

Representation Theorems for Actual Powers over Two-Agent General Concurrent Game Frames

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One of the most well-known connections between logic and games is Pauly’s representation theorem: that the induced powers of individuals and coalitions in concurrent game frames correspond, in a precise sense, to a certain class of neighborhood frames. The precise sense here is what is called *alpha effectivity* (or *alpha power*): the power of a coalition is characterized by the sets of states which it can ensure the outcome to lie in by taking some joint action. This definition is inherently monotonic, and, as pointed out by [3], that fact can obscure relevant information about the power structure in the game: we don’t know whether two sets a coalition has the power to enforce correspond to the same or different joint actions. An alternative is to characterise the power of a coalition by its *actual power* (called *basic power* in [3]): the set of sets of states where each corresponds to one joint action by the coalition and all possible joint actions by the other agents. It has recently been argued [8, 9] that standard concurrent game frames rely on three assumptions that in some cases may be too strong: seriality, independence of agents, and determinism. This gives a total of eight different classes of *general* concurrent game frames. In this paper, assuming two agents, we prove that for actual powers, the eight classes of general concurrent game frames are represented by eight corresponding classes of neighborhood frames. This generalises a result in [3].

1 Introduction

Concurrent game frames are a standard semantic framework for logics of strategic reasoning. Many influential logics in this area are interpreted over concurrent game frames, including Coalition Logic CL [11] and Alternating-time Temporal Logic ATL [2]. Roughly speaking, a concurrent game frame consists of states and agents (forming coalitions), where, at each state, every coalition has available joint actions, and each such action induces a set of possible outcome states.

Two notions of coalition power arise from concurrent game frames: *alpha powers* and *actual powers*. An *alpha power* of a coalition is a set of possible futures such that the coalition has an action that forces every resulting outcome to lie in that set. Alpha power is also known as alpha effectivity. An *actual power* of a coalition is a set of possible futures such that the coalition has an action satisfying both of the

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following conditions: (1) the action forces every resulting outcome to lie in the set, and (2) every future state in the set is realizable by the complement coalition (and the environment). Thus, unlike alpha powers, actual powers explicitly involve agents outside the coalition. Note that actual powers contain more information about the power structure than alpha powers: we can get the latter from the former, but not the other way around.

Correspondingly, one obtains two kinds of neighborhood frames: *alpha neighborhood frames* and *actual neighborhood frames*. In an alpha neighborhood frame, each coalition is assigned an *alpha neighborhood function* that, at every state, specifies an *alpha neighborhood*, i.e., a set of alpha powers. In an actual neighborhood frame, each coalition is assigned an *actual neighborhood function* that, at every state, specifies an *actual neighborhood*, i.e., a set of actual powers.

Alpha powers are widely used in game theory; see, e.g., [10]. By contrast, there appears to be comparatively little work on *actual* powers. To the best of our knowledge, the main reference is van Benthem, Bezhanishvili, and Enqvist [3]. Working primarily from the perspective of game equivalence, they investigate *basic* powers of singleton coalitions in turn-based two-agent extensive games with imperfect information, defined in terms of (uniform) strategies. Conceptually, this notion of basic power coincides with what we call *actual* power.

Alpha and actual powers abstract away from the underlying actions in different ways. Alpha powers record guarantees: if a coalition has an action whose possible outcomes are contained in a set X , then the same action also guarantees any superset of X . Hence alpha powers are upward closed. Actual powers, by contrast, record the precise outcome set generated by an available coalition action: the set of states left possible once that action is fixed and the choices of the remaining agents are left open. They are therefore generally not upward closed. As observed in [3], describing a game solely in terms of its alpha powers can therefore lose information about its power structure, whereas actual powers provide a finer-grained account.

Because actual powers retain the outcomes left open to agents outside the coalition, they are useful for modeling social scenarios in which a coalition's ability depends not only on what it can guarantee, but also on what its action leaves open to others. A central logical example is Socially Friendly Coalition Logic [6]. Its operator $[C](\phi; \psi_1, \dots, \psi_k)$ expresses that coalition C has a collective action σ_C that guarantees ϕ while allowing the complementary coalition $\bar{C}$ to realize each of the alternatives $\psi_1, \dots, \psi_k$ by a suitable response. This semantic clause is the coalition-indexed, action-based counterpart of the instantial neighborhood operator $\Box(\phi; \psi_1, \dots, \psi_k)$ of [4]. Equivalently, in actual-neighborhood terms, there must be an actual power X of C such that every state in X satisfies ϕ and, for every $i \leq k$, X contains some state satisfying ψ_i . The latter requirement depends on the precise outcome range and cannot be captured by alpha powers alone.

Every concurrent game frame induces an alpha neighborhood frame. Pauly [11] and Goranko, Jamroga, and Turrini [7] showed that the class of concurrent game frames is *represented* by a class of alpha neighborhood frames characterized by certain *basic* properties of neighborhood functions, in the following sense: (1) every alpha neighborhood frame induced by a concurrent game frame satisfies these properties, and (2) every alpha neighborhood frame satisfying these properties is induced by some concurrent game frame. This result is the *alpha representation theorem* for concurrent game frames.

Every concurrent game frame also induces an actual neighborhood frame. Do there exist *basic* properties of neighborhood functions such that (1) every actual neighborhood frame induced by a concurrent game frame satisfies these properties, and (2) every actual neighborhood frame satisfying these properties is induced by some concurrent game frame? If so, this would be an *actual representation theorem* for concurrent game frames.

This question has not yet been fully settled in the literature. As far as we are aware, the only closely related result is due to van Benthem, Bezhanishvili, and Enqvist [3]. There, the authors establish a representation theorem for *basic* (strategy-based) powers in turn-based two-agent extensive games with imperfect information, under the restriction to singleton coalitions. Via a standard power-invariance transformation between the class of two-agent concurrent game frames and an appropriate class of turn-based two-agent extensive games with imperfect information, their proof readily yields a representation theorem for actual powers in two-agent concurrent game frames, again restricted to singleton coalitions (see [5] for details).

Let us briefly discuss the significance of representation theorems for alpha and actual powers. What are the powers of players, and coalitions of players, in games? Which sets of outcomes – known as *events* in probability theory or *propositions* in logic – is it in their power to make come about, in the sense that they can choose a (joint) action such that no matter what the other agents do, this is what will happen? This abstracts away the particulars of a game, and leaves an abstract representation of power which is exactly what we are often interested in when we analyse games. In order to understand power in games, the central question then is: what are the *properties* (or *axioms* or *laws*) of individual and coalitional power induced from games in this way? That is exactly what Pauly’s representation theorem [11] answers, for alpha powers. This is the main motivation, and the main significance of also the representation theorems for actual power: giving us a deeper understanding of the power in games.

Representation theorems can also yield alternative neighborhood semantics for some logics originally interpreted over concurrent game models. This makes the whole toolchest developed to deal with neighbourhood semantics available. Such semantics are also often more economical and convenient. Whereas a concurrent game frame explicitly records action names, availability functions, and outcome functions, many modal languages for strategic reasoning depend only on the powers induced by actions: what coalitions can force in the case of alpha powers, or the exact outcome ranges generated by coalition actions in the case of actual powers. Representation theorems therefore justify replacing action-based models with neighborhood-based models, without changing truth conditions. Pauly’s representation theorem for alpha powers provides a prominent example: it allows the semantics of CL over concurrent game frames to be transferred to alpha neighborhood frames. Pauly [11] used such frames to prove completeness of CL, while Ågotnes and Alechina [1] employed the same semantic framework to extend CL with knowledge.

Li and Ju [8, 9] argued that standard concurrent game frames impose three assumptions that may be too strong in practice: *seriality*, *independence of agents*, and *determinism*. For motivating examples, see [8]. They introduced eight classes of general concurrent game frames, corresponding to the eight possible combinations of these assumptions, and studied the generalized coalition logic associated with each class.

In this paper, assuming two agents, we establish the following result: for actual powers, the eight classes of general concurrent game frames are represented by eight corresponding classes of actual neighborhood frames. Our proof method relies essentially on the two-agent setting, and does not extend directly to settings with three or more agents. We briefly explain this at the end of Section 4.

The remainder of the paper is organized as follows. In Section 2, we introduce the general framework used throughout the paper, including action frames, alpha and actual powers, actual neighborhood frames, and representability of action frames by actual neighborhood frames. In Section 3, we define the eight classes of general concurrent game frames. In Section 4, we define eight classes of actual neighborhood frames via eight finite sets of properties of actual neighborhood functions, and show that, for actual powers, the eight classes of general concurrent game frames (with two agents) are represented by

the corresponding eight classes of actual neighborhood frames. In Section 5, we conclude the paper with final remarks.

Due to space limitations, we omit some material that can be found in the full paper [5]. Moreover, building on the representation result for actual powers established in this paper, the full paper shows that, for alpha powers, the same eight classes are represented by eight corresponding classes of alpha neighborhood frames.

2 Preliminaries

In this section, we proceed as follows. First, we introduce *action frames* in an abstract form; specific constraints will be imposed later. Second, we discuss *alpha powers* and *actual powers*, and briefly compare them. Third, we present *actual neighborhood frames*. These are again introduced at an abstract level, with additional constraints to be added in subsequent sections. Finally, we define how action frames are represented by actual neighborhood frames.

2.1 Action frames

Let AG be a finite nonempty set of agents. Subsets of AG are called **coalitions**, and AG itself is called the **grand coalition**. In what follows, whenever no confusion arises, we write a instead of $\{a\}$ for $a \in AG$.

Let AC be a nonempty set of actions. For each $C \subseteq AG$, define $JA_C = \{\sigma_C \mid \sigma_C : C \rightarrow AC\}$, the set of **joint actions** of C . Note $JA_\emptyset = \{\emptyset\}$. For $C, D \subseteq AG$ such that $C \subseteq D$, and for every $\sigma_D \in JA_D$, we denote by $\sigma_D|_C$ the **restriction** of σ_D to C , which is an element of JA_C . In the sequel, we sometimes represent joint actions of coalitions as action sequences, implicitly assuming a fixed order of agents.

Definition 2.1 (Action frames). An **action frame** is a tuple

$$AF = (ST, AC, \{av_C \mid C \subseteq AG\}, \{out_C \mid C \subseteq AG\}),$$

where:

- ST is a nonempty set of states.
- AC is a nonempty set of actions.
- For each $C \subseteq AG$, $av_C : ST \rightarrow \mathcal{P}(JA_C)$ is an **availability function** for C .
Here, $av_C(s)$ is the set of all joint actions of coalition C that are available at state s .
- For each $C \subseteq AG$, $out_C : ST \times JA_C \rightarrow \mathcal{P}(ST)$ is an **outcome function** for C .
Here, $out_C(s, \sigma_C)$ is the set of possible outcome states when coalition C performs joint action σ_C at state s .

Note that the action set AC may vary across action frames.

Different constraints may be imposed on action frames.

Intuitively, $out_\emptyset(s, \emptyset)$ contains all states to which s may evolve, i.e., all possible successors of s .

2.2 Alpha and actual powers

Fix an action frame

$$AF = (ST, AC, \{av_C \mid C \subseteq AG\}, \{out_C \mid C \subseteq AG\}),$$

a state $s \in \text{ST}$, a coalition $C \subseteq \text{AG}$, and an available joint action $\sigma_C \in \text{av}_C(s)$. We say that a set of states $X \subseteq \text{ST}$ is **safe** for σ_C at s if $\text{out}_C(s, \sigma_C) \subseteq X$, that is, performing σ_C at s guarantees that the next state lies in X . We say that X is **tight** for σ_C at s if $X \subseteq \text{out}_C(s, \sigma_C)$, that is, every state in X can occur as an outcome compatible with performing σ_C at s .

An **alpha power** of C at s is a set of states $X \subseteq \text{ST}$ that is safe for some available joint action of C at s . An **actual power** of C at s is a set of states $X \subseteq \text{ST}$ that is both safe and tight for some available joint action of C at s . Equivalently, actual powers of C at s are exactly the sets of the form $\text{out}_C(s, \sigma_C)$ for $\sigma_C \in \text{av}_C(s)$. We say that an (alpha or actual) power X **enables** a state $t \in \text{ST}$ at s if $t \in X$.

Alpha powers are monotonic: if X is an alpha power, then every superset $X' \supseteq X$ is also an alpha power. By contrast, actual powers are generally non-monotonic, since tightness rules out adding states that are not actually reachable.

The following example illustrates the distinction between alpha and actual powers.

Example 2.2. Consider two AI-driven robots, a and b , controlling a warehouse gate. The robots send their commands to the gate controller simultaneously, but robot a has higher authority than robot b .

There are two relevant states, $W = \{w_1, w_2\}$, where w_1 represents the gate being closed and w_2 represents the gate being open. Robot a has three actions: open_a , close_a , and delegate_a . The first opens the gate, the second closes it, and the third delegates the decision to robot b . Robot b has two actions: open_b and close_b .

We consider two scenarios. In both, if a chooses open_a , the gate opens, and if a chooses close_a , the gate closes, regardless of b 's action. The scenarios differ only in the effect of delegate_a . In the first scenario, delegation works as intended: when a delegates, b 's action determines whether the gate opens or closes. In the second scenario, delegation is ineffective: when a delegates, the gate closes regardless of b 's action.

The two scenarios induce the same alpha powers for every coalition:

coalition	alpha powers
$\emptyset$	$\{W\}$
$\{a\}$	$\{\{w_1\}, \{w_2\}, W\}$
$\{b\}$	$\{W\}$
$\{a, b\}$	$\{\{w_1\}, \{w_2\}, W\}$.

Their actual powers differ. In the first scenario, the actual powers of $\{a\}$ are $\{\{w_1\}, \{w_2\}, W\}$, corresponding to closing, opening, and delegating, respectively. In the second scenario, they are only $\{\{w_1\}, \{w_2\}\}$. Indeed, delegation no longer leaves both outcomes open, but instead has the same exact outcome set as close_a .

2.3 Actual neighborhood frames

Definition 2.3 (Actual neighborhood frames). An **actual neighborhood frame** is a tuple

$$\text{NF}^{\text{ac}} = (\text{ST}, \{N_C^{\text{ac}} \mid C \subseteq \text{AG}\}),$$

where:

- ST is a nonempty set of states;
- For each $C \subseteq \text{AG}$, $N_C^{\text{ac}} : \text{ST} \rightarrow \mathcal{P}(\mathcal{P}(\text{ST}))$ is an **actual neighborhood function** of C .

Different constraints may be imposed on actual neighborhood frames. The set $N_C^{\text{ac}}(s)$ is called the **actual neighborhood** of C at s . Intuitively, $N_C^{\text{ac}}(s)$ consists of the actual powers of coalition C at state s . In the intended interpretation, $\bigcup N_\emptyset^{\text{ac}}(s)$ consists of all states to which s may evolve, i.e., the possible successors of s .

Structurally, actual neighborhood frames are ordinary coalition-indexed neighborhood frames. The qualifier “actual” merely emphasizes that their neighborhoods are interpreted as representing the actual powers of coalitions.

2.4 Representation of action frames by actual neighborhood frames

Definition 2.4 (Actual effectivity functions of action frames). Let $\text{AF} = (\text{ST}, \text{AC}, \{\text{av}_C \mid C \subseteq \text{AG}\}, \{\text{out}_C \mid C \subseteq \text{AG}\})$ be an action frame. For each $C \subseteq \text{AG}$, define:

- the **actual effectivity function** EF_C^{ac} for C in AF by setting, for every $s \in \text{ST}$,

$$\text{EF}_C^{\text{ac}}(s) = \{\text{out}_C(s, \sigma_C) \mid \sigma_C \in \text{av}_C(s)\}.$$

Definition 2.5 (AC-representability of action frames). Let $\text{AF} = (\text{ST}, \text{AC}, \{\text{av}_C \mid C \subseteq \text{AG}\}, \{\text{out}_C \mid C \subseteq \text{AG}\})$ be an action frame.

We say that AF is **AC-represented** by an actual neighborhood frame

$$\text{NF}^{\text{ac}} = (\text{ST}, \{N_C^{\text{ac}} \mid C \subseteq \text{AG}\})$$

if, for every $C \subseteq \text{AG}$, we have $\text{EF}_C^{\text{ac}} = N_C^{\text{ac}}$.

Intuitively, AC-representability means that the actual neighborhood frame captures all information about the actual powers of coalitions in the action frame.

Definition 2.6 (AC-representability of classes of action frames). Let $\mathbf{AF}$ be a class of action frames. We say that $\mathbf{AF}$ is **AC-represented** by a class of actual neighborhood frames $\mathbf{AC-NF}$ if:

- Every action frame in $\mathbf{AF}$ is AC-represented by some actual neighborhood frame in $\mathbf{AC-NF}$;
- Every actual neighborhood frame in $\mathbf{AC-NF}$ AC-represents some action frame in $\mathbf{AF}$.

Suppose that a class of action frames is AC-represented by a class of actual neighborhood frames. Intuitively, this means that, as far as actual powers are concerned, one can translate the information carried by action frames into the corresponding neighborhood frames.

Now fix a class of action frames $\mathbf{AF}$ and a collection of properties of actual neighborhood frames. To show that $\mathbf{AF}$ is AC-represented by the class of actual neighborhood frames satisfying these properties, it suffices to establish the following two statements:

- **Havingness** (necessity): every actual neighborhood frame that AC-represents an action frame in $\mathbf{AF}$ satisfies these properties;
- **Enoughness** (sufficiency): every actual neighborhood frame satisfying the properties AC-represents an action frame in $\mathbf{AF}$.

Later we will follow exactly this pattern.

3 General concurrent game frames and eight classes

In this section, we present the eight classes of general concurrent game frames considered by Li and Ju [8, 9]. Later, assuming two agents, we establish their AC-representability.

Definition 3.1 (General concurrent game frames). An action frame

$$\text{GCGF} = (\text{ST}, \text{AC}, \{\text{av}_C \mid C \subseteq \text{AG}\}, \{\text{out}_C \mid C \subseteq \text{AG}\})$$

is a **general concurrent game frame** if the following conditions hold:

- The **grand-coalition-induced outcome condition** (simply the **GCI-condition**): for every $C \subseteq \text{AG}$, every $s \in \text{ST}$, and every $\sigma_C \in \text{JA}_C$,

$$\text{out}_C(s, \sigma_C) = \bigcup \{ \text{out}_{\text{AG}}(s, \sigma_{\text{AG}}) \mid \sigma_{\text{AG}} \in \text{JA}_{\text{AG}} \text{ and } \sigma_{\text{AG}}|_C = \sigma_C \}.$$

- The **outcome-driven availability condition** (simply the **ODA-condition**): for every $C \subseteq \text{AG}$ and every $s \in \text{ST}$,

$$\text{av}_C(s) = \{ \sigma_C \in \text{JA}_C \mid \text{out}_C(s, \sigma_C) \neq \emptyset \}.$$

Thus, under the GCI-condition and the ODA-condition, the functions out_C and av_C (for all $C \subseteq \text{AG}$) are determined by the grand-coalition outcome function out_{AG} .

Definition 3.2 (Seriality, independence, and determinism). Let $\text{GCGF} = (\text{ST}, \text{AC}, \{\text{av}_C \mid C \subseteq \text{AG}\}, \{\text{out}_C \mid C \subseteq \text{AG}\})$ be a general concurrent game frame. We say that:

- GCGF is **serial** if, for all $C \subseteq \text{AG}$ and all $s \in \text{ST}$, we have $\text{av}_C(s) \neq \emptyset$;
- GCGF is **independent** if, for all $s \in \text{ST}$ and all disjoint coalitions $C, D \subseteq \text{AG}$, whenever $\sigma_C \in \text{av}_C(s)$ and $\sigma_D \in \text{av}_D(s)$, we have $\sigma_C \cup \sigma_D \in \text{av}_{C \cup D}(s)$;
- GCGF is **deterministic** if, for all $s \in \text{ST}$ and all $\sigma_{\text{AG}} \in \text{av}_{\text{AG}}(s)$, the set $\text{out}_{\text{AG}}(s, \sigma_{\text{AG}})$ is a singleton.

Let S, I, and D denote seriality, independence, and determinism, respectively. Let

$$\text{ES} = \{\varepsilon, \text{S}, \text{I}, \text{D}, \text{SI}, \text{SD}, \text{ID}, \text{SID}\}$$

be the set of the eight strings encoding all combinations of these three properties. Here ε denotes the empty combination. For $X \in \text{ES}$, we say that a general concurrent game frame is an **X-frame** if it satisfies the properties indicated by X.

The following facts will be used implicitly throughout the sequel.

Fact 3.3. *Let*

$$\text{GCGF} = (\text{ST}, \text{AC}, \{\text{av}_C \mid C \subseteq \text{AG}\}, \{\text{out}_C \mid C \subseteq \text{AG}\})$$

be a general concurrent game frame. Then the following hold:

1. **Outcome monotonicity:** for all $s \in \text{ST}$, all $C \subseteq D \subseteq \text{AG}$, all $\sigma_C \in \text{JA}_C$, and all $\sigma_D \in \text{JA}_D$ such that $\sigma_D|_C = \sigma_C$, we have

$$\text{out}_D(s, \sigma_D) \subseteq \text{out}_C(s, \sigma_C).$$

2. **Alternative GCI-condition:** for all $s \in \text{ST}$, all $C \subseteq \text{AG}$, and all $\sigma_C \in \text{JA}_C$,

$$\text{out}_C(s, \sigma_C) = \bigcup \{ \text{out}_{\text{AG}}(s, \sigma_{\text{AG}}) \mid \sigma_{\text{AG}} \in \text{av}_{\text{AG}}(s) \text{ and } \sigma_{\text{AG}}|_C = \sigma_C \}.$$

The first item is an immediate consequence of the GCI-condition, and the second is a reformulation of the GCI-condition using the ODA-condition (since unavailable grand-coalition actions have empty outcome sets).

4 Representing actual powers

In this section, we proceed as follows. First, we define eight classes of AC-representative actual neighborhood frames and establish several basic facts about them. Second, under the assumption that there are exactly two agents, we show that each of the eight classes of general concurrent game frames introduced in the previous section is AC-represented by the corresponding class of AC-representative actual neighborhood frames. The representability result has two directions: *havingness* and *enoughness*.

4.1 Eight classes of AC-representative actual neighborhood frames

The following definitions are stated for an arbitrary finite set of agents AG . Later, we prove the representation theorem under the assumption that $|AG| = 2$ for the enoughness direction. The havingness direction does not require the two-agent restriction.

Definition 4.1 (AC-representative actual neighborhood frames). Let $NF^{ac} = (ST, \{N_C^{ac} \mid C \subseteq AG\})$ be an actual neighborhood frame. We say that NF^{ac} is **AC-representative** if the following conditions are satisfied:

1. **Actual triviality of the empty coalition:** for all $s \in ST$, if $N_{\emptyset}^{ac}(s)$ is nonempty, then it is a singleton.
Intuitively, this condition says that the empty coalition has at most one actual power. Having more than one actual power would amount to having more than one choice, hence the name.
2. **Liveness:** for all $s \in ST$ and all $C \subseteq AG$, we have $\emptyset \notin N_C^{ac}(s)$.
Intuitively, this condition rules out the degenerate actual power $\emptyset$ for any coalition. Note that liveness does not force coalitions to have powers; it only rules out treating the empty set as a “power” when powers exist.
3. **Actual power inclusion:** for all $s \in ST$ and all $C, D \subseteq AG$ with $C \subseteq D$, for all $X \in N_D^{ac}(s)$ there exists $Y \in N_C^{ac}(s)$ such that $X \subseteq Y$.
Intuitively, every actual power of a larger coalition is included in some actual power of a smaller coalition.
4. **Actual power decomposition:** for all $s \in ST$ and all $C, D \subseteq AG$ with $C \subseteq D$, for every $X \in N_C^{ac}(s)$ there exists $\Delta \subseteq N_D^{ac}(s)$ such that $X = \bigcup \Delta$.
Intuitively, every actual power of a smaller coalition can be decomposed into (possibly many) actual powers of a larger coalition.

We use “AC-representative” for actual neighborhood frames satisfying the intrinsic conditions intended to characterize those that AC-represent general concurrent game frames. Note “AC-representative” and “AC-representable” are defined in different ways.

Definition 4.2 (Seriality, independence, and determinism of AC-representative actual neighborhood frames). Let $NF^{ac} = (ST, \{N_C^{ac} \mid C \subseteq AG\})$ be an AC-representative actual neighborhood frame. We say that:

- NF^{ac} is **AC-serial** if for all $s \in ST$ and all $C \subseteq AG$, we have $N_C^{ac}(s) \neq \emptyset$.
- NF^{ac} is **AC-independent** if, for all $s \in ST$ and all $C, D \subseteq AG$ with $C \cap D = \emptyset$, whenever $X \in N_C^{ac}(s)$ and $Y \in N_D^{ac}(s)$, there exists $Z \in N_{C \cup D}^{ac}(s)$ such that $Z \subseteq X \cap Y$.
Intuitively, for disjoint coalitions, any two actual powers can be jointly refined to an actual power of their union that is compatible with both.

- NF^{ac} is **AC-deterministic** if for all $s \in ST$ and all $X \in N_{AG}^{ac}(s)$, X is a singleton.

As before, we let the three symbols S , I , and D denote AC-seriality, AC-independence, and AC-determinism, respectively. We also reuse the set

$$ES = \{\epsilon, S, I, D, SI, SD, ID, SID\},$$

whose elements serve as labels for the eight possible combinations of these three properties.

For each $X \in ES$ and each AC-representative actual neighborhood frame NF^{ac} , we say that NF^{ac} is an **X-frame** if it satisfies the properties corresponding to X .

What follows are several basic facts about AC-representative actual neighborhood frames. These facts are useful both for intuition and for later technical developments. Again, we refer to [5] for their proofs.

The first fact shows that the set of states reachable from a given state is independent of the coalition: all coalitions share the same set of successors of that state.

Fact 4.3. *Let $NF^{ac} = (ST, \{N_C^{ac} \mid C \subseteq AG\})$ be an AC-representative actual neighborhood frame. For all $s \in ST$ and all $C \subseteq AG$, we have*

$$\bigcup N_C^{ac}(s) = \bigcup N_{\emptyset}^{ac}(s).$$

The next fact shows that emptiness of the set of actual powers at a state does not depend on the coalition: either every coalition has actual power at s , or none does.

Fact 4.4. *Let $NF^{ac} = (ST, \{N_C^{ac} \mid C \subseteq AG\})$ be an AC-representative actual neighborhood frame. For all $s \in ST$ and all $C \subseteq AG$,*

$$N_C^{ac}(s) = \emptyset \quad \text{iff} \quad N_{\emptyset}^{ac}(s) = \emptyset.$$

The following fact shows that if the empty coalition has any actual power at a state s , then it has exactly one: namely, the overall successor set of s .

Fact 4.5. *Let $NF^{ac} = (ST, \{N_C^{ac} \mid C \subseteq AG\})$ be an AC-representative actual neighborhood frame. For all $s \in ST$, if $N_{\emptyset}^{ac}(s) \neq \emptyset$, then*

$$N_{\emptyset}^{ac}(s) = \left\{ \bigcup N_{AG}^{ac}(s) \right\}.$$

The final fact states that, when moving from a smaller coalition to a larger one, every actual power X of the smaller coalition can be recovered as the union of those actual powers of the larger coalition that are included in X .

Fact 4.6. *Let $NF^{ac} = (ST, \{N_C^{ac} \mid C \subseteq AG\})$ be an AC-representative actual neighborhood frame. For every $s \in ST$, all $C, D \subseteq AG$ with $C \subseteq D$, and every $X \in N_C^{ac}(s)$,*

$$\bigcup \{Z \in N_D^{ac}(s) \mid Z \subseteq X\} = X.$$

4.2 Representation theorems for actual powers

Theorem 4.7 (Actual havingness theorem). *Let $X \in ES$. For every actual neighborhood frame, if it AC-represents a general concurrent game X-frame, then it is an AC-representative actual neighborhood X-frame.*

Proof. Let $NF^{ac} = (ST, \{N_C^{ac} \mid C \subseteq AG\})$ be an actual neighborhood frame. Suppose that NF^{ac} AC-represents a general concurrent game X-frame

$$GCGF = (ST, AC, \{av_C \mid C \subseteq AG\}, \{out_C \mid C \subseteq AG\}).$$

Then for every coalition $C \subseteq AG$ and every state $s \in ST$, we have $N_C^{ac}(s) = EF_C^{ac}(s)$.

We show that NF^{ac} is an X-frame. We verify that each of the properties S, I, and D is preserved under AC-representation. It then follows, since GCGF is an X-frame, that NF^{ac} is an X-frame.

- **Seriality.** Assume that GCGF is serial. Fix $s \in \text{ST}$ and $C \subseteq \text{AG}$. Then $\text{av}_C(s) \neq \emptyset$, and hence

$$\text{EF}_C^{\text{ac}}(s) = \{\text{out}_C(s, \sigma_C) \mid \sigma_C \in \text{av}_C(s)\} \neq \emptyset.$$

Therefore $\text{N}_C^{\text{ac}}(s) \neq \emptyset$, and thus NF^{ac} is AC-serial.

- **Independence.** Assume that GCGF is independent. Fix $s \in \text{ST}$ and disjoint coalitions $C, D \subseteq \text{AG}$ with $C \cap D = \emptyset$. Let $X \in \text{N}_C^{\text{ac}}(s)$ and $Y \in \text{N}_D^{\text{ac}}(s)$. Since $\text{N}_C^{\text{ac}} = \text{EF}_C^{\text{ac}}$ and $\text{N}_D^{\text{ac}} = \text{EF}_D^{\text{ac}}$, there exist $\sigma_C \in \text{av}_C(s)$ and $\sigma_D \in \text{av}_D(s)$ such that $\text{out}_C(s, \sigma_C) = X$ and $\text{out}_D(s, \sigma_D) = Y$.

By independence of GCGF, we have $\sigma_C \cup \sigma_D \in \text{av}_{C \cup D}(s)$. Let $Z = \text{out}_{C \cup D}(s, \sigma_C \cup \sigma_D)$. Then $Z \in \text{EF}_{C \cup D}^{\text{ac}}(s) = \text{N}_{C \cup D}^{\text{ac}}(s)$. Moreover, since $\sigma_C \subseteq \sigma_C \cup \sigma_D$ and $\sigma_D \subseteq \sigma_C \cup \sigma_D$, by outcome monotonicity, we have $Z \subseteq X$ and $Z \subseteq Y$, and hence $Z \subseteq X \cap Y$. Therefore NF^{ac} is AC-independent.

- **Determinism.** Assume GCGF is deterministic. Fix $s \in \text{ST}$ and $X \in \text{N}_{\text{AG}}^{\text{ac}}(s)$. Then $X \in \text{EF}_{\text{AG}}^{\text{ac}}(s)$, so there exists $\sigma_{\text{AG}} \in \text{av}_{\text{AG}}(s)$ such that $\text{out}_{\text{AG}}(s, \sigma_{\text{AG}}) = X$. By determinism of GCGF, $\text{out}_{\text{AG}}(s, \sigma_{\text{AG}})$ is a singleton; hence so is X . Therefore NF^{ac} is AC-deterministic.

We show that NF^{ac} satisfies the four AC-representativeness conditions.

1. **Actual triviality of the empty coalition.** Fix $s \in \text{ST}$. Note that

$$\text{out}_{\emptyset}(s, \emptyset) = \bigcup \{\text{out}_{\text{AG}}(s, \sigma_{\text{AG}}) \mid \sigma_{\text{AG}} \in \text{JA}_{\text{AG}}\}.$$

If $\text{out}_{\emptyset}(s, \emptyset) \neq \emptyset$, then by definition of $\text{EF}_{\emptyset}^{\text{ac}}$ we have

$$\text{N}_{\emptyset}^{\text{ac}}(s) = \text{EF}_{\emptyset}^{\text{ac}}(s) = \{\text{out}_{\emptyset}(s, \emptyset)\},$$

so $\text{N}_{\emptyset}^{\text{ac}}(s)$ is a singleton. If $\text{out}_{\emptyset}(s, \emptyset) = \emptyset$, then $\text{av}_{\emptyset}(s) = \emptyset$ and hence $\text{N}_{\emptyset}^{\text{ac}}(s) = \text{EF}_{\emptyset}^{\text{ac}}(s) = \emptyset$.

2. **Liveness.** Fix $s \in \text{ST}$ and $C \subseteq \text{AG}$. Since

$$\text{EF}_C^{\text{ac}}(s) = \{\text{out}_C(s, \sigma_C) \mid \sigma_C \in \text{av}_C(s)\}$$

and each $\sigma_C \in \text{av}_C(s)$ satisfies $\text{out}_C(s, \sigma_C) \neq \emptyset$, we obtain $\emptyset \notin \text{EF}_C^{\text{ac}}(s)$. Thus $\emptyset \notin \text{N}_C^{\text{ac}}(s)$.

3. **Actual power inclusion.** Fix $s \in \text{ST}$ and coalitions $C \subseteq D \subseteq \text{AG}$. Let $X \in \text{N}_D^{\text{ac}}(s) = \text{EF}_D^{\text{ac}}(s)$. Then there exists $\sigma_D \in \text{av}_D(s)$ such that $\text{out}_D(s, \sigma_D) = X$. Write $\sigma_D = \sigma_C \cup \sigma_{D \setminus C}$ with $\sigma_C \in \text{JA}_C$ and $\sigma_{D \setminus C} \in \text{JA}_{D \setminus C}$. Since σ_D is available for D , its restriction σ_C is available for C , i.e., $\sigma_C \in \text{av}_C(s)$. Moreover, $\text{out}_D(s, \sigma_D) \subseteq \text{out}_C(s, \sigma_C)$. Let $Y = \text{out}_C(s, \sigma_C)$. Then $Y \in \text{EF}_C^{\text{ac}}(s) = \text{N}_C^{\text{ac}}(s)$ and $X \subseteq Y$, as required.

4. **Actual power decomposition.** Fix $s \in \text{ST}$ and coalitions $C \subseteq D \subseteq \text{AG}$. Let $X \in \text{N}_C^{\text{ac}}(s) = \text{EF}_C^{\text{ac}}(s)$. Then there exists $\sigma_C \in \text{av}_C(s)$ such that $\text{out}_C(s, \sigma_C) = X$. Let $\Theta = \{\sigma_{\text{AG}} \in \text{av}_{\text{AG}}(s) \mid \sigma_C \subseteq \sigma_{\text{AG}}\}$. By the GCI-condition of GCGF and Fact 3.3,

$$X = \text{out}_C(s, \sigma_C) = \bigcup \{\text{out}_{\text{AG}}(s, \sigma_{\text{AG}}) \mid \sigma_{\text{AG}} \in \Theta\}.$$

Consider the family

$$\Delta = \{\text{out}_D(s, \sigma_{\text{AG}}|_D) \mid \sigma_{\text{AG}} \in \Theta\}.$$

For each $\sigma_{AG} \in \Theta$, we have $\text{out}_{AG}(s, \sigma_{AG}) \neq \emptyset$ and $\text{out}_{AG}(s, \sigma_{AG}) \subseteq \text{out}_D(s, \sigma_{AG|D})$, hence $\sigma_{AG|D} \in \text{av}_D(s)$ and therefore $\text{out}_D(s, \sigma_{AG|D}) \in \text{EF}_D^{\text{ac}}(s) = \text{N}_D^{\text{ac}}(s)$. Thus $\Delta \subseteq \text{N}_D^{\text{ac}}(s)$.

It remains to show that $X = \bigcup \Delta$.

($\subseteq$) Let $u \in X$. Then $u \in \text{out}_{AG}(s, \sigma_{AG})$ for some $\sigma_{AG} \in \Theta$, hence $u \in \text{out}_D(s, \sigma_{AG|D})$ and thus $u \in \bigcup \Delta$.

($\supseteq$) Let $u \in \bigcup \Delta$. Then $u \in \text{out}_D(s, \sigma_{AG|D})$ for some $\sigma_{AG} \in \Theta$. By the GCI-condition of GCGF, there exists $\lambda_{AG} \in \text{JA}_{AG}$ such that $\sigma_{AG|D} \subseteq \lambda_{AG}$ and $u \in \text{out}_{AG}(s, \lambda_{AG})$. Since $\sigma_C \subseteq \sigma_{AG|D} \subseteq \lambda_{AG}$ and λ_{AG} is available, we have $\lambda_{AG} \in \Theta$. Hence

$$u \in \bigcup \{\text{out}_{AG}(s, \tau_{AG}) \mid \tau_{AG} \in \Theta\} = X.$$

This completes the proof. □

Theorem 4.8 (Actual enoughness theorem). *Assume $AG = \{a, b\}$. Let $X \in \text{ES}$. Every AC-representative actual neighborhood X-frame AC-represents some general concurrent game X-frame.*

The proof is somewhat involved; we refer to [5] for an illustrative example.

Proof.

Analysis of the task. Let $\text{NF}^{\text{ac}} = (\text{ST}, \{\text{N}_C^{\text{ac}} \mid C \subseteq AG\})$ be an AC-representative actual neighborhood X-frame. We aim to construct a general concurrent game X-frame

$$\text{GCGF} = (\text{ST}, \text{AC}, \{\text{av}_C \mid C \subseteq AG\}, \{\text{out}_C \mid C \subseteq AG\})$$

such that GCGF is AC-represented by NF^{ac} .

All conditions defining general concurrent game X-frames are statewise: for each source state s , they involve only the availability sets $\text{av}_C(s)$ and the outcome maps $\text{out}_C(s, \cdot)$, and do not relate these information to those at a distinct source state t . Consequently, the required availability and outcome information can be constructed independently at each state. The resulting local constructions can then be combined by using pairwise disjoint, state-tagged action sets and taking their union as the global action set.

Therefore, it suffices to proceed as follows: for each $s \in \text{ST}$, starting from the local neighborhood structure

$$\mathcal{N} = \{\text{N}_C^{\text{ac}}(s) \mid C \subseteq AG\},$$

we construct a local structure

$$\mathcal{G} = (\text{AC}_s, \{\text{av}_C(s) \mid C \subseteq AG\}, \{\text{out}_C(s, \cdot) \mid C \subseteq AG\})$$

satisfying the following conditions.

1. $\mathcal{G}$ is a local component (called a **local game**) of a general concurrent game frame:
 - (a) AC_s is a set of actions.
 - (b) For every $C \subseteq AG$, $\text{av}_C(s) \subseteq \text{JA}_C$.
 - (c) For every $C \subseteq AG$ and every $\sigma_C \in \text{JA}_C$, $\text{out}_C(s, \sigma_C) \subseteq \text{ST}$.

(d) For every $C \subseteq AG$:

$$\text{av}_C(s) = \{\sigma_C \in \text{JA}_C \mid \text{out}_C(s, \sigma_C) \neq \emptyset\}.$$

(e) For every $C \subseteq AG$ and every $\sigma_C \in \text{JA}_C$, the action σ_C satisfies the GCI-condition at s :

$$\text{out}_C(s, \sigma_C) = \bigcup \{\text{out}_{AG}(s, \sigma_{AG}) \mid \sigma_{AG} \in \text{av}_{AG}(s) \text{ and } \sigma_C \subseteq \sigma_{AG}\}.$$

2. $\mathcal{G}$ is AC-represented by $\mathcal{N}$, i.e., for every $C \subseteq AG$,

$$\{\text{out}_C(s, \sigma_C) \mid \sigma_C \in \text{av}_C(s)\} = N_C^{\text{ac}}(s).$$

3. $\mathcal{G}$ has the properties corresponding to $\mathbf{X}$.

Fix $s \in \text{ST}$. In the sequel, to simplify notation, we drop the explicit mention of s . More precisely:

- We write N_C^{ac} for $N_C^{\text{ac}}(s)$.
- We write AC for AC_s .
- We write av_C for $\text{av}_C(s)$.
- We write $\text{out}_C(\sigma_C)$ for $\text{out}_C(s, \sigma_C)$.

The trivial case. Assume $N_\emptyset^{\text{ac}} = \emptyset$. By Fact 4.4, we have $N_C^{\text{ac}} = \emptyset$ for every $C \subseteq AG$. Define a game $\mathcal{G} = (\text{AC}, \{\text{av}_C \mid C \subseteq AG\}, \{\text{out}_C \mid C \subseteq AG\})$ as follows:

- Let AC be any nonempty set (e.g., a singleton).
- For every $C \subseteq AG$, let $\text{av}_C = \emptyset$.
- For every $C \subseteq AG$ and every $\sigma_C \in \text{JA}_C$, let $\text{out}_C(\sigma_C) = \emptyset$.

It is straightforward to verify that $\mathcal{G}$ satisfies the requirements in the analysis above.

Assume now that $N_\emptyset^{\text{ac}} \neq \emptyset$.

Construction of a local game. By Fact 4.4, none of N_a^{ac} , N_b^{ac} , and N_{AG}^{ac} is empty. Assume

$$N_a^{\text{ac}} = \{X_i \mid i \in I\}, \quad N_b^{\text{ac}} = \{Y_j \mid j \in J\}, \quad N_{AG}^{\text{ac}} = \{Z_k \mid k \in K\},$$

where I, J, K are index sets.

Initialize $\mathcal{G} = (\text{AC}, \{\text{av}_C \mid C \subseteq AG\}, \{\text{out}_C \mid C \subseteq AG\})$ by setting $\text{AC} = \emptyset$ and leaving all av_C and out_C empty. We will add actions, availability, and outcomes step by step until $\mathcal{G}$ satisfies conditions 1–3.

Before the construction, we fix some terminology and conventions, and also give some quick preview of what we are going to do next:

- By **individual powers** we mean powers of a or of b .
- Actions in AC will have the form α_{n-X-Z} and β_{n-Y-Z} , where $n \in \{1, 2, 3\}$, $X \in N_a^{\text{ac}}$, $Y \in N_b^{\text{ac}}$, and $Z \in N_{AG}^{\text{ac}}$ satisfy $Z \subseteq X$ and $Z \subseteq Y$. We regard the index n as indicating the *level* (three groups). Joint actions of AG will have the form $(\alpha_{n-X-Z}, \beta_{m-Y-Z'})$

All α_{n-X-Z} will be available actions of a , and all β_{n-Y-Z} will be available actions of b . We will stipulate

$$\text{out}_a(\alpha_{n-X-Z}) = X \quad \text{and} \quad \text{out}_b(\beta_{n-Y-Z}) = Y.$$

We say that α_{n-X-Z} is a **name** of X , and that β_{n-Y-Z} is a name of Y .

Available joint actions of AG will be introduced gradually. Not every such pair must be available, since independence need not hold. Whenever we declare $(\alpha_{n-X-Z}, \beta_{m-Y-Z'})$ to be available, we will simultaneously specify its outcome set $\text{out}_{\text{AG}}(\alpha_{n-X-Z}, \beta_{m-Y-Z'}) = Z''$. In that case we say that the joint action is a name of Z'' .

- For any available action α of a , saying that we *make α satisfy the GCI-condition* means that we ensure

$$\text{out}_a(\alpha) = \bigcup \{ \text{out}_{\text{AG}}(\alpha, \beta) \mid \beta \text{ is available for } b \text{ and } (\alpha, \beta) \text{ is available for AG} \}.$$

The same interpretation applies to any available action β of b .

- Suppose α is an available action of a and β is an available action of b . Saying that we *pair α and β* means that we make (α, β) an available action of AG and specify its outcome set.
- Let α be an available action of a and β an available action of b . Declaring (α, β) to be an available joint action with outcome Z is called **safe** (for α and β) if

$$Z \subseteq \text{out}_a(\alpha) \cap \text{out}_b(\beta).$$

In that case, adding this pair cannot destroy the GCI-condition for α or for β : if either action already satisfies its GCI-condition before the addition, it still satisfies it afterwards.

- Any joint action not explicitly declared available will be treated as unavailable, with outcome $\emptyset$.

The construction consists of three steps.

Step 1: introducing names for individual powers. For every $X \in \mathcal{N}_a^{\text{ac}}$, every $Z \in \mathcal{N}_{\text{AG}}^{\text{ac}}$ with $Z \subseteq X$, and every $n \in \{1, 2, 3\}$, let α_{n-X-Z} be an available action of a such that $\text{out}_a(\alpha_{n-X-Z}) = X$.

For every $Y \in \mathcal{N}_b^{\text{ac}}$, every $Z \in \mathcal{N}_{\text{AG}}^{\text{ac}}$ with $Z \subseteq Y$, and every $n \in \{1, 2, 3\}$, let β_{n-Y-Z} be an available action of b such that $\text{out}_b(\beta_{n-Y-Z}) = Y$.

The reason that we need three levels of individual actions will be given later.

Step 2: enforcing the GCI-condition for names of individual powers. For each $X \in \mathcal{N}_a^{\text{ac}}$ define $\Delta_X = \{Z \in \mathcal{N}_{\text{AG}}^{\text{ac}} \mid Z \subseteq X\}$, and for each $Y \in \mathcal{N}_b^{\text{ac}}$ define $\Delta_Y = \{Z \in \mathcal{N}_{\text{AG}}^{\text{ac}} \mid Z \subseteq Y\}$.

Note that by Fact 4.6, $\bigcup \Delta_X = X$ for each $X \in \mathcal{N}_a^{\text{ac}}$ and $\bigcup \Delta_Y = Y$ for each $Y \in \mathcal{N}_b^{\text{ac}}$.

1. **Make all Group 1 names of a satisfy the GCI-condition.** Pick an available action α_{1-X-Z} of a . For every $Z' \in \Delta_X$, choose some $Y \in \mathcal{N}_b^{\text{ac}}$ such that $Z' \subseteq Y$, and declare $(\alpha_{1-X-Z}, \beta_{2-Y-Z'})$ to be an available action of AG with

$$\text{out}_{\text{AG}}(\alpha_{1-X-Z}, \beta_{2-Y-Z'}) = Z'.$$

Note that such a Y exists for every $Z' \in \Delta_X$, by actual power inclusion in $\mathcal{N}$. Note that for every $Z' \in \Delta_X$, $\beta_{2-Y-Z'}$ is a Group 2 action of b . After this, every $Z' \in \Delta_X$ has a name extending α_{1-X-Z} , and α_{1-X-Z} satisfies the GCI-condition. Repeat this procedure for any remaining Group 1 action of a not yet satisfying the GCI-condition.

2. **Make all Group 1 names of b satisfy the GCI-condition.** Proceed analogously for Group 1 actions of b not yet satisfying the GCI-condition. Here we only pair such actions of b with Group 2 actions of a .

3. **Make all Group 2 names of a satisfy the GCI-condition.** Proceed analogously. *Here we only pair such actions of a with Group 3 actions of b . Note that the pairing is safe.*
4. **Make all Group 2 names of b satisfy the GCI-condition.** Proceed analogously. *Here we only pair such actions of b with Group 3 actions of a . Note that the pairing is safe.*
5. **Make all Group 3 names of a satisfy the GCI-condition.** Proceed analogously. *Note that at this point, no Group 1 action of b has been paired with a Group 3 action of a . Here we only pair Group 3 actions of a with Group 1 actions of b . Note that the pairing is safe.*
6. **Make all Group 3 names of b satisfy the GCI-condition.** Proceed analogously. *Note that at this point, no Group 1 action of a has been paired with a Group 3 action of b . Here we only pair Group 3 actions of b with Group 1 actions of a . Note that the pairing is safe.*

We now explain why three levels of individual actions are used. The idea behind Step 2 is to make the names of individual powers satisfy the GCI-condition one level at a time. For a name α of an individual power, this means ensuring that its outcome set is exactly the union of the outcome sets of its available joint extensions. To achieve this for names at one level, we add joint extensions using names of the other agent at the next level: Group 1 names are handled using Group 2 names, and Group 2 names are handled using Group 3 names. If this procedure were continued linearly, it would seem to require a further level to handle Group 3 names. The point of the three-level construction is that the process closes cyclically. When Group 3 names are treated, the corresponding Group 1 names of the other agent are already available and already satisfy their own GCI-condition, and they have not yet been paired with the Group 3 names currently under consideration. We can therefore pair Group 3 names with Group 1 names. These new pairings are safe, so they do not destroy the GCI-condition already achieved for the Group 1 names. Thus no fourth level is needed.

Step 3: pair names of individual powers whenever possible. Pick any available actions $\alpha_{n-X-Z'}$ of a and $\beta_{m-Y-Z''}$ of b (at arbitrary levels $n, m \in \{1, 2, 3\}$) such that:

- They have not been paired yet, and
- There exists $Z \in N_{AG}^{ac}$ with $Z \subseteq \text{out}_a(\alpha_{n-X-Z'}) \cap \text{out}_b(\beta_{m-Y-Z''})$.

Choose such a Z , and declare $(\alpha_{n-X-Z'}, \beta_{m-Y-Z''})$ to be an available action of AG with

$$\text{out}_{AG}(\alpha_{n-X-Z'}, \beta_{m-Y-Z''}) = Z.$$

Repeat for all remaining pairs satisfying the conditions above.

Final local game. Finally, define $\mathcal{G} = (AC, \{\text{av}_C \mid C \subseteq AG\}, \{\text{out}_C \mid C \subseteq AG\})$ as follows:

- Let AC be the set of all actions specified at Step 1.
- Let $\text{av}_\emptyset = \{\emptyset\}$ and $\text{out}_\emptyset(\emptyset) = \bigcup N_{AG}^{ac}$.
- Let $\text{av}_a, \text{out}_a$ be as specified above; similarly for $\text{av}_b, \text{out}_b$; and let $\text{av}_{AG}, \text{out}_{AG}$ be as specified above.

We verify the ODA-condition and GCI-condition for $\mathcal{G}$.

- For the ODA-condition, first consider the nontrivial case. Every individual action declared available in Step 1 has an outcome $X \in N_a^{\text{ac}}$ or $Y \in N_b^{\text{ac}}$, and every grand-coalition action declared available in Steps 2 and 3 has an outcome $Z \in N_{\text{AG}}^{\text{ac}}$. All these outcome sets are nonempty by liveness. Moreover,

$$\text{out}_\emptyset(\emptyset) = \bigcup N_{\text{AG}}^{\text{ac}} \neq \emptyset,$$

so $\emptyset \in \text{av}_\emptyset$. Conversely, every action or joint action not explicitly declared available is assigned the empty outcome. Hence, for every $C \subseteq \text{AG}$,

$$\text{av}_C = \{\sigma_C \in \text{JA}_C \mid \text{out}_C(\sigma_C) \neq \emptyset\}.$$

In the trivial case, all availability sets and outcome sets are empty, so the same equality holds.

- For the GCI-condition, the case $C = \text{AG}$ is immediate, since a grand-coalition action has only itself as a grand-coalition extension. Now let $\alpha_{n-X-Z} \in \text{av}_a$. In Step 2, extensions of this action are introduced whose outcome sets cover every member of

$$\Delta_X = \{Z' \in N_{\text{AG}}^{\text{ac}} \mid Z' \subseteq X\}.$$

Since $\bigcup \Delta_X = X$ by Fact 4.6, the union of the outcomes of these extensions is $\text{out}_a(\alpha_{n-X-Z}) = X$. Every extension added in Step 3 is safe, and therefore has its outcome contained in X ; such extensions do not change this union. The argument for available actions of b is symmetric. If an individual action is unavailable, no available grand-coalition action extends it, and both sides of the GCI equality are empty.

Finally, consider $C = \emptyset$. Every available grand-coalition action has an outcome in $N_{\text{AG}}^{\text{ac}}$. Conversely, every $Z \in N_{\text{AG}}^{\text{ac}}$ occurs as the outcome of some available grand-coalition action: by actual power inclusion, choose $X \in N_a^{\text{ac}}$ with $Z \subseteq X$; Step 2 assigns outcome Z to an extension of α_{1-X-Z} . Consequently,

$$\bigcup_{\sigma_{\text{AG}} \in \text{JA}_{\text{AG}}} \text{out}_{\text{AG}}(\sigma_{\text{AG}}) = \bigcup N_{\text{AG}}^{\text{ac}} = \text{out}_\emptyset(\emptyset).$$

Thus the GCI-condition holds for every coalition. Hence $\mathcal{G}$ satisfies the defining conditions of a local general concurrent game frame.

Hence $\mathcal{G}$ is a game.

We show that $\mathcal{G}$ is AC-represented by $\mathcal{N}$. It suffices to show that for every $C \subseteq \text{AG}$,

$$\{\text{out}_C(\sigma_C) \mid \sigma_C \in \text{av}_C\} = N_C^{\text{ac}}.$$

The inclusion “ $\subseteq$ ” is immediate from the construction. For $C = \emptyset$, by the definition of the final local game, we have the following equalities: $\{\text{out}_\emptyset(\sigma_\emptyset) \mid \sigma_\emptyset \in \text{av}_\emptyset\} = \{\text{out}_\emptyset(\emptyset)\} = \left\{ \bigcup N_{\text{AG}}^{\text{ac}} \right\} = N_\emptyset^{\text{ac}}$, where the last equality follows from Fact 4.5. For $C = \{a\}$ or $C = \{b\}$, the reverse inclusion is also immediate from Step 1. It remains to show the reverse inclusion for $C = \text{AG}$, i.e., that every $Z \in N_{\text{AG}}^{\text{ac}}$ has a name.

Let $Z \in N_{\text{AG}}^{\text{ac}}$. By actual power inclusion, $Z \subseteq X$ for some $X \in N_a^{\text{ac}}$, hence $Z \in \Delta_X$. Consider the action α_{1-X-Z} introduced in Step 1. In Step 2 we choose some $Y \in N_b^{\text{ac}}$ with $Z \subseteq Y$ and declare $(\alpha_{1-X-Z}, \beta_{2-Y-Z})$ to be an available action of AG with $\text{out}_{\text{AG}}(\alpha_{1-X-Z}, \beta_{2-Y-Z}) = Z$. Thus Z has a name, as required. Therefore, $\mathcal{G}$ is AC-represented by $\mathcal{N}$.

We show that $\mathcal{G}$ has the properties corresponding to X .

- **Seriality.** Assume $S \in X$. Then NF^{ac} is AC-serial, hence $N_C^{ac} \neq \emptyset$ for all $C \subseteq AG$. In particular, pick $X \in N_a^{ac}$ and choose $Z \in N_{AG}^{ac}$ with $Z \subseteq X$. Then α_{1-X-Z} is available for a , so $av_a \neq \emptyset$. Similarly, $av_b \neq \emptyset$. Also, since $N_{AG}^{ac} \neq \emptyset$, there exists $\sigma_{AG} \in av_{AG}$ with $out_{AG}(\sigma_{AG}) \in N_{AG}^{ac}$, hence $av_{AG} \neq \emptyset$. Finally, $av_\emptyset = \{\emptyset\}$ by definition. Therefore $av_C \neq \emptyset$ for all $C \subseteq AG$.
- **Independence.** Assume $I \in X$. Then NF^{ac} is AC-independent. Let $C, D \subseteq AG$ with $C \cap D = \emptyset$, and let $\sigma_C \in av_C$ and $\sigma_D \in av_D$. We show that $\sigma_C \cup \sigma_D \in av_{C \cup D}$.
If $C = \emptyset$ or $D = \emptyset$, the claim is immediate. Otherwise, since $AG = \{a, b\}$, both C and D are singletons; without loss of generality let $C = \{a\}$ and $D = \{b\}$. Write $\sigma_a = \alpha$ and $\sigma_b = \beta$, and let $out_a(\alpha) = X$ and $out_b(\beta) = Y$.
By AC-independence of NF^{ac} , there exists $Z \in N_{AG}^{ac}$ such that $Z \subseteq X \cap Y$.
If (α, β) was already declared available in Step 2, then $(\alpha, \beta) \in av_{AG}$. Otherwise, the existence of such a Z shows that the unpaired actions α and β satisfy the criterion of Step 3. Hence Step 3 declares (α, β) available and assigns it some outcome $Z' \in N_{AG}^{ac}$ with $Z' \subseteq X \cap Y$. Thus, in either case, $\sigma_a \cup \sigma_b = (\alpha, \beta) \in av_{AG} = av_{C \cup D}$. Therefore $\mathcal{G}$ satisfies independence.
- **Determinism.** Assume $D \in X$. Then NF^{ac} is AC-deterministic, so every $Z \in N_{AG}^{ac}$ is a singleton. By construction, for every $\sigma_{AG} \in av_{AG}$ we have $out_{AG}(\sigma_{AG}) \in N_{AG}^{ac}$, hence $out_{AG}(\sigma_{AG})$ is a singleton.

□

Remark The conditions used above remain necessary for arbitrary finite agent sets, but they cease to be sufficient once three agents are present. Actual power inclusion and actual power decomposition constrain powers only along inclusions of coalitions. With three agents, however, incomparable coalitions may overlap. For example, the restrictions of a full joint action to $\{a, b\}$ and $\{a, c\}$ must agree on their common a -component; consequently, their outcome sets must both be contained in the outcome set induced by the same action of a . The conditions used here do not ensure that the witnesses along these two inclusion chains can be chosen compatibly. This obstruction does not arise with two agents, since no two incomparable proper nonempty coalitions have a nonempty intersection. Thus, a general finite-agent representation theorem requires additional coherence conditions for the actual powers of overlapping coalitions. A concrete three-agent counterexample is given in the full paper [5].

5 Concluding remarks

Concurrent game frames are the most widely used semantic frames for logics of strategic reasoning. Li and Ju [8, 9] argued that standard concurrent game frames rely on three assumptions that may be overly strong, namely seriality, independence of agents, and determinism. They therefore proposed eight classes of general concurrent game frames, distinguished by which of these three properties are satisfied. In this work, under the assumption that there are two agents, we showed that with respect to actual powers, the eight classes of general concurrent game frames are represented by eight corresponding classes of actual neighborhood frames. This generalizes the result in [3], which proves a partial representation theorem, for singleton coalitions only and not for the empty or grand coalitions, for the most standard case.

The preceding remark identifies the obstruction to extending the enoughtness direction beyond two agents. A natural next step is to formulate appropriate general finite-agent coherence conditions and seek a genuine general finite-agent representation theorem for actual powers.

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