA-2(k = 50, alg=$FFDk$)	306225.401(39.784)	0.663(0.0003)	0.337(0.0003)
HA-3(k = 5, alg=FFk)	303138.896(145.748)	0.732(0.0007)	0.267(0.0007)
HA-3(k = 5, alg=$FFDk$)	302895.789(82.562)	0.734(0.0012)	0.266(0.0012)
HA-4(k = 50, alg=FFk)	**313024.588**(**30.674**)	0.7686(0.0001)	0.231(0.0001)
HA-4(k = 50, alg=$FFDk$)	313015.52(9.495)	**0.7688(0.0001)**	**0.231(0.0001)**

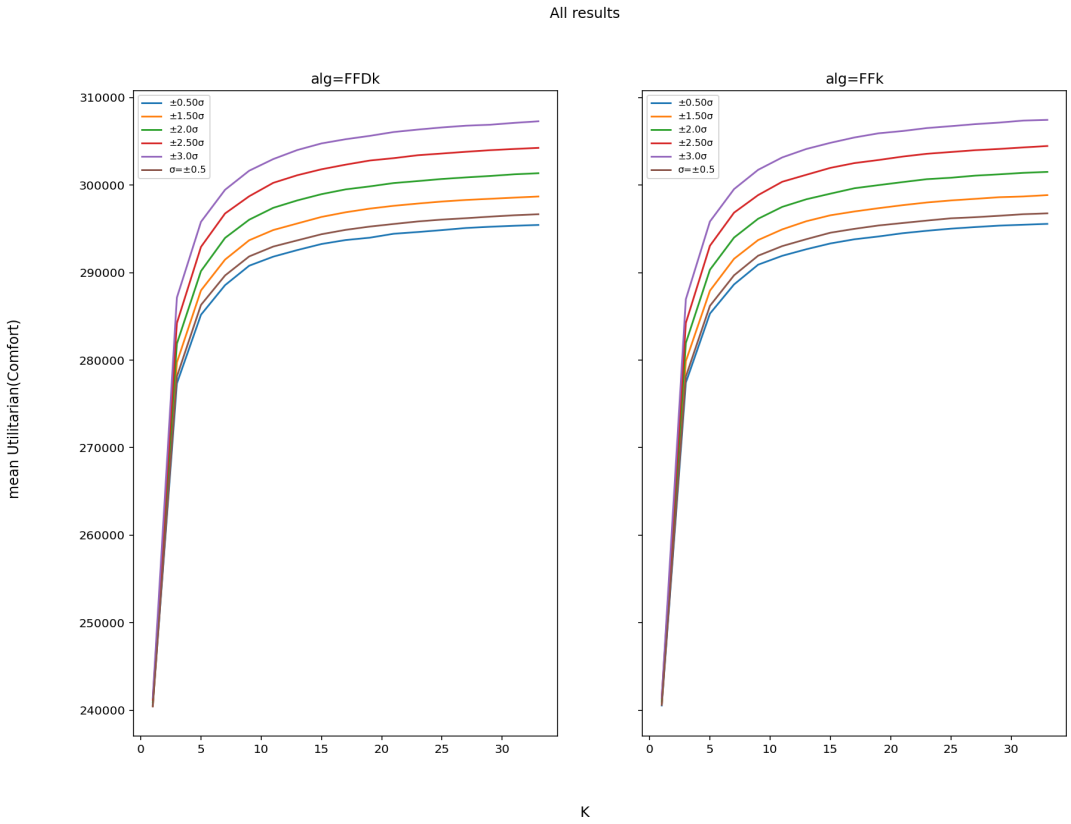


Fig. 7: Values of k and mean utilitarian value of comfort are shown at the x- and y-axis, respectively. Figure shows the increase in sum of comfort delivered by the algorithm to agents with increasing value of k for varying levels of uncertainty σ.

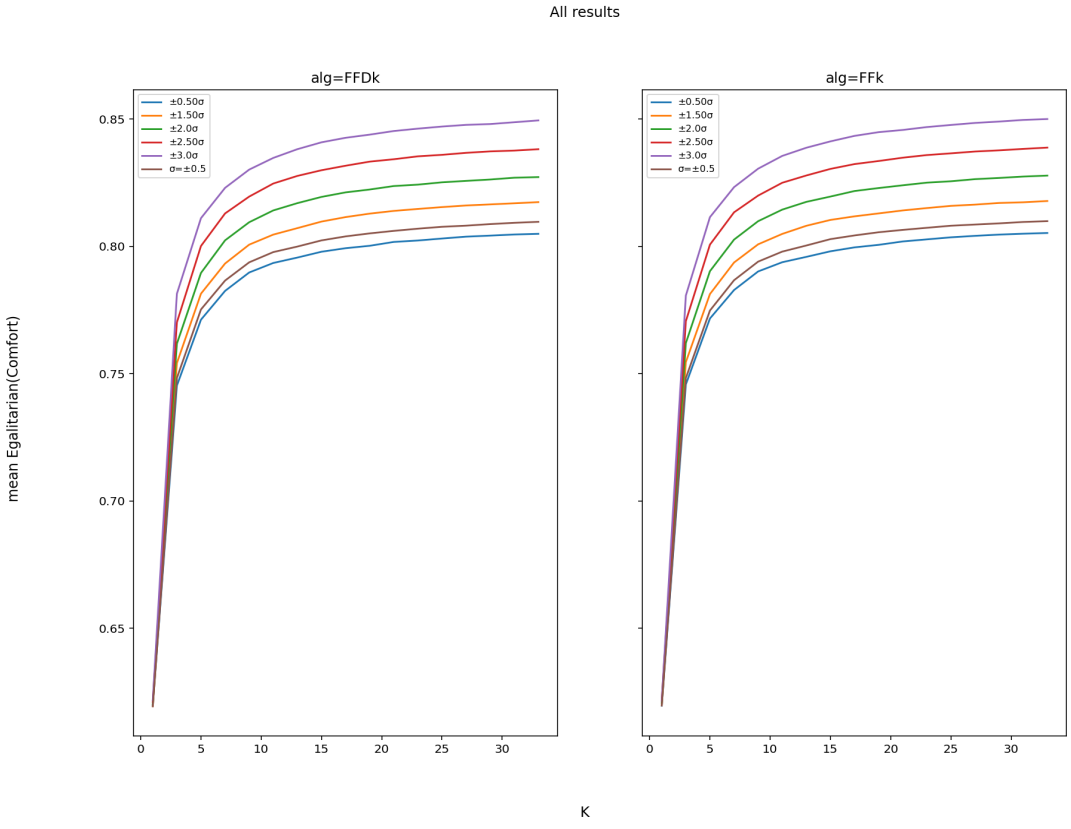


Fig. 8: Values of k and minimum egalitarian value of comfort are shown at the x- and y-axis, respectively. Figure shows the increase in minimum comfort delivered to agents individually with increasing value of k for varying levels of uncertainty σ.

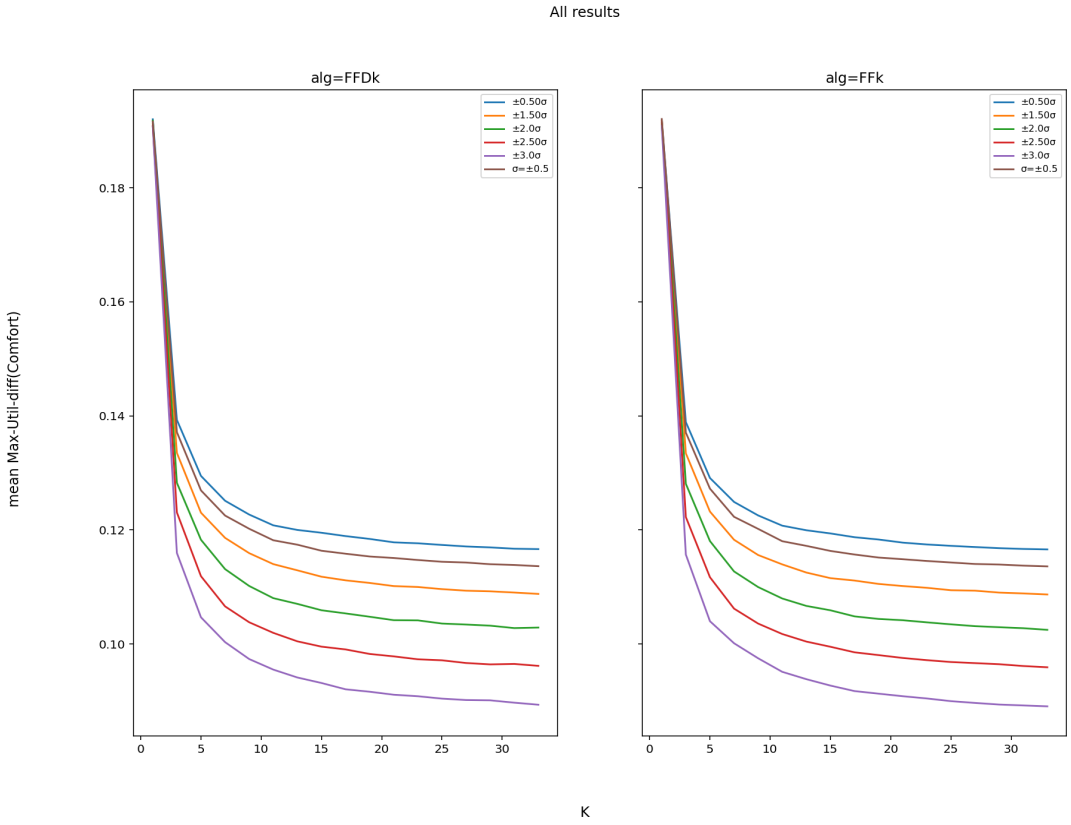


Fig. 9: Values of k and maximum utility difference of comfort are shown at the x- and y-axis, respectively. Figure shows the decrease in maximum utility difference with increasing value of k for varying levels of uncertainty σ.

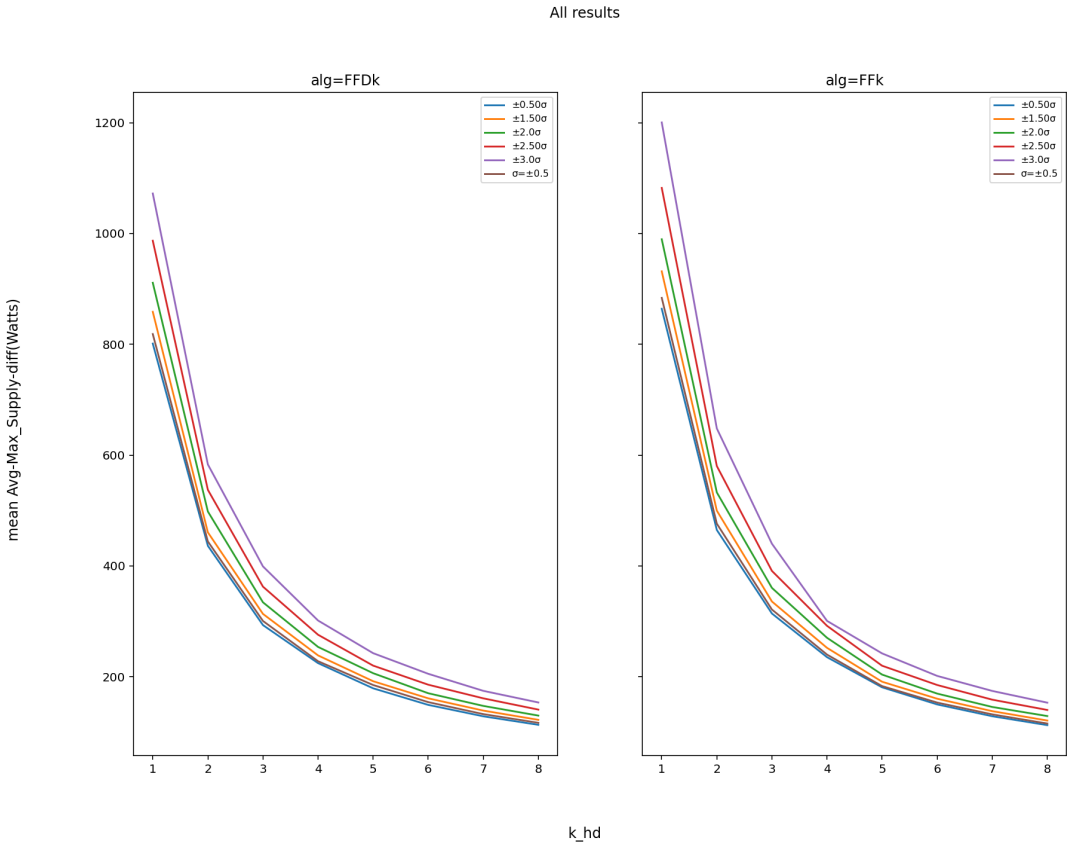


Fig. 10: Values of k and maximum watts difference are shown at the $x-$ and $y-$ axis, respectively. Figures shows that the maximum supply difference decreases with increasing value of k for varying levels of uncertainty σ.