

Distributed Harmonic-Covariance Feedback for Hole Closure in Planar Robot Swarms

Jitian Liu¹ and Subhrajit Bhattacharya¹

Abstract—We consider hole closure in a planar multi-agent system whose sensing disks form a time-varying Čech complex. We propose a harmonic-covariance feedback that uses harmonic edge signals to locate holes and converts those signals into agent forces. For distributed evaluation, we use a fixed-degree polynomial of the regularized Hodge Laplacian. This polynomial is evaluated through local operator applications. The closed loop has a unique global trajectory from every safe initial state. The model data and initial state are supplied with procedures for approximation to any requested accuracy with explicit error bounds. We assume that hole area is measurable and bounded, holes evolve continuously without splitting or merging, and only finitely many temporary new holes appear. Under these conditions, offline finite calculations produce rational lower bounds for the cumulative hole area on the time horizon without querying trajectory data. These bounds converge to that cumulative area, whether finite or infinite. If it is finite, holes close in the time-averaged sense: the average normalized hole area over every fixed-length time window tends to zero as time tends to infinity. In a 15-second numerical demo, all holes close by 9.9s and remain closed through the end.

Index Terms—multi-agent systems, distributed control, algebraic topology, simplicial complexes, coverage control.

I. INTRODUCTION

In nature, among groups of animals such as bird flocks and fish schools, local interaction rules can generate collective motions [1], [2]. The same principle motivates groups of mobile robots known as *swarms*. For multi-robot systems, sensor coverage is a basic problem: each robot senses a disk around its position, and together the robots should cover their operating region. As the robots move, the covered set can develop holes: regions enclosed by the sensing disks that receive no coverage at all. These holes are exactly the holes of the Čech complex built from the current robot positions [3], [4]. The holes of a moving cover are detected by persistent homology [3]–[5] and verified without localization or with switched higher-order Laplacians [6], [7].

Related work on repair and coverage includes simplicial coverage repair for robot teams [8], gradient descent on weighted combinatorial Laplacians [9], homology-based repair of agricultural coverage [10], conformal mapping of domains with holes onto multi-hole disks [11], and diffusive coverage of bounded domains [12]. Related tools include cellular sheaves, which generalize weighted Laplacians for distributed coordination [13], [14]; simplicial filters for processing on complexes [15], [16]; and occupation-measure relaxations, which express dynamical constraints and integral costs through measures and lead to linear optimization problems over measures [17], [18]. The coverage-repair methods above do not provide a criterion for hole closure based solely on the model and initial state. We derive such a criterion for our distributed *harmonic-covariance* controller. The contributions of this paper are as follows:

- A feedback controller that converts the swarm Čech complex’s harmonic edge flows, which localize the holes, into locally Lipschitz force laws on the agents (such that the controlled trajectory is unique), together with a distributed realization of the controller (Sections III and IV).

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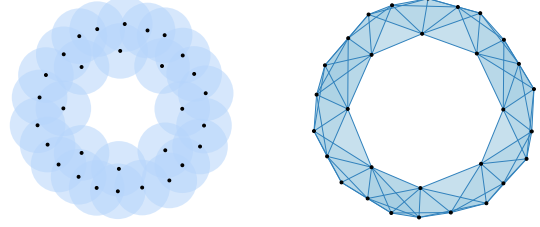


Fig. 1. Left: the union of sensing disks of the 30-agent annulus; right: its strict Čech complex.

- Under this paper’s controlled dynamics and assumptions, we construct an offline sequence of rational lower bounds $\{c_T\}_T$. Each bound is produced by explicit finite calculations without querying trajectory data. The sequence converges to the cumulative hole area over the infinite time horizon. If its limit is finite, the hole area averaged over any fixed time window tends to zero as the window moves forward (Section IV).
- A numerical demo in which two holes close within 10 seconds under the regularized harmonic feedback controller (Section V).

The paper is organized as follows. Section II sets up the task and model. Section III designs the controller and its distributed realization. Section IV proves uniqueness of the state space trajectory under the regularized controller, and constructs $\{c_T\}_T$ from the swarm’s initial state. Section V reports the numerical simulation. Appendix A records details to approximate the normalized hole areas in the swarm and the vector field yielded by the control law, whereas Appendix B supplies proofs to main theoretical results in this paper.

II. PROBLEM FORMULATION AND SENSING MODEL

We consider N mobile sensing agents. Agent i has position $q_i(t) \in \mathbb{R}^2$ and velocity $v_i(t) \in \mathbb{R}^2$; we stack these variables as $q, v \in \mathbb{R}^{2N}$ and write the state as $z = (q, v)$. Each agent possesses a sensing disk centered at itself, and the control objective is to remove the bounded regions left uncovered by the union of those disks. Section IV-A defines these regions, which we call holes, and their cumulative cost.

A. Sensing, the Čech complex, and the Hodge Laplacian

Each agent senses the open disk of radius $r_s > 0$ centered at its position. The *Čech 2-complex* records how these disks overlap. For a configuration q , $\mathcal{C}(q) = \mathcal{C}(q; r_s)$ has vertices $[N] = \{1, \dots, N\}$, an edge $\{i, j\}$ whenever $\|q_i - q_j\| < 2r_s$, and a triangle $\{i, j, k\}$ whenever the intersection of the three disks has strictly positive measure, equivalently whenever the minimal enclosing circle of the three points has radius strictly smaller than r_s [5] (see Fig. 1). At a fixed configuration, call the edges and triangles in $\mathcal{C}(q)$ *active*. Orient each edge by $i < j$ and each triangle consistently with its boundary edges. These choices define incidence (boundary) matrices $B_1 \in \mathbb{R}^{N \times E}$ and $B_2 \in \mathbb{R}^{E \times F}$, where E and F count the active edges and triangles. Because B_1 and B_2 represent consecutive boundary maps, they satisfy $B_1 B_2 = 0$. The one-dimensional Hodge Laplacian is $L_1 = B_1^\top B_1 + B_2 B_2^\top \in \mathbb{R}^{E \times E}$, and $\dim \ker L_1 = \beta_1(q)$, the first Betti number of $\mathcal{C}(q)$, equals the number of holes of the cover [16]. As the disks move,

their overlap pattern changes at edge and triangle thresholds. To make the controller vary smoothly across these changes, fix $0 < \varepsilon_E < 2r_s$ and define the scalar cutoff

$$\eta(x) = \begin{cases} \exp(-1/x), & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (1)$$

For $e = \{i, j\}$, the edge weight is

$$a_e = a_{ij} = \frac{\eta(s_{ij})}{\eta(s_{ij}) + \eta(1 - s_{ij})}, \quad s_{ij} = \frac{2r_s - \|q_i - q_j\|}{\varepsilon_E}. \quad (2)$$

The weight a_{ij} in (2) lies in $[0, 1]$ and equals one for $\|q_i - q_j\| \leq 2r_s - \varepsilon_E$ and zero for $\|q_i - q_j\| \geq 2r_s$. Between these transition endpoints, it decreases smoothly as the agents separate. It is locally Lipschitz, with all distance derivatives vanishing at both thresholds. Thus crossing the edge threshold creates no jump in the weighted operators.

To keep matrix dimensions fixed as simplices enter and leave the active complex, use an *ambient roster* containing every possible edge and triangle. Its sizes are $\bar{E} = \binom{N}{2}$ and $\bar{F} = \binom{N}{3}$, with oriented incidence matrices $\bar{B}_1 \in \mathbb{R}^{N \times \bar{E}}$ and $\bar{B}_2 \in \mathbb{R}^{E \times \bar{F}}$. For each ambient triangle f , let $R_f(q) := \min_{c \in \mathbb{R}^2} \max_{\ell \in f} \|q_\ell - c\|$ be the radius of the minimal enclosing circle of its three vertices. Fix $0 < \varepsilon_F < r_s$ and, using η from (1), define

$$b_f(q) = \frac{\eta(s_f)}{\eta(s_f) + \eta(1 - s_f)}, \quad s_f = \frac{r_s - R_f(q)}{\varepsilon_F}.$$

The weight b_f is globally Lipschitz in q . Both a_e in (2) and b_f can be evaluated to any prescribed accuracy with known error bounds. The active restrictions of $\bar{B}_1, \bar{B}_2$ are B_1, B_2 .

To preserve the corresponding identity $\bar{B}_1 \bar{B}_2 = 0$ under smooth weights, define $D_1 = \bar{B}_1 \text{diag}(a_e) \in \mathbb{R}^{N \times \bar{E}}$ and define $D_2 \in \mathbb{R}^{\bar{E} \times \bar{F}}$ entrywise by

$$D_2[e, f] = \bar{B}_2[e, f] b_f a_e \prod_{e' \subset f, e' \neq e} a_{e'}.$$

For a triangle f and a node i , entry (i, f) is

$$\begin{aligned} (D_1 D_2)[i, f] &= \sum_{e \subset f} D_1[i, e] D_2[e, f] \\ &= b_f \sum_{e \subset f} \bar{B}_1[i, e] \bar{B}_2[e, f] a_e^2 \prod_{e' \subset f, e' \neq e} a_{e'}^2 \\ &= b_f \left(\prod_{e' \subset f} a_{e'}^2 \right) (\bar{B}_1 \bar{B}_2)[i, f] = 0. \end{aligned}$$

The regularized Hodge Laplacian is

$$L_w = D_1^\top D_1 + D_2 D_2^\top \in \mathbb{R}^{\bar{E} \times \bar{E}}.$$

A harmonic edge form is a vector in the kernel of the active-edge restriction of L_w . When the weights are binary and match the current Čech complex, this restriction of L_w is L_1 , so its kernel has dimension $\beta_1(q)$. Figure 2 shows a numerical orthonormal pair from the weighted active-edge kernel of a connected two-hole complex.

B. Agents and dynamics

The agents have second-order dynamics with three acceleration terms: repulsion prevents hard collisions, damping reduces relative velocity along sensed edges [19]–[21], and the controller supplies harmonic-covariance feedback. For a damping gain $\alpha \geq 0$, define

$$L_0^{\text{reg}}(q) = \bar{B}_1 \text{diag}(a_e(q)^2) \bar{B}_1^\top, \quad D(q) = \alpha L_0^{\text{reg}}(q) \otimes I_2.$$

The matrix $D(q)$ is symmetric positive semidefinite. An absent edge has $a_e(q) = 0$ and contributes nothing to the damping. Fix a hard distance $a > 0$, a repulsion activation

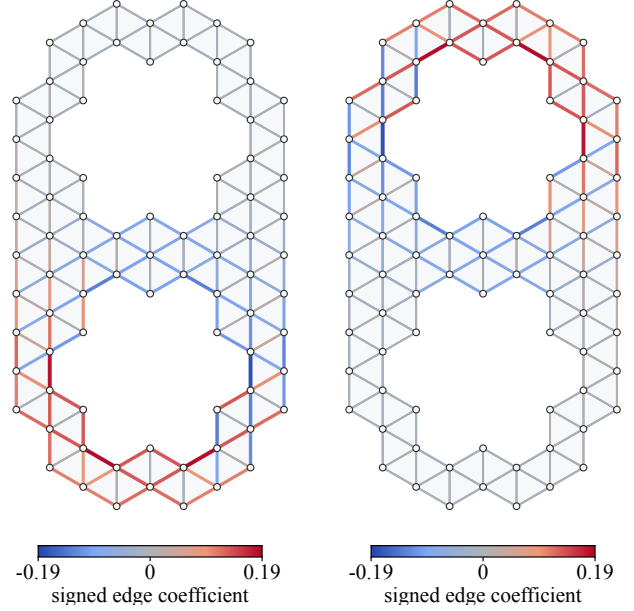


Fig. 2. Two orthonormal harmonic forms of the weighted active-edge restriction of L_w for the same connected two-hole complex. Triangles are shown lightly; colored edges give signed coefficients on the common scale, while gray denotes the zero level. Orientations are fixed by $i < j$.

distance $b > a$, a gain $g > 0$, and a smoothing parameter $\tau_U > 0$, and define the pairwise repulsive potential

$$U(q) = \sum_{i < j} \frac{g}{2} \frac{\rho_{ij}^4}{\tau_U^2 + \rho_{ij}^2}, \quad \rho_{ij} = \frac{(b - \|q_i - q_j\|)_+}{\|q_i - q_j\| - a}. \quad (3)$$

The ratio ρ_{ij} combines finite-range activation, vanishing at and beyond b , with a barrier as the distance decreases to a . The factor $\rho_{ij}^4 / (\tau_U^2 + \rho_{ij}^2)$ turns the force on softly at b ; the potential retains the hard-boundary divergence used in Section IV-C. The agents obey

$$\dot{q} = v, \quad \dot{v} = -\nabla U(q) - D(q)v + u(q),$$

where $u(q) \in \mathbb{R}^{2N}$ is the control input designed in Section III.

III. CONTROLLER DESIGN

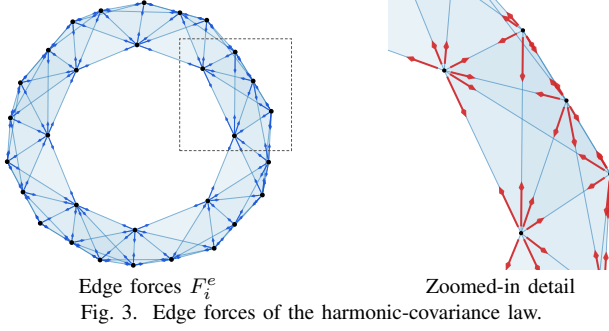
A. The harmonic-covariance force

Harmonic information is stored as scalar values on edges, while each agent needs a two-dimensional force at a node. We first convert edge values into node vectors and use these vectors to define force laws on each agent. The ideal law uses the exact harmonic projector of the binary Čech complex; Section III-B replaces it with a fixed-degree polynomial of L_w from Section II-A. Here “exact” and “ideal” refer only to the projector.

Let $u(q) = F^H(q)$ be the ideal harmonic-covariance force. At a fixed configuration with E active edges, let P be the orthogonal projector onto $\ker L_1$ defined in Section II-A and choose an orthonormal basis $\{\omega^{(k)}\}$ for that space. A harmonic edge vector ω assigns a scalar ω_e to every edge. Orient $e = (p, q)$ by $p < q$ and set $d_e = q - p$. At endpoint i , let $\hat{t}_e$ be the unit vector pointing along the edge away from i and read ω_e with that local orientation (i.e., for $e = (p, q)$, $\omega_{(p, q)} = -\omega_{(q, p)}$).

Fix $\lambda_R > 0$ and define the *lift* vector $y_i(\omega)$ of ω for agent i as the unique minimizer of the following least-squares problem

$$y_i(\omega) = \arg \min_{w \in \mathbb{R}^2} \lambda_R \|w\|^2 + \sum_{e \ni i} \left(a_e w^\top \hat{t}_e - \frac{\omega_e}{\|d_e\|} \right)^2, \quad (4)$$



Edge forces F_i^e
Fig. 3. Edge forces of the harmonic-covariance law.

where the factor $a_e \in [0, 1]$ is the edge weight in (2), and $\lambda_R \|w\|^2$ makes the problem strictly convex. Hence (4) has a unique solution $y_i = G_i^{-1} T_i$ with

$$G_i = \lambda_R I_2 + \sum_{e \ni i} a_e^2 \hat{t}_e \hat{t}_e^\top, \quad T_i = \sum_{e \ni i} a_e \hat{t}_e \frac{\omega_e}{\|d_e\|}. \quad (5)$$

In global edge coordinates, let $R_i \in \mathbb{R}^{2 \times E}$ with

$$(R_i)_{:,e} = \begin{cases} a_e G_i^{-1} d_e / \|d_e\|^2, & i \in e, \\ 0, & i \notin e. \end{cases}$$

It then follows that $y_i(\omega^{(k)}) = R_i \omega^{(k)}$. Because a single harmonic vector may concentrate around one hole and miss others, we aggregate the lifts of the full orthonormal basis in the node ‘‘covariance’’

$$C_i = \sum_k y_i(\omega^{(k)}) y_i(\omega^{(k)})^\top = R_i P R_i^\top \succeq 0, \quad (6)$$

where $\sum_k \omega^{(k)} \omega^{(k)\top} = P$, $P \in \mathbb{R}^{E \times E}$; thus $C_i \in \mathbb{R}^{2 \times 2}$. Division by $\|d_e\|$ puts the edge values on a common distance scale, while the outer-product sum removes the sign and basis choices. Reading the endpoint covariances along an edge gives a nonnegative scalar score, which sets the strength of an attractive pair force on that edge. For $e = \{i, j\}$, gain $\kappa > 0$, and common force limit $c_H > 0$, define

$$s_e = (1/2) \hat{t}_e^\top (C_i + C_j) \hat{t}_e \geq 0, \quad (7)$$

$$F_i^e = \kappa a_e s_e (q_j - q_i), \quad F_j^e = -F_i^e, \quad F_i^{\text{raw}} = \sum_{e \ni i} F_i^e,$$

$$\lambda = \frac{c_H}{\max\{c_H, \max_\ell \|F_\ell^{\text{raw}}\|\}}, \quad F_i^H = \lambda F_i^{\text{raw}}. \quad (8)$$

Equation (7) gives the edge score, while (8) assembles the incident pair forces and scales every node by the same factor λ . Since $C_i, C_j \succeq 0$, the score is nonnegative and independent of edge orientation. The weight a_e attenuates the force near loss of support (see Fig. 3). Each pair is equal and opposite and collinear with its edge, so the harmonic force has zero net force and zero torque. The common scaling by λ in (8) preserves these properties and enforces $\|F_i^H\| \leq c_H$ at every node.

Lemma 1. For every $\bar{q} \in \mathbb{R}^2$, $\sum_i (q_i - \bar{q})^\top F_i^H \leq 0$.

What Lemma 1 implies, in plain language, is that the net harmonic force tends to pull all agents together. Yet the lemma guarantees neither monotonic decrease of the hole areas nor their eventual closure.

B. Distributed realization of harmonic-covariance forces

Direct computation of P in (6) requires eigenvectors in the kernel of the full matrix L_1 , which requires centralized computation. For a fixed binary L_1 , $P = \lim_{\tau \rightarrow \infty} e^{-\tau L_1}$. The exponential leaves the zero eigenspace unchanged and attenuates every other subspace. This limit also applies to L_w and one may use $e^{-\tau L_w}$ as an approximate projector onto $\ker L_w$. Evaluating $e^{-\tau L_w}$ directly, however, still requires centralized computation. We therefore replace it by a polynomial of degree d , which requires d local applications of

L_w [15], [22], [23]. The factor λ in (8) is computed separately. We choose a Chebyshev expansion because it treats the full spectral interval uniformly. A Taylor expansion is centered at a single point and requires a separate remainder estimate to control its error over the full interval.

The Chebyshev interval requires a state-independent spectral upper bound, i.e., some $B \geq \lambda_{\max}(L_w)$. Since the entries of D_1 and D_2 have magnitude at most one, for $N \geq 2$ the row-sum bounds give $\|D_1^\top D_1\|_\infty \leq 2N - 2$ and $\|D_2 D_2^\top\|_\infty \leq 3(N - 2)$. Consequently, $\lambda_{\max}(L_w) \leq \|L_w\|_\infty \leq \|D_1^\top D_1\|_\infty + \|D_2 D_2^\top\|_\infty \leq 5N - 8 =: B$. Since L_w is positive semidefinite, its spectrum lies in $[0, B]$, and the affine map $2\lambda/B - 1$ sends this interval to $[-1, 1]$. Accordingly, define $\tilde{L}_w = 2L_w/B - I$. Put $\sigma = \tau B/2$. Let T_j be the Chebyshev polynomial defined by $T_j(\cos \theta) = \cos(j\theta)$, and let I_j be the modified Bessel function of the first kind. Define

$$c_j = \begin{cases} e^{-\sigma} I_0(\sigma), & j = 0, \\ 2e^{-\sigma} (-1)^j I_j(\sigma), & j \geq 1, \end{cases}$$

In the Jacobi–Anger expansion of $e^{-\tau \lambda}$ [24], its degree- d truncation is

$$p_d(\lambda) = \sum_{j=0}^d c_j T_j \left(\frac{2\lambda}{B} - 1 \right).$$

The symmetric matrix $H_d = p_d(L_w)$ approximates the exponential. Its parameters τ, B, d, c_j are fixed and can be evaluated offline, so no online spectral estimate is needed.

We aim to span $\ker L_w$ and generate edge signals by approximately projecting random initial vectors onto $\ker L_w$. We initialize $Z \in \mathbb{R}^{\bar{E} \times K}$ once at random with $K > 1$ columns on the fixed ambient roster of Section II-A, and set $Y = e^{-\tau L_w} Z \approx H_d Z =: Y_d$. Computationally, Y_d may be evaluated by the following recursive procedure:

$$\begin{aligned} X_0 &= Z, \quad X_1 = \tilde{L}_w Z \quad (d \geq 1), \\ X_{r+1} &= 2\tilde{L}_w X_r - X_{r-1}, \quad 1 \leq r < d, \\ Y_d &= \sum_{r=0}^d c_r X_r = p_d(L_w) Z = H_d Z. \end{aligned}$$

All these computations may be performed locally.

To convert the columns of Y_d back to node quantities, extend the active-edge map R_i to $\mathbb{R}^{2 \times \bar{E}}$ by assigning zero columns to absent edges. The lift of Y_d ’s k -th column is $y_{i,k} = R_i(Y_d)_{:,k}$, and their outer products give the probe covariance

$$\hat{C}_i = \sum_k y_{i,k} y_{i,k}^\top = R_i H_d Z Z^\top H_d^\top R_i^\top \succeq 0. \quad (9)$$

Here $y_{i,k}$ is analogous to $y_i(\omega^{(k)})$ in Section III-A. Since $\hat{C}_i \succeq 0$, substituting it into (7) and (8) preserves nonnegative edge scores. The rules in (8) are otherwise unchanged, except $(C_i + C_j)$ is replaced by $(\hat{C}_i + \hat{C}_j)$. The factor λ uses the maximum raw force norm over all nodes, so the distributed realization assumes a connected communication graph over which this maximum can propagate. Damping is likewise local because $L_0^{\text{reg}}(q)$ couples each agent only to its current one-hop neighbors.

IV. WELL-POSEDNESS AND BOUNDS ON CUMULATIVE HOLE AREAS

After the regularized controller is defined, it is natural to raise the following questions:

- 1) Given a safe initial state, does the controller yield *one unique* trajectory?
- 2) Given an initial state, without simulation, can we determine whether the holes will eventually close under the controlled dynamics?

Section IV-C gives the first question a positive answer, and Section IV-D answers the second question with an offline construction of a rational sequence $\{c_T\}_T$ converging to the all-time cumulative hole area, with each term obtained by finite calculation.

A. Holes and the hole-area cost

All sensing disks are open, and their union at configuration q is $A(q) = \bigcup_{i \in [N]} B(q_i, r_s)$. The *holes* are the bounded connected components of $\mathbb{R}^2 \setminus A(q)$. Let $H(q)$ be their union and let $|\cdot|$ denote two-dimensional Lebesgue measure. A terminal hole count does not record how much uncovered area persisted before the terminal time. We therefore use the normalized instantaneous area and its all-time cumulative cost:

$$h(q) = \frac{|H(q)|}{(8Nr_s)^2}; \quad C = \int_0^\infty h(q(t)) dt \in [0, \infty). \quad (10)$$

If our proposed controller is successful, our objective will be to show that C is finite. Every hole lies in the convex hull of one sensing-graph component. A component with k_j vertices has center diameter at most $2r_s(k_j - 1)$ and fits in a box of area at most $4r_s^2(k_j - 1)^2$; the normalization by $(8Nr_s)^2$ therefore gives $0 \leq h(q) \leq H_N$, where $H_N := (N - 1)^2 / (16N^2) < 1/16$. Section II-A records the correspondence between the holes and $\beta_1(q)$.

B. Assumptions

The admissible motion class satisfies the following standing assumptions:

- A1 $\|q_i(t) - q_j(t)\| > a$ for all $i < j$ and all $t \geq 0$.
- A2 The normalized hole area $h(q(t))$ is well defined, measurable, and bounded for all $t \geq 0$.
- A3 Hole components grow and shrink continuously and do not split or merge.
- A4 At most finitely many holes appear after time zero, and every such hole is *spurious* (present only over a bounded time interval).

Without splitting or merging, each hole keeps a well-defined identity over its lifetime, and since every hole appearing after time zero is spurious, their cumulative total area is finite, and therefore will not affect the finiteness of C . The measurability and boundedness assumptions make the all-time integral (10) well defined.

C. Well-posedness

For our controlled dynamics, the natural state space is the safe open set $\Omega = \{(q, v) : \min_{i < j} \|q_i - q_j\| > a\}$.

Proposition 2. *For every safe initial state $z_0 \in \Omega$, the controlled dynamics of Section II-B with the regularized law of Section III-B has a unique solution $z : [0, +\infty) \rightarrow \Omega$. That means no hard collision occurs in finite time.*

Proof. The proof has two parts. We first show that the vector field is locally Lipschitz. Indeed, the edge weights and entries of L_w have this property on Ω , and the fixed-degree polynomial filter preserves it. In (8), the maximum raw-force norm has the same regularity, while the denominator is bounded below by $c_H > 0$. On the safe open set, the denominators in the repulsive potential are bounded away from zero locally, so U is locally $C^{1,1}$; the damping term $D(q)v$ is a finite sum using the same weights. Hence the full vector field $z \mapsto (v, F^H(q) - \nabla U(q) - D(q)v)$ is locally Lipschitz. Picard-Lindelöf therefore gives a unique solution for a sufficiently short time after $t = 0$.

We next show that this solution can be continued to any time. Fix any finite horizon $T > 0$. Define the energy by $E(q, v) = \|v\|^2/2 + U(q)$. Its damping dissipation satisfies

$v^\top D(q)v = \alpha \sum_{\{i,j\}} a_{ij}(q)^2 \|v_i - v_j\|^2 \geq 0$. Moreover, (8) gives, via $\|F_i^H\| \leq c_H$, that

$$v^\top F^H = \sum_i v_i^\top F_i^H \leq c_H \sum_i \|v_i\| \leq \frac{1}{2} \|v\|^2 + \frac{Nc_H^2}{2},$$

where the last inequality comes from $(c_H - \|v_i\|)^2 \geq 0$. Thus, with $C_H = Nc_H^2/2$,

$$\frac{dE}{dt} = v^\top F^H - v^\top D(q)v \leq \frac{1}{2} \|v\|^2 + C_H \leq E + C_H.$$

Consequently, for $t \leq T$, $E(t) \leq e^t (E(0) + C_H) - C_H \leq e^T (E(0) + C_H) - C_H$. Hence E , and therefore $\|v\|$, is bounded up to time T , and q will not escape to infinity before T . The finiteness of E also enforces that $U(q)$ does not blow up, i.e., one always has $\|q_i - q_j\| > a$. Hence the state stays in a compact subset of Ω before any finite endpoint. The vector field is bounded on that compact set, so the state has a limit in Ω at the endpoint. Starting from this limiting state, the same local existence result continues the solution. This gives a unique solution on every finite interval $[0, T]$. Since T is arbitrary, letting $T \rightarrow \infty$ gives existence for all $t \geq 0$. $\square$

D. Construction of cumulative hole area lower bounds

We now fix the feedback law and examine whether its cumulative hole area can be determined from the model and initial state. Fix the safe initial state $z_0 = (q_0, v_0)$ and write the controlled dynamics as

$$\begin{aligned} \dot{z} &= F(z), \quad F(q, v) = (v, f(q, v)), \\ f(q, v) &= -\nabla U(q) - D(q)v + F^H(q). \end{aligned} \quad (11)$$

For a positive integer horizon T , the physical quantity of interest is the actual hole-area cost from Section IV-A,

$$J_T = \int_0^T h(q(t)) dt.$$

1) *Outline of bound construction:* Directly using h is difficult because it need not vary continuously with the agent positions. A small displacement can connect a hole to the exterior, so that region is no longer counted by h , even though it remains uncovered. We therefore approach h from below by continuous functions. For each positive integer T , it is possible to construct $q_T : \Omega \rightarrow [0, H_N]$ independent of v such that

$$0 \leq q_T(z) \leq q_{T+1}(z) \leq h(q), \quad q_T(z) \uparrow h(q). \quad (12)$$

Here $\uparrow$ denotes non-decreasing convergence. Appendix A constructs this graph-based family q_T . Define

$$A_T = \int_0^T q_T(z(t)) dt \leq J_T \leq C \quad (13)$$

such that $A_T \uparrow C$. For fixed T , it is possible to compute rational bounds

$$L_{T,m} \leq A_T \leq U_{T,m}. \quad (14)$$

The index m will tighten a fixed- T finite calculation, whereas increasing T both extends the time horizon and improves the lower approximation q_T of h . Consequently, the stopping tolerance controls the upper and lower bounds of A_T . The details proceed in three steps. Section IV-D2 constructs integral constraints that recover the cumulative behavior of the unique trajectory. Section IV-D3 replaces the controlled field in those constraints by finite rational data. Section IV-D4 then computes $L_{T,m}$ and $U_{T,m}$ in (14).

2) *Recovering the trajectory through integral constraints:* The construction begins by enclosing every possible state that the true trajectory can visit before time T . The energy, displacement, and repulsion estimates of Section IV-C give rational bounds Q_T, V_T and $r_T > a$, depending only on F, z_0, T , such that the trajectory lies in a compact subset

$$K_T = \left\{ (q, v) \in \Omega : \begin{array}{l} \|q\| \leq Q_T, \quad \|v\| \leq V_T, \\ \min_{i < j} \|q_i - q_j\| \geq r_T \end{array} \right\} \subset \Omega. \quad (15)$$

Write $X_T = [0, T] \times K_T$.

We now add constraints to rule out candidate trajectories that are inconsistent with the controlled dynamics. We use scalar test functions to impose the controlled dynamics as integral constraints and narrow the feasible set. By Lemma A and the uniqueness established in Proposition 2, the full family of constraints recovers the cumulative integrals of the true trajectory.

Choose a scalar rational function $\psi(t, q, v)$ with rational coefficients, C^1 on a neighborhood of X_T , and satisfying $\psi(T, \cdot) = 0$, independently of the unknown trajectory. By the chain rule, $d\psi(t, z(t))/dt = \partial_t \psi + v^\top \nabla_q \psi + f^\top \nabla_v \psi$, with the right-hand side evaluated at $(t, z(t))$. The fundamental theorem of calculus then yields the integral constraints where the right hand side only involves the initial state:

$$\int_0^T (\partial_t \psi + F^\top \nabla_z \psi)(t, z(t)) dt = \underbrace{\psi(T, z(T))}_{=0} - \psi(0, z_0) = -\psi(0, z_0). \quad (16)$$

To obtain a sequence of finite constraint sets, we enumerate all polynomials with rational coefficients as $p_1, p_2, \dots$ and define $\psi_\ell(t, z) = p_\ell(t, z) - p_\ell(T, z)$, so $\psi_\ell(T, z(T)) = 0$. For each index ℓ , define

$$a_\ell = \partial_t \psi_\ell + F^\top \nabla_z \psi_\ell \implies \int_0^T a_\ell(t, z(t)) dt = -\psi_\ell(0, z_0) =: -\beta_\ell. \quad (17)$$

3) *Approximating the controlled vector field:* The integral identities in (17) constrain cumulative behavior, but they are not yet finite rational data: a_ℓ contains the generally non-polynomial field F , and β_ℓ may not be rational. We make both quantities finite rational data by approximating the field and the initial-value term. The integer k below records the requested field accuracy.

Proposition 3. *Suppose uniform rational approximations of the model data, with explicit error bounds, can be constructed on compact sets to any prescribed accuracy. For every integer $k \geq 1$, a finite construction from the model data alone produces a continuous piecewise-affine field F_k with rational coefficients on a neighborhood of K_T , satisfying $\|F_k - F\|_{\infty, K_T} \leq 2^{-k}$.*

Appendix A-A proves Proposition 3 by explicit construction. For this F_k , choose a rational $\Gamma_\ell \geq \|\nabla_z \psi_\ell\|_{\infty, X_T}$ and a rational $\beta_{\ell k}$ satisfying $|\beta_{\ell k} - \beta_\ell| \leq 2^{-k}$. Set

$$a_{\ell k} = \partial_t \psi_\ell + F_k^\top \nabla_z \psi_\ell, \quad \|a_{\ell k} - a_\ell\|_{\infty, X_T} \leq \Gamma_\ell 2^{-k}. \quad (18)$$

At stage m , compute $a_{\ell k}$ and $\beta_{\ell k}$ for every pair $1 \leq \ell, k \leq m$. We use these quantities to construct constraints for the finite optimization problem below to compute the lower and upper bounds of A_T .

4) *Finite optimization of the area bounds:* To formulate this optimization problem, we approximate q_T by a continuous rational piecewise polynomial $\hat{q}_{T,m}$ with

$$\|\hat{q}_{T,m} - q_T\|_{\infty, K_T} \leq 2^{-m}. \quad (19)$$

Appendix A-C gives this construction. Stage m now has m^2 constraints and one approximate-area objective. For the true trajectory, the exact identity $\int_0^T a_\ell(t, z(t)) dt = -\beta_\ell$ gives, for each $1 \leq \ell, k \leq m$,

$$\begin{aligned} \int_0^T a_{\ell k}(t, z) dt + \beta_{\ell k} &= \int_0^T (a_{\ell k} - a_\ell)(t, z) dt + \beta_{\ell k} - \beta_\ell \\ \implies \left| \int_0^T a_{\ell k}(t, z) dt + \beta_{\ell k} \right| &\leq (T\Gamma_\ell + 1) 2^{-k}. \end{aligned} \quad (20)$$

Lemma B in Appendix B shows that there exists a collection of points $x_r = (t_r, z_r) \in X_T$ and weights $w_r \geq 0$, with total weight T , that can simultaneously represent the m^2 constraints and the one objective value, using at most $M = m^2 + 2$ points. For every $1 \leq \ell, k \leq m$, this gives

$$\begin{aligned} \int_0^T a_{\ell k}(t, z(t)) dt &= \sum_{r=1}^M w_r a_{\ell k}(x_r), \\ \int_0^T \hat{q}_{T,m}(z(t)) dt &= \sum_{r=1}^M w_r \hat{q}_{T,m}(z_r), \end{aligned} \quad (21)$$

where r indexes the common representing points. The desired optimization problems then follow from the constraints and approximated cost $\hat{q}_{T,m}$, for each $1 \leq \ell, k \leq m$:

$$\begin{aligned} S_{T,m}(x, w) &= \sum_{r=1}^M w_r \hat{q}_{T,m}(z_r), \\ j_{T,m}^- &= \min_{x, w} S_{T,m}(x, w), \quad j_{T,m}^+ = \max_{x, w} S_{T,m}(x, w), \\ \text{s.t. } x_r &= (t_r, z_r) \in X_T, \quad w_r \geq 0, \quad \sum_{r=1}^M w_r = T, \end{aligned} \quad (22)$$

$$\left| \sum_{r=1}^M w_r a_{\ell k}(x_r) + \beta_{\ell k} \right| \leq (T\Gamma_\ell + 1) 2^{-k}.$$

The last line of (22) directly comes from (20) and (21). Minimizing and maximizing $S_{T,m}$ subject to the constraints in (22) therefore gives the bounds

$$j_{T,m}^- - T2^{-m} \leq A_T \leq j_{T,m}^+ + T2^{-m}. \quad (23)$$

The term $T2^{-m}$ accounts for replacing q_T by $\hat{q}_{T,m}$. Note that (22) should not be solved via ordinary numerical optimization, which may return a feasible or locally optimal candidate without establishing the global bounds needed in (23). A feasible objective value bounds the minimum from above and the maximum from below; (23) requires the opposite. Instead, for fixed T, m , we perform a binary search on a threshold value, θ . Real quantifier elimination [25, Ch. 14] decides feasibility with $S_{T,m} < \theta$ or $S_{T,m} > \theta$, and these answers update rational enclosures of the minimum and maximum. Repeating at rational midpoints encloses both extrema to any demanded accuracy. Round $j_{T,m}^- - T2^{-m}$ downward and $j_{T,m}^+ + T2^{-m}$ upward to rational numbers $L_{T,m}$ and $U_{T,m}$, respectively, with each rounding error at most 2^{-m} . This gives (14). Applying Lemma B with $g(t, z) = q_T(z)$ and $g_m(t, z) = \hat{q}_{T,m}(z)$ shows that these bounds satisfy $U_{T,m} - L_{T,m} \rightarrow 0$ for each fixed T . With $c_0 = 0$, define

$$\begin{aligned} m_T &= \min \{m \geq 1 : U_{T,m} - L_{T,m} < 2^{-T}\}, \\ (\ell_T, u_T) &= (L_{T, m_T}, U_{T, m_T}), \quad c_T = \max \{c_{T-1}, 0, \ell_T\}. \end{aligned} \quad (24)$$

For fixed T, m , the binary search above refines the numerical bounds on the two extrema. If $L_{T,m}, U_{T,m}$ do not satisfy (24), increase m and solve (22) again. The gap convergence above proves that this loop terminates and yields $\ell_T \leq A_T \leq u_T$ with $u_T - \ell_T < 2^{-T}$.

By construction, $\mathbf{1}_{[0,T]}(t)q_T(z(t))$ increases pointwise to $h(q(t))$. Monotone convergence theorem then gives $A_T \uparrow C$, also when $C = +\infty$. If $S \leq T$, then $\ell_S \leq A_S \leq A_T$. Hence the running maximum in (24) is non-decreasing and satisfies $A_T - 2^{-T} < \ell_T \leq c_T \leq A_T \leq C$. Therefore $c_T \uparrow C$. By this point, one completes the proof to the following result:

Theorem 4. *Assume that we can calculate rational approximations of the model data and the safe initial state z_0 to any prescribed accuracy, with explicit error bounds. For*

TABLE I
PARAMETERS FOR THE NUMERICAL DEMO

Setting	Value
Agents; initial number of holes	$N = 60; 2$
Sensing-disk radius	$r_s = 0.234$
Edge/triangle regularization width	2.5×10^{-4}
Local ridge λ_R ; gain κ ; force cap c_H	$0.05; 8; 0.8$
Number of columns in Z ; random seed	$K = 48; 419$
$\tau; d$	$60; 343$
Polynomial error tolerance	5×10^{-3}
Repulsion distances	$a = 0.18; b = 0.24$
Repulsion gain; smoothing	$0.5; \tau_U = 0.25$
Damping gain	$\alpha = 1.2$
Integrator	DOP853
Absolute; relative tolerance	$10^{-9}; 10^{-6}$
Maximum step	0.05
Horizon; saved-frame grid	$15s; 0.1s$

every integer $T \geq 1$, the construction (24) produces a rational number c_T in finitely many steps from these data alone, and

$$c_T \uparrow C = \int_0^\infty h(q(t)) dt, \quad (25)$$

with $C = +\infty$ allowed.

By Theorem 4 and (25), $C < \infty$ exactly when $\{c_T\}$ is bounded. If $C < \infty$, then for every fixed $\Delta > 0$,

$$0 \leq \frac{1}{\Delta} \int_t^{t+\Delta} h(q(s)) ds \leq \frac{1}{\Delta} \int_t^\infty h(q(s)) ds \rightarrow 0.$$

Pointwise convergence of $h(q(t))$ to zero, however, is not guaranteed unless $h(q(t))$ is uniformly continuous, by Barbalat's lemma.

V. NUMERICAL DEMO

This section shows how the controller behaves on a 60-agent configuration with two initial holes. The snapshots show the agent motion and the numerical Čech complex, while the area plot follows the two holes that are present at the initial time.

The initial positions form the two-hole peanut shown on the left of Fig. 4. Table I lists the parameters and integrator settings, where the Chebyshev polynomial degree $d = 343$ is determined by the stated polynomial-tail tolerance. The run integrates the controller in Section III-B. The implementation evaluates the same finite-degree local operators with centralized single-process sparse operations. It emulates the algebra of the distributed recurrence but does not measure message delays or execute a networked message-passing protocol.

A. Results

For the 15-second demo, the saved frames had $\beta_1 = 2$ initially, decreasing to 1 by $t = 8$ s. The first observed zero occurred at $t = 9.9$ s, and every saved frame from that time through the terminal $t = 15$ s frame had $\beta_1 = 0$. Fig. 4 should be read from left to right: the agents move inward around the two uncovered regions, and the displayed complex has no holes by the final saved frame. Fig. 5 reports the raw areas of the two tracked initial holes. The two tracked curves first read zero respectively at 7.8 and 9.8s and remain zero afterwards.

VI. CONCLUSION AND FUTURE WORK

We developed a harmonic-covariance controller that turns harmonic edge information into bounded forces on the agents and admits distributed evaluation. Under the stated assumptions, the regularized closed loop has a unique global trajectory from every safe initial state. For this controlled dynamics, the initial state and model determine an offline sequence of rational numbers c_T . Each stage is a finite calculation and does not query trajectory data. The sequence is non-decreasing and converges to the cumulative normalized hole area. When the cost is finite, the average hole area on every fixed-length window tends to zero as the window

moves forward. Yet this average statement implies neither pointwise convergence to zero nor finite-time hole closure. The numerical demo in Section V illustrates the controller architecture, with both initial holes closing in finite time.

Despite the results in this paper, a key difficulty persists: construction of q_T and quantifier elimination in Section IV-D4 are still heavy in computation, prohibiting an efficient algorithm that tests whether the holes of a given initial state will close. Apart from addressing this problem, the next steps include weakening assumptions (A2), (A3), and (A4) and testing an actual distributed implementation rather than the current synchronous emulation.

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In writing this paper, large language models (LLMs), specifically, the Codex models [26], were used for polishing human-written text, and for enhancing logical continuity and consistency. LLMs were used for verifying the correctness of the mathematical proofs and for increasing clarity of the writing. LLMs were also used in implementation of the numerical demos, including generation of parts of the code for simulation and visualization. All edits suggested by AI were manually reviewed by the authors before implementation. All ideas and technical details originated purely from the authors.

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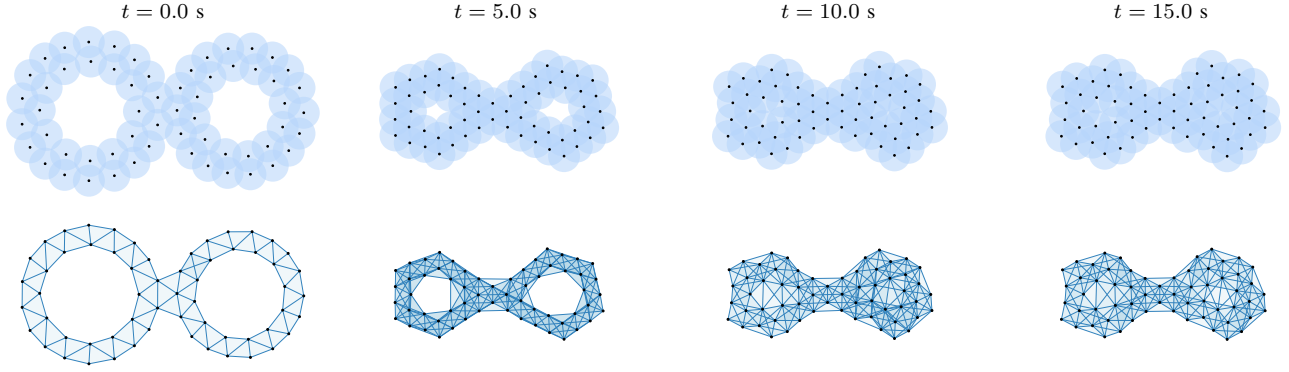


Fig. 4. The two-hole peanut with state-dependent damping. Each column shows the configuration and numerical Čech complex from the same accepted trajectory; the terminal saved frame has $\beta_1 = 0$.

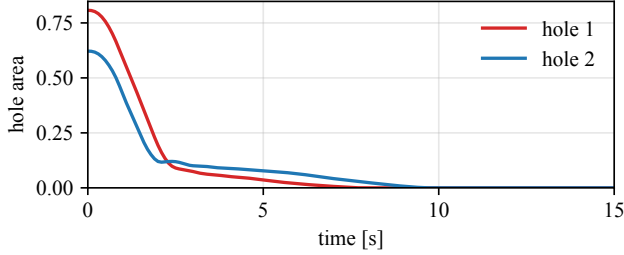


Fig. 5. Areas (coordinate units squared) of the two tracked initial holes over the 15 s run, computed using open disks. Both curves read zero by the end of the run.

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APPENDIX A CONSTRUCTIONS

A. Construction of F_k

Fix T and k , and let $d_z = 4N$. Choose a rational $0 < \gamma_T < r_T - a$. On a rational grid, retain every cube that meets K_T and refine until its diameter is below $\gamma_T/(2\sqrt{2})$. These cubes lie in the safe set, and a consistent subdivision gives a compact rational polyhedral neighborhood $K_T \subset \tilde{P}_T \subset \Omega$ with a finite conforming rational triangulation $\mathcal{T}_k$. At each distinct vertex ζ of $\mathcal{T}_k$, evaluate the known formula $F(\zeta)$ directly and store one shared rational value $c_{\zeta,k} \in \mathbb{Q}^{d_z}$ satisfying $\|c_{\zeta,k} - F(\zeta)\|_2 \leq \eta_k$. For a simplex $\sigma = \text{conv}\{\zeta_0, \dots, \zeta_{d_z}\}$, let $\lambda_{\sigma,j}$ be its rational barycentric coordinates and define

$$F_k(z) = \sum_{j=0}^{d_z} \lambda_{\sigma,j}(z) c_{\zeta_j,k}, \quad z \in \sigma. \quad (26)$$

The same model data and positive safety margin give a rational Lipschitz upper bound $L_T \geq 1$ for F on the retained simplices. With h_k denoting the maximum simplex diameter, refine until

$$h_k \leq 2^{-k-1}/L_T, \quad \eta_k = 2^{-k-1}. \quad (27)$$

Finite refinement achieves these bounds. Shared nodal values make F_k continuous across simplex faces. For each simplex σ and $z \in \sigma \cap K_T$, barycentric nonnegativity and (27) give $\|F_k(z) - F(z)\|_2 \leq \eta_k + L_T h_k \leq 2^{-k}$. Taking the supremum over K_T proves Proposition 3.

B. Construction of q_T

We construct q_T satisfying (12). Start with a fixed graph $\mathcal{E}$ on $[N]$ and draw each edge as the straight segment between its endpoints. Let $\mathcal{W}_{\mathcal{E}}(q)$ be the set of points about which some polygonal cycle in $\mathcal{E}$ has nonzero winding number. The graph segments themselves have zero area. The normalized area enclosed by the graph but left uncovered by the sensing disks is

$$h_{\mathcal{E}}(q) = |\mathcal{W}_{\mathcal{E}}(q) \setminus A(q)| / (8Nr_s)^2. \quad (28)$$

Take two configurations $q, \tilde{q}$ of the same labeled nodes, ordered so that $\|\tilde{q}\|_{\infty} \leq \|q\|_{\infty}$. Set $Q = \|q\|_{\infty}$ and $\delta = \max_i \|q_i - \tilde{q}_i\|$, so both configurations lie in $[-Q, Q]^{2N}$. Take the translation direction of one disk as horizontal. At each height, its left and right boundary points each move by at most δ . These give an area bound $(2r_s)(2\delta) = 4r_s\delta$. Moving all N disks therefore changes coverage over at most $4r_s N \delta$ of area.

Next set $q(s) = (1-s)q + s\tilde{q}$ for $0 \leq s \leq 1$. The graph $\mathcal{E}$ is fixed throughout the position box, including configurations where its edges exceed the sensing range $2r_s$. Each edge is bounded by the diagonal of $[-Q, Q]^2$, namely $2\sqrt{2}Q \leq 3Q$, and every point on it has speed at most δ . Thus each edge sweeps area at most $\int_0^1 3Q\delta ds = 3Q\delta$. Cycle winding numbers are unchanged outside the swept regions, so the graph-enclosed region changes over at most $3Q|\mathcal{E}|\delta$ of area. Adding the two area bounds and dividing by $(8Nr_s)^2$ gives

$$|h_{\mathcal{E}}(q) - h_{\mathcal{E}}(\tilde{q})| \leq L_{\mathcal{E},Q}\delta, \quad L_{\mathcal{E},Q} = \frac{4r_s N + 3Q|\mathcal{E}|}{(8Nr_s)^2}. \quad (29)$$

It remains to approach the sensing graph without making a discontinuous edge decision. For every complete-graph edge $e = \{i, j\}$, define $b_e(q) = 4r_s^2 - \|q_i - q_j\|^2$ and set

$$p_{e,T}(q) = \min(1, \max(0, Tb_e(q))). \quad (30)$$

The number $p_{e,T}$ acts as the inclusion weight of edge e ; its exclusion weight is $1 - p_{e,T}$. Averaging over the finitely many graphs gives

$$q_T(q) = \sum_{\mathcal{E}} \left[\prod_{e \in \mathcal{E}} p_{e,T}(q) \right] \left[\prod_{e \notin \mathcal{E}} (1 - p_{e,T}(q)) \right] h_{\mathcal{E}}(q). \quad (31)$$

For the sensing graph $\mathcal{E}(q)$, a point outside $A(q)$ is encircled by a cycle exactly when it lies in a hole, so $h_{\mathcal{E}(q)}(q) = h(q)$. Since $h_{\mathcal{E}}$ is nondecreasing under edge addition, the convex combination in (31), supported on sensing subgraphs, gives $0 \leq q_T \leq h$. Continuity follows from (29) and (30).

For fixed q , since $p_{e,T}(q)$ is non-decreasing with T , q_T is also non-decreasing. Every nonsensing edge has zero weight. Once $Tb_e(q) \geq 1$ for every sensing edge, (31) reduces to $h_{\mathcal{E}(q)}(q) = h(q)$. Hence $q_T(q) \uparrow h(q)$, proving (12).

C. Construction of $\hat{q}_{T,m}$

Fix T and m . Choose a rational $Q_+ > 0$ so that $P = [-Q_+, Q_+]^{2N}$ contains all q -components of K_T , and take a rational Lipschitz upper bound $\Lambda_T \geq 1$ for q_T on P ; its construction is given below. Take a rational triangulation such that any two configurations $q, \tilde{q}$ in the same simplex satisfy $\max_i \|q_i - \tilde{q}_i\|_2 \leq \rho_m := 2^{-m-1}/\Lambda_T$.

At each shared vertex q^ν , compute $y_\nu \in \mathbb{Q}$ satisfying $|y_\nu - q_T(q^\nu)| \leq 2^{-m-1}$. For each graph, divide a box containing the region in (28) into small squares. Discard squares verified to lie entirely outside the region. The total area of squares verified to lie entirely inside is a lower bound; adding the area of every square that intersects the boundary gives an upper bound. Normalize these bounds and evaluate (31) using rational input bounds. Refine the grid until the resulting gap between upper and lower bounds falls below 2^{-m} , and take its midpoint as y_ν . For $q \in P$, let $\lambda_\nu(q) \geq 0$ be its barycentric coordinate with respect to the vertices q^ν of the containing simplex and define the piecewise affine approximation

$$\hat{q}_{T,m}(q) = \sum_\nu \lambda_\nu(q) y_\nu.$$

Then it follows that

$$|\hat{q}_{T,m}(q) - q_T(q)| \leq \sum_\nu \lambda_\nu(q) (2^{-m-1} + \Lambda_T \rho_m) = 2^{-m}.$$

Thus this construction satisfies (19). To choose Λ_T , note that every graph-enclosed region lies in $[-Q_+, Q_+]^2$, so $0 \leq h_{\mathcal{E}} \leq H$ for a rational $H \geq 4Q_+^2/(8Nr_s)^2$. For $q, \tilde{q} \in P$, set $\delta = \max_i \|q_i - \tilde{q}_i\|$. The edge lengths are at most $3Q_+$, and

$$\begin{aligned} |p_{e,T}(q) - p_{e,T}(\tilde{q})| &\leq T \|q_i - q_j\|^2 - \|\tilde{q}_i - \tilde{q}_j\|^2 \\ &\leq T(3Q_+ + 3Q_+)(\delta + \delta) = 12TQ_+\delta. \end{aligned}$$

(31) then gives

$$\begin{aligned} |q_T(q) - q_T(\tilde{q})| &\leq \max_{\mathcal{E}} |h_{\mathcal{E}}(q) - h_{\mathcal{E}}(\tilde{q})| + H \sum_{\mathcal{E}} |p_{e,T}(q) - p_{e,T}(\tilde{q})| \\ &\leq \left(\max_{\mathcal{E}} L_{\mathcal{E}, Q_+} + 12TQ_+H \binom{N}{2} \right) \delta. \end{aligned}$$

Any rational $\Lambda_T \geq 1$ not smaller than this coefficient suffices.

APPENDIX B PROOFS

Proof of Lemma 1. Summing the pair forces gives

$$\begin{aligned} \sum_i (q_i - \bar{q})^\top F_i^H &= \lambda \sum_e ((q_i - \bar{q})^\top F_i^e + (q_j - \bar{q})^\top F_j^e) \\ &= -\lambda \kappa \sum_e a_e s_e \|q_j - q_i\|^2 \leq 0. \end{aligned}$$

The last inequality uses $\lambda, \kappa, a_e, s_e \geq 0$. $\square$

Lemma A. Fix $T \geq 1$ and use $\bar{K}_T$ and X_T defined in Section IV-D. Among the finite nonnegative measures on X_T , the equations

$$\int_{X_T} (\partial_t \psi + F^\top \nabla_z \psi) d\mu = -\psi(0, z_0)$$

for every C^1 function ψ with $\psi(T, \cdot) = 0$ have the single solution $\mu_T = dt \otimes \delta_{z(t)}$ with $\mu_T(X_T) = T$.

Proof of Lemma A. By definition, for $g \in C(X_T)$, one has that $\int_{X_T} g d\mu_T = \int_0^T g(t, z) dt$. Taking $g = \partial_t \psi + F^\top \nabla_z \psi$ and applying (16) gives the stated equations. Conversely, for any solution μ , taking $\psi(t, z) = \int_t^T \zeta(s) ds$ gives $\int_{X_T} \zeta(t) d\mu = \int_0^T \zeta(t) dt$, with $\zeta \in C([0, T])$. Hence $\pi_t \mu = dt$ and $\mu(X_T) = T$. Meanwhile, local Lipschitzness of F gives a constant L such that $\|F(x) - F(y)\| \leq L\|x - y\|$ for $x, y \in K_T$. Let $z(t)$ be the trajectory under F and take $\Psi(t, x) = (T - t)e^{-2Lt} \|x - z(t)\|^2$. Then $\Psi \in C^1$ and $\Psi(T, \cdot) = 0$. Since $\dot{z} = F(z)$, Cauchy-Schwarz and the Lipschitz bound give

$$\partial_t \Psi + F(x)^\top \nabla_x \Psi \leq -e^{-2Lt} \|x - z(t)\|^2.$$

The integral identity for Ψ has zero right-hand side. Its non-positive integrand must therefore vanish μ -almost everywhere. Hence $\|x - z(t)\| = 0$ almost everywhere. This finally gives $\mu = dt \otimes \delta_{z(t)}$. $\square$

Lemma B. Fix a positive integer T and a bounded continuous function $g : X_T \rightarrow \mathbb{R}$, with uniform rational approximations on X_T with any prescribed error bound and corresponding trajectory integral in Lemma A. For every rational $\varepsilon > 0$, without querying the trajectory, a terminating finite calculation involving the model, the initial state z_0 , the compact set K_T , and these approximation data returns rational numbers $L \leq U$ with

$$U - L < \varepsilon, \quad L \leq \int_{X_T} g d\mu_T \leq U.$$

Proof of Lemma B. Use the terminal-zero functions ψ_ℓ defined in Section IV-D. They are dense in the required C^1 class. For each k , Proposition 3 supplies the field F_k used in $a_{\ell k}$, while Γ_ℓ and $\beta_{\ell k}$ are chosen as in Section IV-D. Equation (20) supplies the tolerance in the finite constraints. Let $\mathcal{P}_{T,m}$ be the set of nonnegative measures μ on X_T satisfying $\mu(X_T) = T$ and

$$\left| \int_{X_T} a_{\ell k} d\mu + \beta_{\ell k} \right| \leq (T\Gamma_\ell + 1) 2^{-k}, \quad 1 \leq \ell, k \leq m.$$

The true measure belongs to every $\mathcal{P}_{T,m}$ by (20). These sets are nested. They are also weakly compact because X_T is compact, every measure is nonnegative with the same finite mass T , and the defining inequalities are weakly closed. For each fixed ℓ , letting $k \rightarrow \infty$ recovers its exact integral identity; density of the ψ_ℓ family and Lemma A therefore give $\bigcap_{m \geq 1} \mathcal{P}_{T,m} = \{\mu_T\}$. Thus

$$\sup_{\mu \in \mathcal{P}_{T,m}} \left| \int_{X_T} g d\mu - \int_{X_T} g d\mu_T \right| \rightarrow 0. \quad (32)$$

We now make a fixed stage finite. Choose a continuous rational piecewise-polynomial g_m on X_T with $\|g_m - g\|_{\infty, X_T} \leq 2^{-m}$, and define $H_m(x) = ((a_{\ell k}(x))_{1 \leq \ell, k \leq m}, g_m(x)) \in \mathbb{R}^{m^2+1}$. For every $\mu \in \mathcal{P}_{T,m}$, the normalized integral of H_m lies in $\text{conv } H_m(X_T)$. Carathéodory's theorem in convex geometry therefore gives points $x_r \in X_T$, coefficients $\lambda_r \geq 0$ with $\sum_r \lambda_r = 1$, and at most $m^2 + 2$ terms such that

$$\frac{1}{T} \int_{X_T} H_m d\mu = \sum_{r=1}^M \lambda_r H_m(x_r).$$

Set $w_r = T\lambda_r$, padding to $M = m^2 + 2$ with zero weights. The extrema $j_m^\pm$ of $\sum_r w_r g_m(x_r)$ under (22) equal those of $\int g_m d\mu$ over $\mathcal{P}_{T,m}$. Since $\mu(X_T) = T$, $|\int (g_m - g) d\mu| \leq T2^{-m}$. Together with (32), this gives $j_m^\pm \rightarrow \int g d\mu_T$ and

$$j_m^- - T2^{-m} \leq \int_{X_T} g d\mu_T \leq j_m^+ + T2^{-m}. \quad (33)$$

The procedure in Section IV-D4 converts (33) into rational bounds with vanishing gap, so refinement terminates. $\square$